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CHARACTERISTIC EQUATIONS OF RECTANGULAR PLATES BY DIFFERENTIAL TRANSFORMATION

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1. INTRODUCTION

The differential transformation method, a transformation technique based on the Taylor series expansion, offers a convenient means for obtaining analytical solutions of the differential equations. Although the technique was introduced in 1986 [1], it seems to be largely unknown to the research community. In this method, following certain rules of transformation, the governing differential equations and the boundary conditions of the system are transformed into a set of algebraic equations in terms of the differential transforms of the field variables (the functions). Subsequently, the solution of the algebraic equations leads to the desired solution of the problem.

From some applications of the method which have come in the literature recently [2–4], the differential transformation method seems to be particularly attractive for eigenvalue problems. In the present paper, the application of the method is demonstrated for yet another eigenvalue problem of the free vibration of classical thin rectangular plates. The plates being considered have two opposite sides simply supported; each of the other two sides may be any of the simply supported, clamped, or free edges.

In the following, first, the basic concepts of the differential transformation method are briefly described. Next, the governing differential equation and the boundary conditions of the types of plates being considered are given from the classical theory. The differential transformation analysis includes development of the transformed algebraic equations with solution details for one type of plate. Finally, the characteristic equations derived using the differential transformation method are given for six types of plates. The correctness of these equations is verified by comparing the results of calculations with those from the corresponding analytical equations available in the literature [5, 6].

2. THE DIFFERENTIAL TRANSFORMATION

The concept of the method of differential transformation has its basis in the Taylor series expansion. The Taylor series expansion of a function $f = f(y)$ about a point $y = y_0$ is given by

$$f(y)=\sum_{n=0}^{\infty}\frac{(y-y_0)^n}{n!}\left[\frac{\mathrm{d}^n f}{\mathrm{d}y^n}\right]_{y=y_0}, \tag{1}$$

which may be written as

$$f(y)=\sum_{n=0}^{\infty}(y-y_0)^n F_n, \tag{2}$$

where F_n, referred to as the nth order differential transform of the function $f=f(y)$ about a point $y=y_0$, is given by

$$F_n=\frac{1}{n!}\left[\frac{\mathrm{d}^n f}{\mathrm{d}y^n}\right]_{y=y_0}. \tag{3}$$

It may be noted here that an upper-case symbol is used to denote the differential transform of a function represented by a corresponding lower-case symbol.

From the above basic definition of the differential transform of a function, one can derive the rules of transformational operations; some of these, which are useful in the following analysis, are as follows:

Functional form	Differential transform	
$f(y)=g(y)+h(y)$,	$F_n=G_n+H_n$,	(4)
$f(y)=cg(y)$,	$F_n=cG_n$,	(5)
$f(y)=\dfrac{\mathrm{d}^k g}{\mathrm{d}y^k}$,	$F_n=\dfrac{(n+k)!}{n!}G_{(n+k)}$,	(6)

where in equation (5), c is a scalar constant.

3. THE CLASSICAL THIN PLATE VIBRATION PROBLEM

The governing differential equation of a thin rectangular plate undergoing free harmonic vibration is given, in a non-dimensional form, as

$$w_{,xxxx}+2\lambda^2 w_{,xxyy}+\lambda^4 w_{,yyyy}=\Omega^2 w \quad (0\leqslant x\leqslant 1;\quad 0\leqslant y\leqslant 1), \tag{7}$$

where $w=w(x,y)$ is the deflection of the plate (the mode function); λ is the ratio of the lengths of the sides along the x- and y-axis, respectively; and Ω is the frequency of vibration.

Assuming the plate to be simply supported along the sides $x=0$ and 1, the plate deflection, consistent with the boundary conditions at these two sides, may be expressed as

$$w=f(y)\sin(m\pi x), \tag{8}$$

where m is an integer and $f(y)$ is the y direction mode function. Substituting equation (8) in equation (7), the differential equation of the vibrating plate becomes

$$f_{,yyyy}-2Rf_{,yy}-Sf=0, \tag{9}$$

where

$$R = (m\pi/\lambda)^2, \quad S = (\Omega^2 - m^4\pi^4)/\lambda^4. \tag{10}$$

The boundary conditions along each of the sides $y = 0$ and 1 may be one of the simply supported, clamped, and free edges. These may be expressed in terms of the function $f(y)$ as

$$\left.\begin{array}{lll} \text{Simply supported} & f(y) = 0, & f_{,yy} = 0 \\ \text{Clamped} & f(y) = 0, & f_{,y} = 0 \\ \text{Free} & f_{,yy} - p_1 f(y) = 0, & f_{,yyy} - p_2 f_{,y} = 0 \end{array}\right\} \text{ at } y = 0, 1, \tag{11}$$

where

$$p_1 = \nu(m\pi/\lambda)^2, \quad p_2 = (2 - \nu)(m\pi/\lambda)^2, \tag{12}$$

in which ν is the Poisson ratio.

4. DIFFERENTIAL TRANSFORMATION ANALYSIS

Using the transformation rules (4)–(6), the governing differential equation (9) may be transformed into the following algebraic equation:

$$(n + 4)!\, F_{(n+4)} - 2(n + 2)!\, RF_{(n+2)} - n!\, SF_n = 0. \tag{13}$$

Also, the boundary conditions, equations (11), may be transformed as

Simply supported
$$\sum_{n=0}^{\infty} (\bar{y} - y_0)^n F_n = 0,$$

$$\sum_{n=0}^{\infty} n(n - 1)(\bar{y} - y_0)^{(n-2)} F_n = 0,$$

Clamped
$$\sum_{n=0}^{\infty} (\bar{y} - y_0)^n F_n = 0,$$

$$\sum_{n=0}^{\infty} n(\bar{y} - y_0)^{(n-1)} F_n = 0,$$

Free
$$\sum_{n=0}^{\infty} [n(n - 1) - p_1(\bar{y} - y_0)^2](\bar{y} - y_0)^{(n-2)} F_n = 0,$$

$$\sum_{n=0}^{\infty} n[(n - 1)(n - 2) - p_2(\bar{y} - y_0)^2](\bar{y} - y_0)^{(n-3)} F_n = 0, \tag{14}$$

where $\bar{y} = 0, 1$.

Assuming

$$F_n = \frac{1}{n!} s^n \tag{15}$$

it can be shown that the general solution of equation (13) is

$$F_n = \frac{1}{n!}(C_1 s_1^n + C_2 s_2^n), \quad n = 0, 1, 2, \ldots, \infty, \tag{16}$$

where

$$s_{1,2}^2 = (m^2\pi^2 \pm \Omega)/\lambda^2. \tag{17}$$

The constants C_1 and C_2, which could be real or complex, may be determined taking $y_0 = 0$ or 1 and using the boundary conditions of the same edge (i.e., taking $\bar{y} = y_0$). The characteristic equation is then derived using the boundary conditions of the other y-edge (i.e., taking the other value of $\bar{y}$). This derivation may be understood taking one particular set of boundary conditions. For this purpose, it is convenient to write equation (16) in terms of even- and odd-order differential transforms,

$$F_{2n} = \frac{1}{(2n)!}(C_1 s_1^{2n} + C_2 s_2^{2n}), \tag{18}$$

$$F_{(2n+1)} = \frac{1}{(2n+1)!}(D_1 s_1^{2n+1} + D_2 s_2^{2n+1}). \tag{19}$$

In terms of these transforms, equation (2) may be written as

$$f(y) = \sum_{n=0}^{\infty} (y - y_0)^{2n} F_{(2n)} + \sum_{n=0}^{\infty} (y - y_0)^{2n+1} F_{(2n+1)}. \tag{20}$$

Now, consider a rectangular plate *simply supported* at $y = 0$ and *free* at $y = 1$. Letting $y_0 = 0$ and using the boundary conditions at $y = \bar{y} = 0$, it is found that $F_0 = 0$ and $F_2 = 0$. Using these in equation (18) for $n = 0$ and $n = 1$ yields $C_1 = C_2 = 0$, implying that all even-order differential transforms are zero,

$$F_{(2n)} = 0. \tag{21}$$

Next, from equation (19) one obtains with $n = 0$ and $n = 1$,

$$D_1 s_1 + D_2 s_2 = F_1, \quad D_1 s_1^3 + D_2 s_2^3 = \tfrac{1}{6} F_3,$$

which give

$$D_1 s_1 = (6F_3 - s_2^2 F_1)\frac{\lambda^2}{2\Omega}, \quad D_2 s_2 = (s_1^2 F_1 - 6F_3)\frac{\lambda^2}{2\Omega}.$$

Substituting these into equation (19),

$$F_{(2n+1)} = \frac{1}{(2n+1)!}[(\alpha_1 \alpha_2^n - \alpha_2 \alpha_1^n) F_1 + 6(\alpha_1^n - \alpha_2^n) F_3]\frac{\lambda^2}{2\Omega}, \tag{22}$$

where

$$\alpha_i = s_i^2, \quad i = 1, 2. \tag{23}$$

The boundary conditions of the free edge at $y = \bar{y} = 1$ are given by

$$\sum_{n=0}^{\infty} [(2n+1)(2n) - p_1] F_{(2n+)} = 0, \quad \sum_{n=0}^{\infty} (2n+1)[(2n)(2n-1) - p_2] F_{(2n+1)} = 0. \quad (24)$$

Using equation (22) in the boundary conditions (24),

$$\sum_{n=0}^{\infty} \frac{(2n+1)(2n) - p_1}{(2n+1)!} (\alpha_1 \alpha_2^n - \alpha_2 \alpha_1^n) F_1 + 6 \sum_{n=0}^{\infty} \frac{(2n+1)(2n) - p_1}{(2n+1)!} (\alpha_1^n - \alpha_2^n) F_3 = 0, \quad (25)$$

$$\sum_{n=0}^{\infty} \frac{(2n)(2n-1) - p_2}{(2n)!} (\alpha_1 \alpha_2^n - \alpha_2 \alpha_1^n) F_1 + 6 \sum_{n=0}^{\infty} \frac{(2n)(2n-1) - p_2}{(2n)!} (\alpha_1^n - \alpha_2^n) F_3 = 0, \quad (26)$$

from which the condition for non-trivial solution yields the characteristics equation.

5. CHARACTERISTICS EQUATIONS

For a rectangular plate having two opposite sides simply supported, six types of plate configurations are possible with the combinations of simply supported (S), clamped (C), and free (F) edge conditions at the other two sides. Following Leissa [5], these are designated as SSSS, SCSS, SCSC, SCSF, SSSF, and SFSF plates where in each case, the four letters indicate the type of edge along the four sides in the other $x = 0$, $y = 0$, $x = 1$, and $y = 1$ respectively. The characteristics equations for these plates, derived using the analysis of the foregoing section, are as follows.

5.1. *SSSS plate*

$$\left\{ \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (\alpha_1 \alpha_2^n - \alpha_2 \alpha_1^n) \right\} \left\{ \sum_{n=0}^{\infty} \frac{1}{(2n-1)!} (\alpha_1^n - \alpha_2^n) \right\}$$
$$- \left\{ \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} (\alpha_1 \alpha_2^n - \alpha_2 \alpha_1^n) \right\} \left\{ \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (\alpha_1^n - \alpha_2^n) \right\} = 0. \quad (27)$$

5.2. *SCSS plate*

$$\left\{ \sum_{n=1}^{\infty} \frac{1}{(2n)!} (\alpha_1^n - \alpha_2^n) \right\} \left\{ \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} (\alpha_1^n - \alpha_2^n) \right\}$$
$$- \left\{ \sum_{n=1}^{\infty} \frac{1}{(2n-2)!} (\alpha_1^n - \alpha_2^n) \right\} \left\{ \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} (\alpha_1^n - \alpha_2^n) \right\} = 0. \quad (28)$$

5.3. *SCSC plate*

$$\left\{ \sum_{n=1}^{\infty} \frac{1}{(2n)!} (\alpha_1^n - \alpha_2^n) \right\}^2$$
$$- \left\{ \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} (\alpha_1^n - \alpha_2^n) \right\} \left\{ \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} (\alpha_1^n - \alpha_2^n) \right\} = 0. \quad (29)$$

5.4. *SCSF plate*

$$\left\{\sum_{n=1}^{\infty}\frac{(2n)(2n-1)-p_1}{(2n)!}(\alpha_1^n-\alpha_2^n)\right\}\left\{\sum_{n=1}^{\infty}\frac{(2n)(2n-1)-p_2}{(2n)!}(\alpha_1^n-\alpha_2^n)\right\}$$

$$-\left\{\sum_{n=1}^{\infty}\frac{(2n-1)(2n-2)-p_2}{(2n-1)!}(\alpha_1^n-\alpha_2^n)\right\}\left\{\sum_{n=1}^{\infty}\frac{(2n+1)(2n)-p_1}{(2n+1)!}(\alpha_1^n-\alpha_2^n)\right\}=0. \quad (30)$$

5.5. *SSSF plate*

$$\left\{\sum_{n=0}^{\infty}\frac{(2n+1)(2n)-p_1}{(2n+1)!}(\alpha_1\alpha_2^n-\alpha_2\alpha_1^n)\right\}\left\{\sum_{n=0}^{\infty}\frac{(2n)(2n-1)-p_2}{(2n)!}(\alpha_1^n-\alpha_2^n)\right\}$$

$$-\left\{\sum_{n=0}^{\infty}\frac{(2n)(2n-1)-p_2}{(2n)!}(\alpha_1\alpha_2^n-\alpha_2\alpha_1^n)\right\}\left\{\sum_{n=0}^{\infty}\frac{(2n+1)(2n)-p_1}{(2n+1)!}(\alpha_1^n-\alpha_2^n)\right\}=0. \quad (31)$$

5.6. *SFSF plate*

$$\left\{\sum_{n=0}^{\infty}\frac{(2n)(2n-1)-p_1}{(2n)!}(\beta_1\alpha_2^n-\beta_2\alpha_1^n)\right\}\left\{\sum_{n=0}^{\infty}\frac{(2n)(2n-1)-p_2}{(2n)!}(\gamma_1\alpha_2^n-\gamma_2\alpha_1^n)\right\}$$

$$-\left\{\sum_{n=0}^{\infty}\frac{(2n-1)(2n-2)-p_2}{(2n-1)!}(\beta_1\alpha_2^n-\beta_2\alpha_1^n)\right\}\left\{\sum_{n=0}^{\infty}\frac{(2n+1)(2n)-p_1}{(2n+1)!}(\gamma_1\alpha_2^n-\gamma_2\alpha_1^n)\right\}=0, \quad (32)$$

where

$$\beta_i=\alpha_i-p_1, \quad \gamma_i=\alpha_i-p_2, \quad i=1,2. \quad (33)$$

6. SAMPLE RESULTS AND DISCUSSION

In reference [5], Lessa has derived analytical characteristic equations for the six types of plates having two opposite sides simply supported and provided calculated frequencies of the first nine modes of free vibration; see also the monograph [6]. With the exception of SSSS plate, the frequency equations of the remaining five plates are of the transcendental form. Compared to the compact form of frequency equations of reference [5], the corresponding equations of the differential transformation method appear to be quite complex. However, these equations are handled quite conveniently on the mathematical softwares. In the present work, calculations were carried out on Mathematica (version 4.0) to determine the frequencies from both the equations of reference [5] and equations (27)–(32) of the preceding section. The results from the two sets of equations were found to be identical. Any comparison of the calculated frequencies with those of reference [5] will be merely a repetition of the available data. However, herein, additional frequencies of the 10th mode of free vibration of the six types of plates calculated from equations (27)–(32) are given in Table 1. These data are for the same aspect ratios as used in reference [5]. It has been verified that the frequencies in Table 1 match exactly with those obtained from the corresponding equations of reference [5].

Table 1 includes the wave numbers, a pair of numbers given in parentheses, for each frequency indicating the number of half-waves in the x and y directions. It may be noted

TABLE 1

Dimensionless frequencies Ω of the 10th mode of free vibration of thin plates obtained by the differential transformation method*

Plate type	Aspect ratio, λ				
	2/5	2/3	1	3/2	5/2
SSSS	66·7185† (1,6)‡	109·662 (2,4)	177·653 (3,3)	268·947 (5,1)	545·296 (7,1)
SCSS	71·1180 (1,6)	116·674 (2,4)	187·437 (1,4)	270·685 (2,3)	454·722 (4,2)
SCSC	75·7369 (1,6)	124·289 (2,4)	199·811 (3,3)	280·161 (1,2)	483·611 (7,1)
SCSF	59·2944 (2,4)	98·7162 (3,2)	152·773 (3,3)	210·231 (4,2)	368·048 (6,1)
SSSF	57·8502 (2,4)	90·2662 (1,5)	145·638 (3,3)	202·896 (4,2)	324·572 (1,3)
SFSF	52·2282 (2,4)	88·2363 (3,1)	110·025 (2,4)	159·165 (4,1)	202·162 (4,2)

*$\Omega = \omega a^2\sqrt{(\rho h/D)}$, where ω = circular frequency (rad/s); a, h, ρ = side length along the x-axis, thickness, and mass density of the plate; $D = Eh^3/12(1-\nu^2)$; E and ν are elastic modulus and the Poisson ratio of the plate material.

†This value is quoted as the ninth mode frequency in reference [5]; the ninth mode value is actually 64·7446 (2,4).

‡The pair of numbers in parentheses indicates the number of half-waves in the x and y directions.

that the function $f(y)$ may be obtained quite easily from equation (20) once equations of the form (25) and (26) are set in. Consequently, the mode shape may be plotted using the mode-function equation (8). The equations for the y direction mode functions $f(y)$ are not included here; however, the wave numbers given in Table 1 were actually obtained by plotting the mode shapes (using Mathematica 4.0).

7. CLOSURE

As one would expect, due to its very basis in the Taylor series, the analytical solutions obtained by the differential transformation method are in the form of infinite series. These solutions can be handled quite conveniently on mathematical softwares. In this paper, the method was illustrated through its application to the eigenvalue problem of vibration of thin plates. It is believed that applications of the method to more challenging eigenvalue problems will follow in future.

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