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# STABILITY SWITCHES OF TIME-DELAYED DYNAMIC SYSTEMS WITH UNKNOWN PARAMETERS

Z. H. WANG AND H. Y. HU

*Institute of Vibration Engineering Research, Nanjing University of Aeronautics and Astronautics,* 210016 *Nanjing, People's Republic of China*

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The paper presents a systematic method of stability analysis for high-dimensional dynamic systems involving a time delay and some unknown parameters. Here, the term "unknown" means that the parameters are constants but yet to be determined. The analysis focuses on the stability switches of those systems with increase of the time delay from zero to infinity. On the basis of the generalized Sturm criterion, the parameter space of concern is divided into several regions determined by a discrimination sequence and the Routh–Hurwitz conditions. It is found, as the time delay increases, that the system may undergo no stability switch, exactly one stability switch, or more than one stability switches when the parameters are chosen from different regions. To demonstrate the approach, a detailed analysis of the stability switches is made in the paper for an active vehicle suspension equipped with a delayed "sky-hook" damper and a four-wheel steering vehicle with time delay in driver's response, respectively.



## 1. INTRODUCTION

Over the last few decades, a great number of studies have been carried out on the stability analysis of dynamic systems involving various time delays, which are usually caused by controllers, actuators, human-beings, etc. The studies indicate that on the one hand, the time delay may give rise to retardation or even instability for a system. On the other hand, an appropriate time delay can improve system performance, or even stabilize an unstable system, see for example, references [1–5]. Thus, it is very desirable to gain an insight into the relations between the stability and the time delay for any practical systems.

From a practical point of view, a system usually involves a number of parameters, besides a time delay, to be designed so as to render the systems asymptotically stable. If the time delay is given, some approaches, say, the D-subdivision method, are available to determine the stable regions of the system in the parameter space. However, an unknown time delay would make the stability analysis very difficult.

Alternatively, one can focus on a relatively simple problem, i.e., delay-independent stability analysis. As shown in the authors' earlier paper [1], practical and concise criteria do exist for the delay-independent stability of a delayed dynamic system, even if the system is high dimensional. On the basis of the generalized Sturm theory developed in references [6, 7], a systematic approach is proposed in paper [1] to derive the sufficient and necessary conditions for the delay-independent stability of linear dynamic systems of multiple degrees of freedom. A graphical method is also presented to achieve the delay-independent stable region in the parameter space of concern. Once the so-called "discrimination sequence" is

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obtained by using a short MAPLE routine, the sufficient and necessary conditions for delay-independent stability can be characterized by the sign tables of the discrimination sequence. In addition, the delay-independent stable region can be obtained by plotting the graphs, which divide the parameter space of concern into several regions, of some functions related to the discrimination sequence and by checking the number of variations of sign tables of the discrimination sequence in each region. In practical applications, the delay-independent stable region is usually a very small part in the parameter space. Then, a question arises. How does a delayed dynamic system behave if the system parameters are chosen from other regions in the parameter space?

The aim of the present paper is to address this question. In what follows, the characteristic roots of the system will be regarded as the functions of a time delay. As the delay increases, the stability of the system changes. Such a phenomenon is often referred to as stability switches and has been discussed for some delayed dynamic systems of low order [2]. The contribution of this paper is the idea of using stability switches to analyze the stability of delayed systems with unknown parameters. What will be emphasized in the paper is, as the time delay increases from zero to infinity, how to gain a good understanding for the parameter space concerned and to give a simple approach to complete the stability analysis as done by using the D-subdivision method. The method proposed in the paper enables one to achieve some analytical results for the stability switches of high-dimensional systems with a time delay.

The paper is organized as follows. Section 2 begins with an analysis of stability switches of linear, high-dimensional, dynamic systems with a time delay. The number of stability switches is dependent on the number of real roots of a polynomial, which can be derived by using the classical Sturm criterion if the system parameters are given. However, the classical Sturm criterion does not work if the system involves any unknown parameters. In this case, the generalized Sturm criterion proves to be a useful tool. Hence, a brief introduction of the generalized Sturm criterion is made in Section 2. Then in Section 3, two examples in the dynamics of ground car are given to demonstrate the method. Finally in Section 4, several conclusions are drawn from the discussion.

## 2. THE METHOD

To make the exposition as simple as possible, attention hereafter is paid to the linear dynamic systems of multiple degrees of freedom, involving a single constant time delay only. Those systems are governed by the characteristic equation in a compact form

$$D(\lambda, \tau) \equiv P(\lambda) + Q(\lambda)\exp(-\lambda\tau) = 0, \tag{1}$$

where $\tau \geqslant 0$ is the time delay, $P(\lambda)$ and $Q(\lambda)$ are two polynomials of real coefficients, and $\deg(P) = n > \deg(Q)$. As is well known, the system concerned is asymptotically stable for given $\tau \geqslant 0$ if and only if each of the characteristic roots has a negative real part. Thus the marginal stability is determined by $D(\mathrm{i}\omega, \tau) = 0$. Let

$$P_R(\omega) = \mathrm{Re}(P(\mathrm{i}\omega)), \quad P_I(\omega) = \mathrm{Im}(P(\mathrm{i}\omega)), \tag{2a}$$

$$Q_R(\omega) = \mathrm{Re}(Q(\mathrm{i}\omega)), \quad Q_I(\omega) = \mathrm{Im}(Q(\mathrm{i}\omega)). \tag{2b}$$

From $D(\mathrm{i}\omega, \tau) = 0$, one arrives at

$$Q_R\cos(\omega\tau) + Q_I\sin(\omega\tau) = -P_R, \quad Q_I\cos(\omega\tau) - Q_R\sin(\omega\tau) = -P_I. \tag{3}$$

In order that equation (1) has a pair of pure imginary roots $\pm i\omega$ for given $\tau \geqslant 0$, it is necessary that $|P(i\omega)| = |Q(i\omega)|$ or

$$P_R^2 + P_I^2 - (Q_R^2 + Q_I^2) = 0 \tag{4}$$

has a positive root $\omega$. The left-hand side of equation (4) can simply be rewritten in the more explicit form

$$F(\omega) \equiv \omega^{2n} + b_1\omega^{2(n-1)} + b_2\omega^{2(n-2)} + \cdots + b_{n-1}\omega^2 + b_n. \tag{5}$$

Once a positive root $\omega$ is found, the corresponding values of critical time delay are given by

$$\tau_k = \frac{\theta}{\omega} + \frac{2k\pi}{\omega}, \quad k = 0, 1, 2, \ldots \tag{6a}$$

for a $\theta \in [0, 2\pi)$, which should satisfy

$$\sin\theta = \frac{-P_R Q_I + P_I Q_R}{Q_R^2 + Q_I^2}, \quad \cos\theta = \frac{-P_R Q_R - P_I Q_I}{Q_R^2 + Q_I^2}, \tag{6b}$$

where $Q_R^2 + Q_I^2 \neq 0$ is assumed.

### 2.1. STABILITY ANALYSIS VIA STABILITY SWITCHES

If $F(\omega) = 0$ has no positive roots, the system does not undergo any stability switch. That is, the system is delay-independent stable if it is asymptotically stable when the time delay disappears, or unstable for an arbitrary time delay if the system free of delay is unstable. A detailed analysis of delay-independent stability has been made in reference [1].

When $F(\omega) = 0$ has any positive roots, we regard the root $\lambda$ of equation (1) as a function of delay $\tau$. Differentiating equation (1) with respect to $\tau$ gives

$$\frac{d\lambda}{d\tau} = \frac{\lambda Q(\lambda)}{P'(\lambda)\exp(\lambda\tau) + Q'(\lambda) - \tau Q(\lambda)}. \tag{7}$$

Once we find a pair of conjugate pure imaginary characteristic roots $\pm i\omega$ with critical time delays $\tau$ satisfying equation (6), we can determine the moving direction of its real part as $\tau$ is varied, namely, the sign $S = \operatorname{sgn}\{\operatorname{Re}(d\lambda(\tau)/d\tau)|_{\lambda = i\omega}\}$. In order that $S$ is well defined, we require $P'(i\omega)\exp(i\omega\tau) + Q'(i\omega) - \tau Q(i\omega) \neq 0$, i.e., $\pm i\omega$ are not repeated characteristic roots. Note that

$$\operatorname{sgn}\left(\operatorname{Re}\left(\frac{1}{a + bi}\right)\right) = \operatorname{sgn}\left(\left(\frac{a - bi}{a^2 + b^2}\right)\right) = \operatorname{sgn}(\operatorname{Re}(a + bi)), \quad (a, b \in R), \tag{8a}$$

$$\frac{d\tau}{d\lambda} = \left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{P'(\lambda)}{\lambda P(\lambda)} + \frac{Q'(\lambda)}{\lambda Q(\lambda)} - \frac{\tau}{\lambda}, \tag{8b}$$

we have

$$S = \operatorname{sgn}\left\{\operatorname{Re}\left(\frac{d\lambda(\tau)}{d\tau}\right)\bigg|_{\lambda = i\omega}\right\} = \operatorname{sgn}\left\{\operatorname{Re}\left(\frac{d\tau}{d\lambda}\right)\bigg|_{\lambda = i\omega}\right\}$$

$$= \operatorname{sgn}\left\{\operatorname{Re}\left(-\frac{P'(\mathrm{i}\omega)}{\mathrm{i}\omega P(\mathrm{i}\omega)}+\frac{Q'(\mathrm{i}\omega)}{\mathrm{i}\omega Q(\mathrm{i}\omega)}\right)\right\} = -\operatorname{sgn}\left\{\operatorname{Im}\left(\frac{P'(\mathrm{i}\omega)}{\omega P(\mathrm{i}\omega)}-\frac{Q'(\mathrm{i}\omega)}{\omega Q(\mathrm{i}\omega)}\right)\right\}$$

$$= -\operatorname{sgn}\left\{\operatorname{Im}\left(\frac{P'(\mathrm{i}\omega)\bar{P}(\mathrm{i}\omega)}{\omega|P(\mathrm{i}\omega)|^2}-\frac{Q'(\mathrm{i}\omega)\bar{Q}(\mathrm{i}\omega)}{\omega|Q(\mathrm{i}\omega)|^2}\right)\right\}$$

$$= -\operatorname{sgn}\{\operatorname{Im}(P'(\mathrm{i}\omega)\bar{P}(\mathrm{i}\omega)-Q'(\mathrm{i}\omega)\bar{Q}(\mathrm{i}\omega))\}, \tag{9}$$

where " $^{-}$ " denotes complex conjugate. Observe that

$$-\{\operatorname{Im}(P'(\mathrm{i}\omega)\bar{P}(\mathrm{i}\omega)-Q'(\mathrm{i}\omega)\bar{Q}(\mathrm{i}\omega))\} = P_R P_R' + P_I P_I' - Q_R Q_R' - Q_I Q_I', \tag{10a}$$

$$F'(\omega) = 2(P_R P_R' + P_I P_I' - Q_R Q_R' - Q_I Q_I'), \tag{10b}$$

we have

$$S = \operatorname{sgn} F'(\omega). \tag{11}$$

Now, we study the case when $F(\omega)=0$ has exactly one simple positive $\omega_0$ with the critical delay values given by equations (6a) and (6b). Since the leading coefficient of $F(\omega)$ is positive, it follows that $F(\omega) > F(\omega_0) = 0$ for all $\omega > \omega_0$, and $F(\omega) < F(\omega_0) = 0$ for $\omega \in [0, \omega_0)$. Thus, one must have $F'(\omega_0) > 0$. This indicates that each crossing of the real part of characteristic roots at $\tau_k$'s (corresponding to $\pm \mathrm{i}\omega_0$) must be from left to right. Thus, the characteristic equation of system has a new pair of conjugate roots with positive real parts when the time delay $\tau$ is crossing each critical value $\tau_k$ of time delay, and the number of characteristic roots with positive real parts cannot decrease as the time delay increases. Hence, if the system without time delay is asymptotically stable, the numbers of characteristic roots with positive real parts are $0, 2, 4, \ldots, 2i, \ldots$ in the intervals $[0, \tau_0)$, $(\tau_0, \tau_1)$, $(\tau_1, \tau_2)$, $\ldots, (\tau_{i-1}, \tau_i)$, $\ldots$ respectively. This means that the system is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for all $\tau \geqslant \tau_0$. If the system which is free of time delay is unstable, then there exists at least one pair of conjugate characteristic roots with positive real part for $\tau \in [0, \tau_0)$. As a result, the system must be unstable for all $\tau \geqslant 0$.

Then, we discuss the case when $F(\omega)=0$ has simple positive roots $\omega_1 > \omega_2 > \cdots > \omega_p > 0$. The difference between two critical values of time delay corresponding to a given pair of roots $\pm \mathrm{i}\omega_j$ satisfies

$$\tau_{j,k+1} - \tau_{j,k} = \frac{2\pi}{\omega_j} < \frac{2\pi}{\omega_{j+1}} = \tau_{j+1,k+1} - \tau_{j+1,k} \quad (k = 0, 1, 2, \ldots, \quad j = 1, 2, \ldots, p-1). \tag{12}$$

The crossing of real parts of characteristic roots at two adjacent simple roots $\omega_j$ and $\omega_{j+1}$ must be in opposite directions, since $F'(\omega_j)$ and $F'(\omega_{j+1})$ have opposite signs. Actually, we must have $\operatorname{sgn} F'(\omega_{2j-1}) > 0$ and $\operatorname{sgn} F'(\omega_{2j}) < 0$ $(j \geqslant 1)$ since $F(\omega) > F(\omega_1) = 0$ for all $\omega \in (\omega_1, +\infty)$ and all possible $(\omega_{2k+1}, \omega_{2k})$, but $F(\omega) < F(\omega_1) = 0$ for all possible $(\omega_{2k}, \omega_{2k-1})$. That is to say, the crossing of real parts of characteristic roots at $\tau_{2j-1,k}$ (corresponding to $\pm \mathrm{i}\omega_{2j-1}$) must be from left to right, and the crossing at $\tau_{2j,k}$ (corresponding to $\pm \mathrm{i}\omega_{2j}$) must be from right to left. Therefore, as the time delay varies from zero to infinity, the characteristic equation of the system always adds a new pair of conjugate characteristic roots with positive real parts for each crossing of time delay at $\tau_{2j-1,k}$, but removes such a pair for each crossing of time delay at $\tau_{2j,k}$. Considering equation (12), we can find that more characteristic roots change their sign of real parts from negative to positive at $\tau_{1,k}$ (corresponding to $\pm i\omega_1$) than those from positive to negative at $\tau_{2,l}$ (corresponding to $\pm i\omega_2$). A similar assertion holds true for the delay crossing at $\tau_{3,m}$ and $\tau_{4,n}$ (corresponding to $\pm i\omega_3$ and $\pm i\omega_4$), and so on. Hence, the characteristic equation of system must have eventually some roots of positive real parts when the time delay increases. That is, the system must finally be unstable with increase in time delay, and the number of stability switches must be finite.

In summary, we have the following theorem.

**Theorem 1.** *Assume that equation* (1) *has no pure imaginary characteristic roots satisfying* $Q(i\omega) = 0$; *then, the following statements are true.*

(a) *If* $F(\omega) = 0$ *has no positive root, the system is delay-independent stable or unstable for any given time delay, depending on whether the system which is free of time delay is stable or not.*
(b) *Assume that* $F(\omega) = 0$ *has only on simple positive root* $\omega$. *If the system free of time delay is asymptotically stable, then there exists exactly one* $\tau_c > 0$ *such that the system remains asymptotically stable when* $\tau \in [0, \tau_c)$, *and becomes unstable when* $\tau \geqslant \tau_0$. *If the system is unstable for* $\tau = 0$, *it is unstable for arbitrary time delay.*
(c) *If* $F(\omega) = 0$ *has at least two positive roots* $\omega_1 > \omega_2 > \cdots > \omega_p > 0$ *and the roots are simple, then a finite number of stability switches may occur as* $\tau$ *increases from zero to infinity, and the system becomes unstable finally.*

If the system is of one or two dimensions, the analysis on the number of real roots of $F(\omega)$ is considerably easy. When it is high dimensional and involves some unknown parameters, however, pure numerical consideration is time-consuming. In this case, it is better to use some analytical tools to determine the number of real roots of $F(\omega)$ with unknown parameters. The generalized Sturm criterion serves this purpose effectively.

### 2.2. A BRIEF REVIEW ON THE GENERALIZED STURM CRITERION

The generalized Sturm criterion is a useful tool to determine the number of real roots of polynomial $F(\omega)$ with unknown parameters. In the generalized Sturm criterion, the so-called "discrimination sequence" plays a role as important as that of the Sturm sequence, which works effectively for the polynomials with given parameters.

Let $f(x)$ be a real polynomial of order $n$. We first define the Bezout matrix (cf. references [6, 7]) of $f(x)$ and $f'(x)$ as the discrimination matrix of $f(x)$ and denote it by $\mathrm{discr}(f)$. Then, we define the discrimination sequence of $f(x)$ as the principal sub-determinant sequence of $\mathrm{discr}(f)$ taken in order, and denote it by

$$D_1(f),\ D_2(f),\ \ldots,D_n(f). \tag{13}$$

This sequence can be simply obtained by using a short MAPLE routine *discr* in Appendix A.

Given a real number sequence $x_1,\ x_2,\ \ldots,x_n$ $(x_1 \neq 0)$, the sign sequence $[s_1, s_2, \ldots, s_n]$, with $s_i = \mathrm{sign}(x_i)$, $(i = 1, 2, \ldots, n)$, is called the sign table of the original sequence. The modified sign table $[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n]$ can be written down by following the two rules given below:

(1) For any segment $[s_i, s_{i+1}, s_{i+2}, \ldots, s_{i+j}]$ with $s_i \neq 0$, $s_{i+1} = s_{i+2} \cdots = s_{i+j-1} = 0$ and $s_{i+j} \neq 0$ of a given sign table, $[s_{i+1}, s_{i+2}, \ldots, s_{i+j-1}]$ is replaced by $[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, \ldots]$.
(2) All the other terms in the table are kept unchanged.

Then, the generalized Sturm criterion can be stated as follows.

**Theorem 2.** *Let $f(x)$ be a polynomial of order $n$ and $D_1(f)$, $D_2(f)$, $\ldots, D_n(f)$ be the corresponding discrimination sequence. Assume that the number of variations of signs in the modified sign table is $v$, and $l$ is the positive integer satisfying $D_l(f) \neq 0$, $D_m(f) = 0$ $(m > l)$; then* (a) *the number of distinct pairs of conjugate complex roots of $f(x)$ is $v$*; (b) *the number of distinct real roots of $f(x)$ is $l \div 2v$*; (c) *in order that the function $f(x)$ has repeated roots, it is necessary that $l < n$.*

### 2.3. PREDICTION OF STABILITY SWITCHES IN PARAMETER SPACE

On the basis of the above discussion, the analysis on stability switches for delayed dynamic systems with unknown parameters can be completed as follows. First, find out the characteristic function and the corresponding function $F(\omega)$. Then, run the MAPLE routine *discr* to obtain the discrimination sequence of $F(\omega)$ and the factors $d_i$. Afterwards, divide the parameter region concerned into some sub-regions by plotting the graphs of $d_i$ and the curves determined by Routh–Hurwitz conditions corresponding to $\tau = 0$. Finally, use Theorem 2 (the generalized Sturm criterion) to check the number of real roots of $F(\omega)$ and use Theorem 1 to predict the stability switches in the parameter space.

## 3. STABILITY SWITCHES OF GROUND VEHICLES

In this section, the stability of two models of ground vehicle is analyzed to demonstrate how to investigate the stability switches of delayed systems with unknown parameters. One is the active suspension of a quarter car model equipped with a "sky-hook" damper; the other is a four-wheel steering car with a time delay in driver's response taken into account. The stability of these two models has been reported in a number of previous publications, but mainly limited to relatively simple cases. For example, the stability of an undamped quarter car model with active suspension was analyzed for a given time delay in reference [9] by using the D-subdivision method. For the quarter car model equipped with a passive damper and a "sky-hook" damper, the delay-independent stability was given in reference [1]. Numerical analysis was made in reference [10] to investigate the stability and Hopf bifurcation for the four-wheel steering vehicle with given parameters. In this paper, attention is paid to the analysis of stability switches for different parameter combinations.

### 3.1. ACTIVE SUSPENSION FOR A QUARTER CAR MODEL WITH "SKY-HOOK" DAMPER

We first consider the stability switches of a linearized quarter car model equipped with an active suspension. The equation of motion of the system yields

$$m_b x'' + \bar{c}_s(x' - y') + \bar{k}_s(x - y) + u = 0,$$
$$m_t y'' - \bar{c}_s(x' - y') - \bar{k}_s(x - y) - u + \bar{k}_t(y - f) = 0, \tag{14}$$

where $(\ )' \equiv d(\ )/d\bar{t}$, $x$ is the vertical displacement of the vehicle body $m_b$, $y$ the vertical displacement of the unsprung mass $m_t$, $f$ is the road disturbance, $\bar{c}_s \geqslant 0$, the $\bar{k}_s \geqslant 0$ and $\bar{k}_t \geqslant 0$ are the damping coefficient, the stiffness of suspension, and the stiffness of the tyre respectively. The system parameters are given as follows: $m_b = 290$ kg, $m_t = 59$ kg, $\bar{c}_s = 0 \sim 980$ N s/m, $\bar{k}_s = 16\,812$ N/m, $\bar{k}_t = 19\,000$ N/m.

To reduce the vibration of the vehicle body and the suspension deformation as well, an active control force $u$ is introduced through linear feedback of vehicle body velocity. This control strategy is called "sky-hook" damper. From a practical point of view, the above feedback involves a time delay caused mainly by the hydraulic actuator. Thus, the control force generated by the sky-hook damper takes the form

$$u = \bar{g}x'(\bar{t} - \bar{\tau}), \tag{15}$$

where the feedback gain $|\bar{g}|$ is approximately within 0–2000 (N s/m).

Let $w_s = \sqrt{\bar{k}_s/m_b}$, $\beta = m_b/m_t$, $\tau = \bar{\tau}w_s$, $k_t = \bar{k}_t/\bar{k}_s$, $c = \bar{c}_s/\sqrt{m_b\bar{k}_s}$, and $g = \bar{g}/\sqrt{m_b\bar{k}_s}$. One can cast equation (14) into

$$\ddot{x} + c_s(\dot{x} - \dot{y}) + (x - y) + g\dot{x}(t - \tau) = 0,$$

$$\ddot{y} - c\beta(\dot{x} - \dot{y}) - \beta(x - y) + k_t\beta y - g\beta\dot{x}(t - \tau) = 0, \tag{16}$$

where the dot represents the derivative with respect to the new time $t = \bar{t}/w_s$. The dimensionless parameters of the vehicle become

$$\beta = 4{\cdot}9153, \quad k_t = 1{\cdot}1301, \quad c = 0\text{–}0{\cdot}4438, \quad g = 0\text{–}0{\cdot}9078. \tag{17}$$

The characteristic equation of equation (16) reads

$$D(\lambda, \tau) \equiv \lambda^4 + c(1 + \beta)\lambda^3 + (1 + \beta + k_t\beta)\lambda^2 + ck_t\beta\lambda + k_t\beta + g\lambda(\lambda^2 + k_t\beta)\exp(-\lambda\tau) = 0. \tag{18}$$

Obviously, one has

$$D(\lambda, 0) = \lambda^4 + (c(1 + \beta) + g)\lambda^3 + (1 + \beta + k_t\beta)\lambda^2 + k_t\beta(c + g)\lambda + k_t\beta = 0. \tag{19}$$

From the Routh–Hurwitz criterion, one can readily know that $D(\lambda, 0)$ is Hurwitz stable if and only if

$$g + c > 0, \quad g^2 + c\beta(1 + k_t)g + cg + c^2\beta k_t > 0. \tag{20}$$

When $\tau > 0$, the function $F(\omega)$ is in the form

$$F(\omega) = \omega^8 + b_1\omega^6 + b_2\omega^4 + b_3\omega^2 + b_4. \tag{21}$$

By using *discr*, one obtains the discrimination sequence

$$1, \quad d_0, \quad d_0d_1, \quad d_1d_2, \quad d_2d_3, \quad d_3d_4, \quad d_4d_5, \quad d_5^2d_6, \tag{22}$$

where the coefficients in equation (21) and the expressions of $d_i$ in equation (22) are listed in Appendices B and C respectively. In what follows, the stability switches of this quarter car model will be discussed for different combinations of passive damping ratio $c$ and feedback gain $g$.

3.1.1. *Analytic results*

To keep the discussion broad enough, we consider a larger region of $\Omega = \{(g, c): |g| < 1, 0 < c < 0{\cdot}5\}$ than the parameter combinations in equation (22). As $c$ is varied from 0 to 0·5, $d_0 > 0$ and $d_6 > 0$ always hold. In $\Omega$, the graph of $d_5 = 0$ is composed of two V-shaped curves. The curves determined by $d_3 = 0$ and $d_4 = 0$ coincide with the wide V-shaped curve. The graphs of $d_1 = 0$ and $d_2 = 0$ intersect with the narrow V-shaped curve respectively. As shown in Figure 1, the region $\Omega$ is divided into 11 sections, which are numbered from the left to the right and from the top to the bottom by 1, 2, ..., 11, respectively.

The second condition in equation (20) is now in the form: $g > -0{\cdot}5067c$ or $g < -10{\cdot}9634c$. As the conditions $g > -c$ and $g < -10{\cdot}9634c$ cannot hold true at the same time, the Routh–Hurwitz condition is simplified to

$$g > -0{\cdot}5067c. \tag{23}$$

The graph, denoted by RHb, of $g = -0{\cdot}5067c$ coincides with the left boundary of the narrow V-shaped region. This implies that the system which is free of time delay is asymptotically stable when the parameters are chosen from the sub-regions of 2, 3, 5, 6, 8, 9 and 11, and becomes unstable if the parameter combinations are taken from the sub-regions of 1, 4, 7 and 10.

It is easy to find that the number of variations of sign tables of the discrimination sequence, as shown in Table 1, is 2 for all sub-regions, except for those (numbered as 2, 5 and 8) in the narrow V-shaped section, where the number of variations of sign tables is 4. It follows that $F(\omega) = 0$ has exactly $2(= (8 - 2 \times 2)/2)$ distinct positive roots when the parameters are taken from the sub-regions 1, 3, 4, 6, 7, 9, 10 and 11, and has no real roots $(0 = 8 - 2 \times 4)$ if the parameters are in the sub-regions of 2, 5 and 8. Thus, the system is delay-independent stable in the narrow V-shaped region (the sub-regions 2, 5 and 8), and it may exhibit a finite number of stability switches in the other sub-regions.

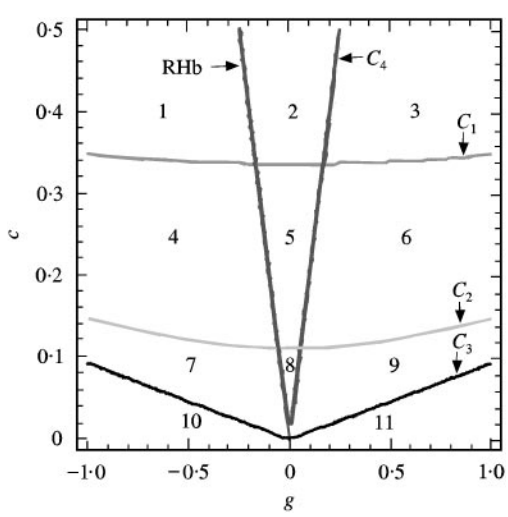


Figure 1. Parameter division by $d_i$ for a quarter car model with a delayed sky-hook damper. The curves, symmetrical to the $c$-axis, in this graph are given by $C_1: d_1 = 0$; $C_2: d_2 = 0$; $C_3: d_3 = 0$, $d_4 = 0$, $d_5 = 0$; and $C_4: d_5 = 0$. The left part of the narrow V-shaped curve $C_4$ is also the boundary of the region determined by the Routh–Hurwitz conditions.

TABLE 1

*Sign tables of the discrimination sequence of a quarter car model with sky-hook damper*

| Sub-regions | $d_0$ $d_1$ $\cdots$ $d_5$ $d_6$ | | | | | | | $D_1$ $D_2$ $\cdots$ $D_7$ $D_8$ (sign tables) | | | | | | | | $l-2v$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1, 3 | + | − | − | − | − | − | + | 1 | 1 | −1 | 1 | 1 | 1 | 1 | 1 | 4 |
| 2 | + | − | − | − | − | + | + | 1 | 1 | −1 | 1 | 1 | 1 | −1 | 1 | 0 |
| 4, 6, 7, 9 | + | + | − | − | − | − | + | 1 | 1 | 1 | −1 | 1 | 1 | 1 | 1 | 4 |
| 5, 8 | + | + | − | − | − | + | + | 1 | 1 | 1 | −1 | 1 | 1 | −1 | 1 | 0 |
| 10, 11 | + | + | − | + | + | + | + | 1 | 1 | 1 | −1 | −1 | 1 | 1 | 1 | 4 |

From Theorem 2, we see that only when the parameters are taken from the boundary of the two V-shaped regions, can the function $F(\omega)$ have probably repeated real roots. In contrast, on the other common boundaries outside the narrow V-shape region, $F(\omega)$ has two distinct simple positive roots. For example, on the common boundary, determined by $d_2 = 0$, of sub-regions 6 and 9 the modified sign table of the discrimination sequence is $[1, 1, 1, -1, -1, 1, 1, 1]$ since the original sign table of the discrimination sequence is $[1, 1, 1, 0, 0, 1, 1, 1]$. As the variation number of the modified sign table is 2, $F(\omega)$ has two distinct simple positive roots. Thus, the system possesses a finite number of stability switches when the parameters are taken from the common boundaries except for the two V-shaped lines.

Though all the sub-regions, except for those numbered as 2, 5 and 8, cause the polynomial $F(\omega)$ to have two distinct positive roots, the dynamic behaviors of the system do have differences in these sub-regions. Let the time delay be varied from 0 to infinity. When the parameters are chosen from the sub-regions 1, 4, 7 and 10, the system may undergo a finite number of stability changes from instability to stability, then to instability and so on, and eventually become unstable. However, if the parameters are taken from the sub-regions 3, 6, 9 and 11, the system first remains stable, then becomes unstable, again becomes stable and so on, and eventually becomes unstable. This will be demonstrated in the next subsection by numerical examples.

### 3.1.2. *Numerical examples*

**Example 1.** $g = 0{\cdot}6$ and $c = 0{\cdot}4$.

The point (0·6, 0·4) falls into the sub-region 3, where the function $F(\omega)$ has exactly two distinct real roots. In fact, the two real roots are $\omega_+ = 0{\cdot}8647$ and $\omega_- = 0{\cdot}5922$ with $F'(\omega_+) > 0$ and $F'(\omega_-) < 0$. The corresponding critical values of time delay are

$$\tau_{1,0} = 1{\cdot}9987, \quad \tau_{1,1} = 9{\cdot}2650, \quad \tau_{1,2} = 16{\cdot}5313, \quad \tau_{1,3} = 23{\cdot}7976, \quad \tau_{1,4} = 31{\cdot}0639, \ldots$$

and

$$\tau_{2,0} = 7{\cdot}1771, \quad \tau_{2,1} = 17{\cdot}7870, \quad \tau_{2,2} = 28{\cdot}3969, \quad \tau_{2,3} = 39{\cdot}0068, \quad \tau_{2,4} = 49{\cdot}6167, \ldots$$

They can be ranked as

$$\tau_{1,0} < \tau_{2,0} < \underline{\tau_{1,1} < \tau_{1,2}} < \tau_{2,1} < \tau_{1,3} < \tau_{2,2} < \cdots \tag{24}$$

We claim that the system is asymptotically stable for $\tau \in [0, \tau_{1,0})$, unstable for $\tau \in [\tau_{1,0}, \tau_{2,0}]$, asymptotically stable again for $\tau \in (\tau_{2,0}, \tau_{1,1})$, and eventually unstable for all $\tau \geqslant \tau_{1,1}$.

The first three conclusions lie in the facts that $F'(\omega_+) > 0$ and $F'(\omega_-) < 0$, and that the system is asymptotically stable for $\tau = 0$.

To show the fourth conclusion, we observe that, in the sequence (24) of critical values of time delay, $\tau_{1,1}$ is followed immediately by $\tau_{1,2}$, but any $\tau_{2,k}$ cannot be followed by $\tau_{2,k+1}$ since $\tau_{1,k+1} - \tau_{1,k} = 2\pi/\omega_+ < 2\pi/\omega_- = \tau_{2,k+1} - \tau_{2,k}$ and $\tau_{1,0} < \tau_{2,0}$. Thus, the system has at least two characteristic roots of positive real parts if $\tau > \tau_{1,1}$. This means that the system is unstable for all $\tau > \tau_{1,1}$.

Special attention should be paid to the case of $\tau > \tau_{1,2}$. For $\tau > \tau_{1,2}$, we cannot simply follow the conditions $F'(\omega_+) > 0$ and $F'(\omega_-) < 0$ to conclude that the system is asymptotically stable for $\tau \in (\tau_{2,1}, \tau_{1,3}), (\tau_{2,2}, \tau_{1,4})$, etc. This can be verified as follows. Take $\tau = 20 \in (\tau_{2,1}, \tau_{1,3})$ as an example, and let $R(\omega) \equiv \mathrm{Re}(D(\mathrm{i}\omega, \tau))$ and $I(\omega) \equiv \mathrm{Im}(D(\mathrm{i}\omega, \tau))$. Then, $R(\omega)$ has exactly two positive roots $\rho_1 = 3{\cdot}3017$ and $\rho_2 = 0{\cdot}7666$, which yields

$I(\rho_1) = -67{\cdot}2561$ and $I(\rho_2) = -1{\cdot}4871$. Thus, one has

$$\sum_{k=1}^{2} (-1)^k \operatorname{sgn} I(\rho_k) = 0, \tag{25a}$$

which contradicts the stability condition (cf. reference [3] or Appendix E)

$$\sum_{k=1}^{2} (-1)^k \operatorname{sgn} I(\rho_k) = (-1)^2 2 = 2. \tag{25b}$$

Therefore, the system is unstable for $\tau \in (\tau_{2,1}, \tau_{1,3}), (\tau_{2,2}, \tau_{1,4})$, etc., and in turn, the system is unstable for all $\tau \geqslant \tau_{1,1}$. As a result, the number of stability switches is 2.

**Example 2.** $g = 0{\cdot}3$ and $c = 0{\cdot}1$.

We have $\omega_+ = 0{\cdot}7863$ and $\omega_- = 0{\cdot}6438$, and the critical time delays are

$$\tau_{1,0} = 2{\cdot}1610, \quad \tau_{1,1} = 10{\cdot}1516, \quad \tau_{1,2} = 18{\cdot}1422, \quad \tau_{1,3} = 26{\cdot}1329, \quad \tau_{1,4} = 34{\cdot}6096, \ldots$$

and

$$\tau_{2,0} = 7{\cdot}0050, \quad \tau_{2,1} = 16{\cdot}7645, \quad \tau_{2,2} = 26{\cdot}5241, \quad \tau_{2,3} = 36{\cdot}2836, \ldots$$

They are ranked as

$$\tau_{1,0} < \tau_{2,0} < \tau_{1,1} < \tau_{2,1} < \underline{\tau_{1,2} < \tau_{1,3}} < \tau_{2,2} < \tau_{1,4} < \tau_{2,3} < \cdots. \tag{26}$$

One can similarly find that the system is asymptotically stable for $[0, \tau_{1,0}), (\tau_{2,0}, \tau_{1,1})$ and $(\tau_{2,1}, \tau_{1,2})$, and unstable for $[\tau_{1,0}, \tau_{2,0}]$, $[\tau_{1,1}, \tau_{2,1}]$ and $[\tau_{1,2}, +\infty)$. Hence, the system exhibits five stability switches.

**Example 3.** $g = -0{\cdot}5$ and $c = 0{\cdot}2$.

We have $\omega_+ = 0{\cdot}8389$ and $\omega_- = 0{\cdot}6034$, and the critical time delays are

$$\tau_{1,0} = 5{\cdot}7548, \quad \tau_{1,1} = 13{\cdot}2451, \quad \tau_{1,2} = 20{\cdot}7353, \quad \tau_{1,3} = 28{\cdot}2256, \ldots$$

and

$$\tau_{2,0} = 2{\cdot}1638, \quad \tau_{2,1} = 12{\cdot}5769, \quad \tau_{2,2} = 22{\cdot}9900, \quad \tau_{2,3} = 33{\cdot}4301, \ldots$$

They are ranked as

$$\tau_{2,0} < \tau_{1,0} < \tau_{2,1} < \underline{\tau_{1,1} < \tau_{1,2}} < \tau_{2,2} < \tau_{1,3} < \tau_{2,3} < \cdots. \tag{27}$$

The system is asymptotically stable for $(\tau_{2,0}, \tau_{1,0})$ and $(\tau_{2,1}, \tau_{1,1})$, and unstable for $[0, \tau_{2,0}]$, $[\tau_{1,0}, \tau_{2,1}]$ and $[\tau_{1,1}, +\infty)$. Hence, the number of stability switches is 4.

The numerical examples show that, in the sense of stability switches, the stability behavior of the quarter car model with a delayed sky-hook damper is rather complicated. The system may change its stability many, but finite, times as the time delay varies. If all the critical values of time delay were increasingly ranked, then the change of stability must be ended as soon as any $\tau_{1,k}$ is followed by $\tau_{1,k+1}$ in this sequence of time delay, and the system is unstable for all $\tau \geqslant \tau_{1,k}$. The increase in time delay usually results in instability of the system, but it also offers a way to stabilize an unstable suspension system.

### 3.2. FOUR-WHEEL STEERING VEHICLE WITH TIME DELAY IN DRIVE'S RESPONSE

Now, we consider the "bicycle model" of a four-wheel steering vehicle. Let $v$ be the lateral velocity, $r$ the yaw angular velocity, $y$ the vertical coordinate in a fixed frame, $\psi$ the heading angle of the vehicle, and $\delta_f$ and $\delta_r$ the steering angles applied on the front and rear wheels respectively. When the time delay in driver's response is taken into account, the motion of system can be described by a set of five-dimensional differential equations with a time delay as follows [10]:

$$\begin{aligned} m(\dot{v} + rU) &= 2F_f \cos\delta_f + 2F_r \cos(k_\delta \delta_f + k_r r), \\ I_z \dot{r} &= 2aF_f \cos\delta_f - 2bF_r \sin(k_\delta \delta_f + k_r r), \\ \dot{y} &= v\cos\psi + U\sin\psi, \\ \dot{\psi} &= r, \\ \dot{\delta}_f &= -\left\{\frac{K_m}{T_s} y(t-\tau) + \frac{L}{U}\dot{y}(t-\tau) + \frac{1}{T_s}\delta_f\right\} + f(t), \end{aligned} \tag{28}$$

where $U$ is the constant moving speed of the vehicle, $I_z$ the inertia moment of rotation of the vehicle body with respect to the vertical axis through the center of gravity, $a$ and $b$ the distances from the center of mass to the front and rear axles respectively, $F_f$ and $F_r$ the lateral forces generated by the contact between the tyre and the road surface at each of the front and the rear wheels respectively, $L$ the preview distance of the driver, $\tau$ the time delay in driver's response, and $f(t)$ the external disturbance. Moreover, we use the geometric relations and the truncated Magic model of tyre force

$$\alpha_f = \arctan\frac{v + ar}{U} - \delta_f, \qquad \alpha_r = \arctan\frac{v - br}{U} - \delta_r, \tag{29a}$$

$$F_f = -C_1\alpha_f + C_3\alpha_f^3, \qquad F_r = -D_1\alpha_r + D_3\alpha_r^3, \tag{29b}$$

as well as the control strategy between the front wheels and the rear wheels [10],

$$k_\delta = -\frac{C_1}{D_1}, \qquad k_r = \frac{2(aC_1 - bD_1) + mU^2}{2D_1 U}. \tag{29c}$$

The vehicle is said to be under-steered, neutral steered and over-steered if $aC_1 - bD_1 < 0$, $aC_1 - bD_1 = 0$ and $aC_1 - bD_1 > 0$ respectively. In the following discussion, the system parameters are chosen as [10]: $m = 1300$ kg, $I = 3000\ \mathrm{kg\,m^2}$, $a = 1{\cdot}0$ m, $b = 1{\cdot}6$ m, $C_1 = 44\,400$ N/rad, $D_1 = 43\,600$ N/rad (or $D_1 = 25\,600$ N/rad for over-steered), $C_3 = 44\,400$ N/rad, $D_3 = 44\,400$ N/rad, $T_s = 0{\cdot}2$ s, $K_m = 0{\cdot}02$.

As analyzed in reference [10], the system has nine steady states including the trivial one. After some necessary manipulation, one can get the characteristic function of the linearized equations corresponding to the trivial solution as follows:

$$D(\lambda, \tau) \equiv \lambda^5 + c_{04}\lambda^4 + c_{03}\lambda^3 + c_{02}\lambda^2 + (c_{13}\lambda^3 + c_{12}\lambda^2 + c_{11}\lambda + c_{10})\exp(-\lambda\tau), \tag{30}$$

where the coefficients are listed in Appendix D. According to the Routh–Hurwitz criterion, the trivial solution for $\tau = 0$ is asymptotically stable if and only if

$$a_0 > 0, \quad a_4 > 0, \quad a_4 a_3 - a_2 > 0, \quad a_4 a_3 a_2 - a_4^2 a_1 - a_2^2 + a_4 a_0 > 0, \tag{31a}$$

$$a_4 a_3 a_2 a_1 - a_4 a_3^2 a_0 - a_4^2 a_1^2 + 2a_4 a_1 a_0 - a_2^2 a_1 + a_3 a_2 a_1 - a_0^2 > 0, \tag{31b}$$

where $a_i = c_{0i} + c_{1i}$ $(i = 0–4)$ are the coefficients of polynomial $D(\lambda, 0)$. The function $F(\omega)$ can be readily written out as

$$F(\omega) = \omega^{10} + (c_{04}^2 - 2c_{03})\omega^8 + (-2c_{04}c_{02} + c_{03}^2 - c_{13}^2)\omega^6 + (c_{02}^2 - c_{12}^2 + 2c_{13}c_{11})\omega^4 + (2c_{12}c_{10} - c_{11}^2)\omega^2 - c_{10}^2, \quad (32)$$

which has obviously at least one positive root since $F(0) = -c_{10}^2 < 0$ and $F(+\infty) \to +\infty$. This means that the system cannot be delay-independent stable. By using *discr*, we get the discrimination sequence of $F(\omega)$ as

$$1, \quad d_0, \quad d_0 d_1, \quad d_1 d_2, \ldots, d_6 d_7, \quad d_7^2 d_8. \quad (33)$$

Since the expressions of $d_6$, $d_7$ are lengthy, all the terms of the discrimination sequence are omitted in this paper. In what follows, we consider the following parameter combinations:

$$\Delta = \{(L, U): 5\ \text{m/s} < U < 40\ \text{m/s},\ 10\ \text{m} < L < 120\ \text{m}\}. \quad (34)$$

In the case of the under-steered vehicle, it is easy to know that the Routh–Hurwitz conditions hold true in the whole given region. This means that the system is asymptotically stable for $\tau = 0$. In addition, we have $d_0 < 0$, $d_1 > 0$, $d_3 > 0$, $d_5 > 0$, $d_6 < 0$, $d_7 > 0$ and $d_8 > 0$. By plotting the graphs of $d_2 = 0$ and $d_4 = 0$, the given region $\Delta$ in the parametric space of $(L, U)$ is divided into five sub-regions as shown in Figure 2. The sign tables of the discrimination sequence are given in Table 2. The numbers of variation of all sign tables are the same, namely, 4. Thus, for each of the parameter combinations in the given region, $F(\omega)$ has exactly $1(= (10 - 2 \times 4)/2)$ simple positive root. Once this positive root is found, it is easy to obtain the minimal time delay $\tau_0$ satisfying equations (6a)–(6b). Then, the system remains asymptotically stable for $0 \leqslant \tau \leqslant \tau_0$ and becomes unstable for all $\tau \geqslant \tau_0$.

When the vehicle is over-steered, the region $\Delta$ is divided into eight sub-regions, which are numbered from the top to the bottom sub-regions by 1, 2, …, 8 respectively as shown in

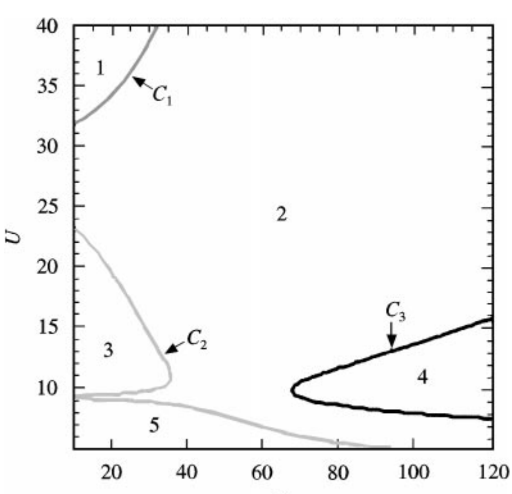


Figure 2. Parameter division of a 4WS vehicle model by $d_i$ in the under-steered case. The equation $d_4 = 0$ gives the curves $C_1$ and $C_2$. The graph of $d_2 = 0$ is denoted by $C_3$.

TABLE 2

*Sign tables of the discrimination sequence of a 4WS vehicle model in the under-steered case*

| Sub-regions | $d_0$ $d_1$ $\cdots$ $d_7$ $d_8$ | | | | | | | | | $D_1$ $D_2$ $\cdots$ $D_9$ $D_{10}$ (sign tables) | | | | | | | | | | $l-2v$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1, 3, 5 | − | + | + | + | − | + | − | + | + | 1 | −1 | −1 | 1 | 1 | −1 | −1 | −1 | −1 | 1 | 2 |
| 2 | − | + | + | + | + | + | − | + | + | 1 | −1 | −1 | 1 | 1 | 1 | 1 | −1 | −1 | 1 | 2 |
| 4 | − | + | − | + | + | + | − | + | + | 1 | −1 | −1 | −1 | −1 | 1 | 1 | −1 | −1 | 1 | 2 |

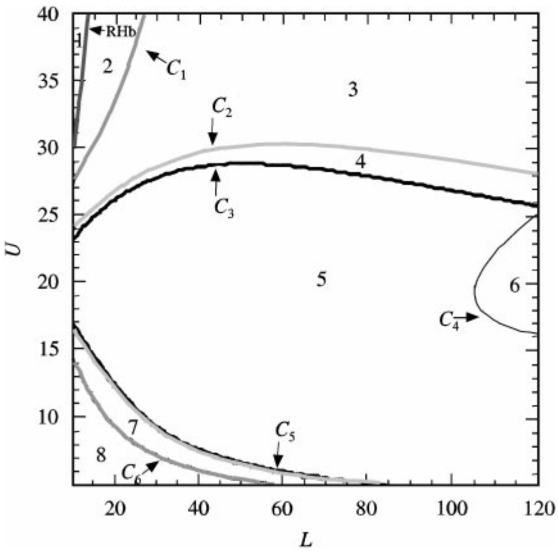


Figure 3. Parameter division of a 4WS vehicle model by $d_i$ in the over-steered case. RHb is determined by the Routh–Hurwitz conditions. The graph of $d_4 = 0$ is composed of two curves $C_1$ and $C_6$. The graph of $d_5 = 0$ is composed of two curves $C_3$ and $C_5$. The graph of $d_6 = 0$ is the same as that of $d_7 = 0$, and is composed of two curves $C_2$ and $C_5$, and $C_4$ is the graph of $d_2 = 0$.

Figure 3. As in Figure 1, the curve RHb denotes the boundary determined by the Routh–Hurwitz stability conditions for the system without time delay. The sub-region 1 is such a region that the system is unstable for $\tau = 0$, while the other sub-regions are those that ensure that the system is asymptotically stable for $\tau = 0$. The sign tables of the discrimination sequence are shown in Table 3.

From Table 3, we see that $F(\omega)$ has exactly $1(= (10 - 2 \times 4)/2)$ simple positive root in every sub-region except for the parameter combinations on the two common boundaries of sub-regions 3 and 4, as well as 5 and 7, where $F(\omega)$ has repeated roots. Therefore, sub-region 1 is a region where the system is unstable for all given time delays. In the other sub-regions, for almost all the parameter combinations in the given region, there exists a $\tau_0$ depending on the parameters so that the system is asymptotically stable for $\tau \in [0, \tau_0)$, and unstable for all $\tau > \tau_0$.

Here are two numerical examples: (1) $U = 30$ and $L = 40$, (2) $U = 30$ and $L = 60$. In the under-steered case, the critical time delays are $\tau_0 = 0{\cdot}2899$ and $0{\cdot}2108$, and the corresponding frequencies of self-excited vibration are $\omega = 2{\cdot}4586$ and $3{\cdot}3027$ respectively. In the over-steered case, the critical time delays are $\tau_0 = 0{\cdot}1943$ and $0{\cdot}1412$, while the corresponding vibration frequencies are $\omega = 2{\cdot}6654$ and $3{\cdot}4454$ respectively.

In summary, for the four-wheel steering vehicle with the time delay in driver's response taken into account, the stability behavior of the system is relatively simple. As the time delay is varied from zero to infinity, only two cases may occur. If the system which is free of time delay is unstable, then the system is unstable for any time delay. Or, there exists a specific value $\tau_0 > 0$ such that the system remains asymptotically stable for any $\tau \in [0, \tau_0)$ and becomes unstable for any $\tau \geqslant \tau_0$ if the system is asymptotically stable when $\tau = 0$. Numerical examples show that the critical time delays are usually very short. Thus, a driver's slow response may cause undesirable instability of the four-wheel steering vehicle.

TABLE 3

*Sign tables of the discrimination sequence of a 4WS vehicle model in the over-steered case*

| Sub-regions | $d_0 \quad d_1 \quad \cdots \quad d_7 \quad d_8$ | | | | | | | | | $D_1 \quad D_2 \quad \cdots \quad D_9 \quad D_{10}$ (sign tables) | | | | | | | | | | $l-2v$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1, 2, 8 | − | + | + | + | − | + | − | + | + | 1 | −1 | −1 | 1 | 1 | −1 | −1 | −1 | −1 | 1 | 2 |
| 3, 7 | − | + | + | + | + | + | − | + | + | 1 | −1 | −1 | 1 | 1 | 1 | 1 | −1 | −1 | 1 | 2 |
| 4 | − | + | + | + | + | + | + | − | + | 1 | −1 | −1 | 1 | 1 | 1 | 1 | 1 | −1 | 1 | 2 |
| 5 | − | + | + | + | + | − | + | − | + | 1 | −1 | −1 | 1 | 1 | 1 | −1 | −1 | −1 | 1 | 2 |
| 6 | − | + | − | + | + | − | + | − | + | 1 | −1 | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 | 2 |

## 4. CONCLUDING REMARKS

With an increase of time delay from zero to infinity, a linear, high-dimensional, dynamic system under different parameter combinations may have various stability switches. If the system has no more than one single stability switch, the stability structure of the system is relatively simple. Otherwise, the system can either be destabilized by decreasing time delay or be stabilized by increasing time delay. The approach proposed in the paper enables one to know easily under what parameter combinations the system has no stability switch, one stability switch, or more than one stability switches. Thus, the approach can be considered as a new method of parameter division for linear dynamic systems involving a time delay.

A great difference in stability switches exists between the two models of ground vehicle under the parameter combinations concerned. For example, there are some parameter combinations for the quarter car model with active suspension such that the system is delay-independent stable, i.e., asymptotically stable for any given time delay. However, the four-wheel steering vehicle does not possess such parameters, and it has no more than one stability switch. Furthermore, under any parameter combination, an unstable four-wheel steering vehicle cannot be stabilized by increasing the time delay of driver's response. While the quarter car model with active suspension exhibits complicated phenomena of stability switches, there are some parameter combinations such that an unstable quarter car model with active suspension can be stabilized by increasing the time delay in velocity feedback.

Finally, one can readily list a number of open problems related to the stability switches. For example, when $F(\omega)$ has repeated real roots, no results on stability switches are available. In addition, if $F(\omega)$ has at least two distinct positive roots, a rough conclusion states that the system has a finite number of stability switches, only no analytical tools are available for predicting the exact number of stability switches.

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## APPENDIX A: MAPLE ROUTINE *discr*

```
> discr:= proc(poly, var)
>     local f, g, tt, d, bz, i, ar, j, mm, dd:
>         f:= expand(poly): d:= degree(f, var):
>         g:= tt*var^d + diff(f, var):
>         with(linalg):
>         bz:= subs(tt = 0, bezout(f, g, var)): ar:= [ ]:
>         for i to d do
>         ar:= [op(ar), row(bz, d + 1 − i..d + 1 − i)] od:
>         mm:= matrix(ar): dd:= [ ]:
>         for j to d do
>         dd:= [op(dd), det(submatrix(mm, 1..j, 1..j))] od:
>         dd:= map(primpart, dd)
> end:
```

## APPENDIX B: THE COEFFICIENTS OF $F(\omega)$ IN EQUATION (21)

$$b_1 = -2 - 2\beta - 2k_t\beta + c^2 + 2c^2\beta + c^2\beta^2 - g^2,$$

$$b_2 = 1 + 4k_t\beta + 2k_t\beta^2 + k_t^2\beta^2 + 2g^2k_t\beta - 2c^2k_t\beta^2 - 2c^2k_t\beta + 2\beta + \beta^2,$$

$$b_3 = -k_t\beta(-c^2k_t\beta + 2k_t\beta + 2\beta + g^2k_t\beta + 2),$$

$$b_4 = k_t^2\beta^2.$$

## APPENDIX C: THE EXPRESSIONS FOR $d_i$ IN EQUATION (22)

$$d_0 = -b_1, \quad d_1 = -8b_2 + 3b_1^2, \qquad d_2 = b_1^2b_2 + 3b_1b_3 - 4b_2^2,$$

$$d_3 = -3b_1^3b_3 + b_1^2b_2^2 - 6b_1^2b_4 + 14b_1b_2b_3 - 4b_2^3 + 16b_2b_4 - 18b_3^2,$$

$$d_4 = -b_1^2b_2^2b_3 - 18b_1b_2b_3^2 + 7b_1^2b_3b_4 + 12b_1b_2^2b_4 - 48b_2b_3b_4 + 4b_2^3b_3 + 16b_1b_4^2 + 27b_3^3 + 4b_1^3b_3^2 - 3b_1^3b_2b_4,$$

$$d_5 = -27b_1^4b_4^2 + 18b_1^3b_2b_3b_4 - 4b_1^3b_3^3 - 4b_1^2b_2^3b_4 + b_1^2b_2^2b_3^2 + 144b_1^2b_2b_4^2 - 6b_1^2b_3^2b_4 - 80b_1b_2^2b_3b_4 + 18b_1b_2b_3^3 - 192b_1b_3b_4^2 + 16b_2^4b_4 - 4b_2^3b_3^2 - 128b_2^2b_4^2 + 144b_2b_3^2b_4 + 256b_4^3 - 27b_3^4,$$

$$d_6 = b_4 > 0.$$

## APPENDIX D: THE COEFFICIENTS IN EQUATION (30)

$$c_{00} = 0, \quad c_{01} = 0, \qquad c_{14} = 0,$$

$$c_{10} = -4\,\frac{K_mD_1C_1(k_d - 1)(a + b)}{mT_sI_z},$$

$$c_{11} = -4\,\frac{K_m C_1 D_1 (a+b)(-ak_d + k_d L - b - k_r U - L)}{m I_z T_s U},$$

$$c_{02} = 2\,\frac{2C_1 D_1 (a+b)^2 + m(bD_1 - aC_1)U^2 + 2(a+b)C_1 D_1 k_r U}{mU^2 I_z T_s},$$

$$c_{12} = 2\left[\frac{K_m(2abL(1+k_d)C_1 D_1 + (C_1 + D_1 k_d)I_z U^2}{mU^2 I_z T_s} + \frac{2(b^2 + a^2 k_d)LC_1 D_1 + 2(a+b)C_1 D_1 L k_r U}{mU^2 I_z T_s}\right],$$

$$c_{03} = 2\left[\frac{I_z(C_1 + D_1)U + (bD_1 - aC_1)mT_s U^2 + (a^2 C_1 + b^2 D_1)mU}{mU^2 I_z T_s} + \frac{bmD_1 k_r U^2 + 2(a+b)^2 C_1 D_1 T_s + 2(a+b)C_1 D_1 k_r T_s}{mU^2 I_z T_s}\right],$$

$$c_{13} = 2\,\frac{K_m L(C_1 + D_1 k_d)}{mT_s U},$$

$$c_{04} = \frac{2(a^2 C_1 + b^2 D_1)mT_s + mI_z U + 2(C_1 + D_1)I_z T_s + 2bD_1 k_r m T_s}{m I_z T_s U}.$$

## APPENDIX E: A STABILITY CRITERION FOR RETARDED SYSTEM WITH TIME DELAY

Consider an $n$-dimensional linear autonomous dynamic system with a time delay. Assume the characteristic function to be in the form of equation (1), and $R(\omega) \equiv \mathrm{Re}(D(\mathrm{i}\omega, \tau))$ and $I(\omega) \equiv \mathrm{Im}(D(\mathrm{i}\omega, \tau))$. Let $\rho_1 \geqslant \rho_2 \geqslant \cdots > \rho_r \geqslant 0$ and $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_s = 0$ denote the non-negative real zeros of $R$ and $I$ respectively. Then, the zero solution of the system is asymptotically stable if and only if

$$n = 2m\colon I(\rho_k) \neq 0, \quad k = 1, 2, \ldots, r$$

$$\sum_{k=1}^{r} (-1)^k \operatorname{sgn} I(\rho_k) = (-1)^m m$$

or

$$n = 2m+1\colon R(\sigma_k) \neq 0, \quad k = 1, 2, \ldots, s$$

$$\sum_{k=1}^{s} (-1)^k \operatorname{sgn} R(\sigma_k) + \frac{1}{2}((-1)^s + (-1)^m) + (-1)^m m = 0.$$