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A NORMAL MODE EXPANSION METHOD FOR THE UNDAMPED FORCED VIBRATION OF LINEAR PIEZOELECTRIC SOLID

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A normal mode expansion method for the vibrational responses of non-homogeneous linear piezoelectric materials without damping is presented. It can be applied directly to arbitrary piezoelectric composites, which are widely used in vibrational and acoustic sensor/actuator/transmitter applications. In the present article it is shown that if the normal modes are given, the displacement field can be expanded as the linear superposition of normal modes, while the modal coefficients can be represented in terms of surface and volume integrals directly over the six types of distributed excitations without solving the quasi-static solution explicitly. The present treatment is a modification of an earlier work by Liu [11] using a different definition of the so-called quasi-static solution, and the damping effect has been neglected for simplicity. A simple example is given to exemplify the application of the present formulation.

1. INTRODUCTION

Since the piezoelectric effect was discovered by the Curie brothers in 1880, piezoelectric materials have been widely used as primary elements in acoustic transmitters, signal processing elements [1], sensors [2], actuators [3], and resonators [4] in both mechanical and electrical applications. The main feature of a piezoelectric material is its ability to convert mechanical energy into electrical energy and *vice versa*. In some cases, the behaviors of a piezoelectric material can practically be modelled as lumped parameter elements with the given electrical or mechanical impedances [4]. For a detailed study of the electromechanical responses of a structure involving piezoelectricity, however, three-dimensional continuum modelling is required for design purposes [5–7]. Mindlin [8] introduced the linear theory for a piezoelectric solid. Tierstein [9] derived the orthogonality relation of vibrational modes for a homogeneous piezoelectric solid. For a pure homogeneous elastic solid without piezoelectricity, it can be shown that, once the normal modes have been given, the modal coefficients for the displacement field can be represented in terms of surface and volume source integrals over the distributed excitations [10]. In the present article it is shown that if the normal modes are given, the displacement field can be expanded as the linear superposition of normal modes, while the modal coefficients can be represented in terms of surface and volume integrals directly over the six types of distributed excitations without solving the quasi-static solution explicitly. The present treatment is a modification of an earlier work by Liu [11] using a different definition of the so-called quasi-static solution, and the damping effect has been neglected for simplicity.

2. BASIC EQUATIONS

Consider a linear piezoelectric material occupying the space domain V with its boundary S, then, the stress tensor τ_{ij} and electric displacement d_j are related linearly to the gradients of displacement vector u_k and electric potential φ, respectively, as

$$\tau_{ij} = C_{ijkl}\partial_l u_k + e_{kij}\partial_k\varphi, \qquad d_j = e_{jkl}\partial_l u_k - \varepsilon_{jk}\partial_k\varphi, \tag{1,2}$$

where C_{ijkl} are elastic constants, e_{kij} are piezoelectric constants, ε_{jk} are dielectric permittivity constants, and ∂_l means $\partial/\partial x_l$, i.e., the partial derivative with respect to the lth Cartesian co-ordinate. As in electrostatics, the negative gradient of the potential field φ equals the electric field intensity. Summation convention over dummy indices is assumed unless otherwise mentioned. The fundamental law of linear momentum gives

$$\partial_i\tau_{ij} + \rho f_j = \rho\ddot{u}_j \tag{3}$$

and the law of electrostatics gives

$$\partial_j d_j = q, \tag{4}$$

where ρ is the mass density, f_j is the body force per unit mass, and q is the density of free charge. By substituting the constitutive equations (1) and (2) into the above fundamental laws, the governing equations for non-homogeneous linear piezoelectric materials will be

$$\partial_i(C_{ijkl}\partial_l u_k) + \partial_i(e_{kij}\partial_k\varphi) + \rho f_j = \rho\ddot{u}_j, \tag{5}$$

$$\partial_j(e_{jkl}\partial_l u_k) - \partial_j(\varepsilon_{jk}\partial_k\varphi) - q = 0. \tag{6}$$

The above two relations constitute four equations with four unknowns u_j and φ. For the displacement field u_j, the initial conditions at time $t = 0$ are

$$\begin{aligned} u_j(\mathbf{x}, 0) &= u_j^0(\mathbf{x}), \\ \dot{u}_j(\mathbf{x}, 0) &= v_j^0(\mathbf{x}). \end{aligned} \tag{7}$$

The initial displacement $u_j^0(\mathbf{x})$ and initial velocity $v_j^0(\mathbf{x})$ must be specified as functions of the space co-ordinates $\mathbf{x} = (x_1, x_2, x_3)$ for each point inside the volume V. The boundary conditions for stress, displacement, normal electric displacement, and electric potential should also be specified on the boundary surface:

$$\begin{aligned} n_i\tau_{ij} &= \hat{t}_j(\mathbf{x}, t), \quad \mathbf{x} \in S_1, \qquad u_j = \hat{u}_j(\mathbf{x}, t), \quad \mathbf{x} \in S_2, \\ n_j d_j &= \hat{d}(\mathbf{x}, t), \quad \mathbf{x} \in S_3, \qquad \varphi = \hat{\varphi}(\mathbf{x}, t), \quad \mathbf{x} \in S_4, \end{aligned} \tag{8}$$

where $S_1 \cup S_2 = S_3 \cup S_4 = S$ is the boundary surface, and $S_1 \cap S_2 = S_3 \cap S_4 = \emptyset$ is empty, i.e., there is no intersection between S_1 and S_2, nor between S_3 and S_4. The prescribed boundary values $\hat{t}_j$, $\hat{u}_j$, $\hat{d}$, and $\hat{\varphi}$ are given functions of the boundary position $\mathbf{x}$ and time t. The initial–boundary value problem formulated as equations (5)–(8) defines the most general case for non-homogeneous linear piezoelectric solid with arbitrary initial conditions and dynamic excitations.

3. METHOD OF NORMAL MODE EXPANSION

Similar to the normal mode expansion method used for an elastic solid [10], we expand the solutions $u_j(\mathbf{x}, t)$ and $\varphi(\mathbf{x}, t)$ of equations (5)–(8) as the summations of two parts:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x}, t) + \sum_n q_n(t)\mathbf{u}^n(\mathbf{x}), \tag{9}$$

$$\varphi(\mathbf{x}, t) = \Phi(\mathbf{x}, t) + \sum_n q_n(t)\varphi^n(\mathbf{x}), \tag{10}$$

where the bold-face letters $\mathbf{u}$, $\mathbf{u}^n$ and $\mathbf{U}$ denote the vectors (u_1, u_2, u_3), (u_1^n, u_2^n, u_3^n) and (U_1, U_2, U_3) respectively. The first part $\{\mathbf{U}(\mathbf{x}, t), \Phi(\mathbf{x}, t)\}$ is the special solution which satisfies the following quasi-static equations without time-derivative terms:

$$\partial_i(C_{ijkl}\partial_l U_k) + \partial_i(e_{kij}\partial_k \Phi) = -\rho f_j, \tag{11}$$

$$\partial_j(e_{jkl}\partial_l U_k) - \partial_j(\varepsilon_{jk}\partial_k \Phi) = q, \tag{12}$$

and the non-homogeneous boundary conditions

$$\begin{aligned} n_i T_{ij} &= \hat{t}_j(\mathbf{x}, t), \quad \mathbf{x} \in S_1, \qquad U_j = \hat{u}_j(\mathbf{x}, t), \quad \mathbf{x} \in S_2, \\ n_j D_j &= \hat{d}(\mathbf{x}, t), \quad \mathbf{x} \in S_3, \qquad \Phi = \hat{\varphi}(\mathbf{x}, t), \quad \mathbf{x} \in S_4, \end{aligned} \tag{13}$$

where T_{ij} and D_j are the stress and electric displacement corresponding to $\{\mathbf{U}, \Phi\}$, respectively, i.e.,

$$T_{ij} = C_{ijkl}\partial_l U_k + e_{kij}\partial_k \Phi, \qquad D_j = e_{jkl}\partial_l U_k - \varepsilon_{jk}\partial_k \Phi. \tag{14}$$

The second parts of equations (9) and (10) consist of superposition of the normal mode solutions $\{\mathbf{u}^n(\mathbf{x}), \varphi^n(\mathbf{x})\}$, $n = 1, 2, \ldots$, which are the non-trivial solutions of the following eigenvalue problem:

$$\partial_i(C_{ijkl}\partial_l u_k^n) + \partial_i(e_{kij}\partial_k \varphi^n) + \rho\omega_n^2 u_j^n = 0, \tag{15}$$

$$\partial_j(e_{jkl}\partial_l u_k^n) - \partial_j(\varepsilon_{jk}\partial_k \varphi^n) = 0, \tag{16}$$

$$n_i\tau_{ij}^n = 0, \quad \mathbf{x} \in S_1, \qquad u_j^n = 0, \quad \mathbf{x} \in S_2, \qquad n_j d_j^n = 0, \quad \mathbf{x} \in S_3, \qquad \varphi^n = 0, \quad \mathbf{x} \in S_4, \tag{17}$$

where τ_{ij}^n and d_j^n are the stress and the electric displacement corresponding to $\{\mathbf{u}^n, \varphi^n\}$, respectively, i.e.,

$$\tau_{ij}^n = C_{ijkl}\partial_l u_k^n + e_{kij}\partial_k \varphi^n, \qquad d_j^n = e_{jkl}\partial_l u_k^n - \varepsilon_{jk}\partial_k \varphi^n. \tag{18}$$

This eigenvalue problem has non-trivial solutions only for discrete values of circular frequencies ω_n, $n = 1, 2, \ldots$. Following the same procedures as Tierstein's [9], it can be easily proved that, in the case of $\omega_m \neq \omega_n$, by using the mass density $\rho(\mathbf{x})$ as weighting function, the mode shapes $\mathbf{u}^m(\mathbf{x})$ and $\mathbf{u}^n(\mathbf{x})$ of two different modes are orthogonal to each other:

$$\int_V \rho(\mathbf{x})\mathbf{u}^m(\mathbf{x}) \cdot \mathbf{u}^n(\mathbf{x})\,\mathrm{d}^3 v = N_m \delta_{mn} \quad \text{(no summation over } m\text{)}, \tag{19}$$

where dot "$\cdot$" denotes the vector inner product, δ_{mn} is Kronecker's symbol and $N_m = \int_V \rho \mathbf{u}^m \cdot \mathbf{u}^m \mathrm{d}^3 v$ (no summation over m) is the generalized modal mass for the mth mode. It should be emphasized that the above result holds for the most general case where all the material properties could be inhomogeneous. In contrast to equation (19), however, there exists no orthogonality relation between φ^m and φ^n. This is due to the fact that the potential field is considered to be static in the usual formulation of linear piezoelectricity. Unlike the displacement field, there is no simple way to get the explicit expression for the quasi-static part of the potential field. A suitable orthogonal basis for the potential field has been proposed in reference [11] by the present author. Following a similar derivation as shown below, the expansion coefficients for the potential field can be expressed analytically only when a suitable orthogonal basis is adopted [11].

The expansion equations (9) and (10) have already satisfied all the boundary conditions in equation (8). Upon substituting equations (9) and (10) into the governing equations (5) and (6), and using relations (11), (12), (15), and (16), we see that equation (6) has been automatically satisfied, and equation (5) results in

$$\sum_{n=1}^{\infty} \{\rho(\mathbf{x})\ddot{q}_n + \rho(\mathbf{x})\omega_n^2 q_n\}\mathbf{u}^n(\mathbf{x}) = -\rho\ddot{\mathbf{U}}. \tag{20}$$

As in a standard treatment of mechanical vibrations, we multiply equation (18) by $\mathbf{u}^m(\mathbf{x})$ and integrating over the volume V, we have the decoupled equation of motion for the mth mode:

$$\ddot{q}_m + \omega_m^2 q_m = -\ddot{F}_m(t), \tag{21}$$

where $F_m(t)$ is defined as

$$F_m(t) = \frac{1}{N_m}\int_V \rho(\mathbf{x}) U_j(\mathbf{x}, t) \cdot u_j^m(\mathbf{x})\, \mathrm{d}^3 v. \tag{22}$$

Note that $F_m(t)$ is the mth Fourier coefficient of $\mathbf{U}(\mathbf{x}, t)$. Thus the quasi-static part $\mathbf{U}(\mathbf{x}, t)$ can be expanded as

$$\mathbf{U}(\mathbf{x}, t) = \sum_m F_m(t)\mathbf{u}^m(\mathbf{x}). \tag{23}$$

The initial conditions for $q_m(t)$ can be easily derived as

$$q_m(0) = \lambda_m - F_m(0), \qquad \dot{q}_m(0) = \kappa_m - \dot{F}_m(0), \tag{24}$$

where λ_m and κ_m are defined as

$$\lambda_m = \frac{1}{N_m}\int_V \rho(\mathbf{x}) u_j^0(\mathbf{x}) \cdot u_j^m(\mathbf{x})\, \mathrm{d}^3 v, \tag{25}$$

$$\kappa_m = \frac{1}{N_m}\int_V \rho(\mathbf{x}) v_j^0(\mathbf{x}) \cdot u_j^m(\mathbf{x})\, \mathrm{d}^3 v, \tag{26}$$

respectively. For later convenience, we introduce the definition of the generalized co-ordinate $\bar{q}_m(t)$:

$$\bar{q}_m(t) = q_m(t) + F_m(t). \tag{27}$$

The governing equation and initial conditions for $\bar{q}_m(t)$ are

$$\ddot{\bar{q}}_m + \omega_m^2 \bar{q}_m = \omega_m^2 F_m(t), \qquad \bar{q}_m(0) = \lambda_m, \qquad \dot{\bar{q}}_m(0) = \kappa_m. \tag{28}$$

According to equation (9), if the normal mode solutions $\mathbf{u}^m(\mathbf{x})$ of equations (15)–(17) are assumed to be given by analysis or experiment, the solution for the displacement field $\mathbf{u}(\mathbf{x}, t)$ can be expressed in terms of the special solution $\mathbf{U}(\mathbf{x}, t)$ and modal coefficients q_m. Referring to equations (21) and (23), both $\mathbf{U}$ and q_m will be given with the knowledge of $F_m(t)$, which, however, depends on the special solution $\mathbf{U}$ itself by equation (22). Fortunately, by substituting equation (15) into equation (22) and taking successive partial integrations with the usage of equations (11), (12), (18), (14), and (16), the integral in equation (22) of the unknown integrand over the volume V can readily be expressed as

$$N_m \omega_m^2 F_m(t) = \int_V u_j^m \rho f_j \mathrm{d}^3 v - \int_V \varphi^m q \mathrm{d}^3 v + \int_S (u_j^m T_j - t_j^m U_j + \varphi^m D - d^m \Phi) \mathrm{d}^2 s, \tag{29}$$

where $t_j^m = n_i \tau_{ij}^m$ and $d^m = n_j d_j^m$ are the surface traction and the normal electric displacement due to the mth mode shape $\{\mathbf{u}^m, \varphi^m\}$, respectively, $T_j = n_i T_{ij}$ and $D = n_j D_j$ are the surface traction and the normal electric displacement due to the special solution $\{\mathbf{U}, \Phi\}$ respectively. That is, the coefficient $F_m(t)$ can be expressed as a volume integral over the body force and free charge excitation plus the boundary integral terms. By using the boundary conditions in equation (17), the above equation leads to

$$\begin{aligned} N_m \omega_m^2 F_m(t) = {} & \int_V u_j^m \rho f_j \mathrm{d}^3 v + \int_{S_1} u_j^m \hat{t}_j(\mathbf{x}, t) \mathrm{d}^2 s - \int_{S_2} t_j^m \hat{u}_j(\mathbf{x}, t) \, \mathrm{d}^2 s \\ & - \int_V \varphi^m q \mathrm{d}^3 v + \int_{S_3} \varphi^m \hat{d}(\mathbf{x}, t) \mathrm{d}^2 s - \int_{S_4} d^m \hat{\varphi}(\mathbf{x}, t) \mathrm{d}^2 s. \end{aligned} \tag{30}$$

Thus, the function $F_m(t)$ can be expressed as integrals over known quantities. Accordingly, the unknown special solution $\mathbf{U}(\mathbf{x}, t)$ is completely determined by equation (23) without solving equations (11)–(13). Combining equations (9), (23) and (27), the response of the displacement field is simply

$$\mathbf{u}(\mathbf{x}, t) = \sum_m \bar{q}_m(t) \mathbf{u}^m(\mathbf{x}). \tag{31}$$

The generalized co-ordinate $\bar{q}_m(t)$ in the above equation is determined by equation (28) with $F_m(t)$ given by equation (30). To further simplify the above formulae and the ensuing derivations, we introduce the following definitions:

$$A_m(t) \equiv \omega_m^2 F_m(t),$$

$$B_m(t) \equiv N_m^{-1} \int_V u_j^m(\mathbf{x}) \rho f_j(\mathbf{x}, t) \mathrm{d}^3 v,$$

$$T_m(t) \equiv N_m^{-1} \int_{S_1} u_j^m(\mathbf{x}) \hat{t}_j(\mathbf{x}, t) \mathrm{d}^2 s,$$

$$U_m(t) \equiv -N_m^{-1} \int_{S_2} t_j^m(\mathbf{x}) \hat{u}_j(\mathbf{x}, t) \mathrm{d}^2 s,$$

$$D_m(t) \equiv N_m^{-1} \int_{S_3} \varphi^m(\mathbf{x})\hat{d}(\mathbf{x}, t)\mathrm{d}^2 s,$$

$$\psi_m(t) \equiv -N_m^{-1} \int_{S_4} d^m(\mathbf{x})\hat{\varphi}(\mathbf{x}, t)\mathrm{d}^2 s,$$

$$Q_m(t) \equiv -N_m^{-1} \int_V \varphi^m(\mathbf{x}) q(\mathbf{x}, t)\mathrm{d}^3 v. \tag{32}$$

The time function $A_m(t)$ is the combined modal excitation function of the mth mode. The mechanical quantities $B_m(t)$, $T_m(t)$ and $U_m(t)$ are referred to as the participation factors to the mth mode due to body force, boundary traction and boundary displacement respectively. Similarly, the electrical quantities $Q_m(t)$, $D_m(t)$ and $\psi_m(t)$ are referred to as the participation factors to the mth mode due to free charge density, boundary electric displacement and applied boundary voltage respectively. With the above definitions, we obtain

$$A_m(t) = B_m + T_m + U_m + D_m + \psi_m + Q_m. \tag{33}$$

From equations (28) and (32), the governing equation and initial conditions for $\bar{q}_m(t)$ are

$$\ddot{\bar{q}}_m + \omega_m^2 \bar{q}_m = A_m(t), \qquad \bar{q}_m(0) = \lambda_m, \qquad \dot{\bar{q}}_m(0) = \kappa_m. \tag{34}$$

The combined modal excitation function $A_m(t)$ can be carried out through equations (33) and (32). The initial values λ_m and κ_m are given by equations (25) and (26) respectively. The solution of equation (34) is

$$\bar{q}_m(t) = \lambda_m \cos\omega_m t + \frac{\kappa_m}{\omega_m}\sin\omega_m t + \frac{1}{\omega_m}\int_0^t A_m(\tau)\sin\omega_m(t-\tau)\,\mathrm{d}\tau. \tag{35}$$

Once the generalized co-ordinate $\bar{q}_m(t)$ is obtained by the above equation, the displacement $\mathbf{u}(x, t)$ can be determined from equation (31) completely. This is the general solution for the vibration of non-homogeneous linear piezoelectric solid under various excitations. Six types of excitations are considered: the body forces, the boundary traction forces, the specified boundary displacements, the free charges, the boundary electric displacements and the applied boundary voltages. For the special case of zero initial conditions, i.e., no velocity and displacement at time $t = 0$, thus $\lambda_m = \kappa_m = 0$, the solution of $\bar{q}_m(t)$ reduces to

$$\bar{q}_m(t) = \frac{1}{\omega_m}\int_0^t A_m(\tau)\sin\omega_m(t-\tau)\,\mathrm{d}\tau. \tag{36}$$

When the excitation $A_m(t)$ is the unit delta function, the impulse response function $h_m(t)$ of $\bar{q}_m(t)$ can be determined from equation (36) as

$$h_m(t) = \frac{1}{\omega_m}\sin\omega_m t \tag{37}$$

and equation (36) can be rewritten simply as a convolution integral of the combined modal excitation function and the impulse response function:

$$\bar{q}_m(t) = \int_0^t A_m(\tau) h_m(t-\tau)\,\mathrm{d}\tau. \tag{38}$$

By the above derivations, a normal mode expansion method of the displacement field for the non-homogeneous linear piezoelectric solid is obtained. It is trivial exercise to check that when the piezoelectric constants approach zero, the formulation reduces to a normal mode expansion fomalism for linear but non-homogeneous elastic solids. Of course, the present method, requires the knowledge of free-vibration solution. If the exact analytical solution of free-vibration problem does not exist, we may resort to the numerical method, e.g., finite element method [7]. Once we get the normal mode solution, either analytically or numerically, the displacement field can be calculated by equations (31) and (35) accompanied with definitions in equations (25), (26), (32), and (33).

4. EXAMPLE

To illustrate the above we consider the forced shear motion of a piezoelectric plate with monoclinic symmetry. In Figure 1, the dimensions in both x_1 and x_3 directions are infinite, i.e., $-\infty < x_1 < \infty, -\infty < x_3 < \infty$. The x_3-axis is chosen to be perpendicular to the plane of mirror symmetry. The thickness of the plate is $2h$, $-h < x_2 < h$. We assume that there are no variations for all field quantities along both x_1 and x_3 directions. The governing equations are [9]

$$C_{66}u_{1,22} + e_{26}\varphi_{,22} = \rho\ddot{u}_1, \qquad C_{22}u_{2,22} + C_{24}u_{3,22} = \rho\ddot{u}_2,$$
$$C_{24}u_{2,22} + C_{44}u_{3,22} = \rho\ddot{u}_3, \qquad e_{26}u_{1,22} - \varepsilon_{22}\varphi_{,22} = 0. \tag{39}$$

The given shear stresses and voltage on two side faces are

$$C_{66}u_{1,2} + e_{26}\varphi_{,2} = \sigma_1(t), \quad x_2 = h,$$
$$C_{66}u_{1,2} + e_{26}\varphi_{,2} = -\sigma_2(t), \quad x_2 = -h,$$
$$\varphi = V_1(t), \quad x_2 = h, \qquad \varphi = V_2(t), \quad x_2 = -h. \tag{40}$$

The signs of stresses σ_1 and σ_2 are chosen such that the positive directions for them are in the positive x_1-axis. It is clear that only u_1 is coupled to potential field φ. We will neglect the motions of u_2 and u_3 in the following.

The normal mode solutions which satisfy the homogeneous BC's corresponding to equation (40) will first be solved. For even modal numbers $n = 0, 2, 4, \ldots$, the normal mode solutions are

$$u^n(x) = \cos\eta_n x, \qquad \varphi^n(x) = e(\cos\eta_n x - \cos\eta_n h), \tag{41}$$

where $\eta_n = n\pi/2h$, $e = e_{26}/\varepsilon_{22}$, $x = x_2$. The subscript 1's for the displacement field and 2's for the x_2-co-ordinate are omitted in equation (41) and the following expressions. For odd modal numbers $n = 1, 3, 5, \ldots$, the normal mode solutions are

$$u^n(x) = \sin\eta_n x, \qquad \varphi^n(x) = e[\sin\eta_n x - (x/h)\sin\eta_n h], \tag{42}$$

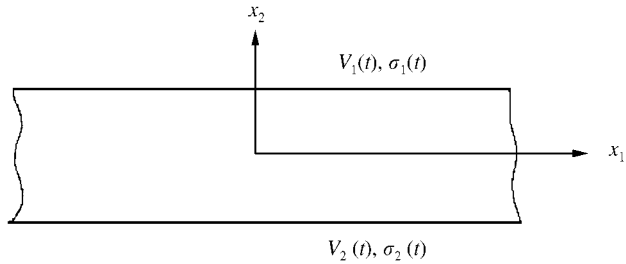

Figure 1. Infinite piezoelectric plate of thickness $2h$.

where η_n satisfies $\tan \eta_n h = (\bar{C}_{66}\varepsilon_{22}/e_{26}^2)\eta_n h$ and $(n-1)\pi/2h < \eta_n < (n+1)\pi/2h$, and $\bar{C}_{66} = C_{66} + e_{26}^2/\varepsilon_{22}$ is the equivalent shear modulus. Following the solution formalism derived in the previous sections, we can get the general solution easily. According to equation (31), the general solution could be expressed as

$$u(x, t) = \sum_0^\infty \bar{q}_n(t) u^n(x). \tag{43}$$

The generalized co-ordinate $\bar{q}_n(t)$ can be calculated by equation (35) as

$$\bar{q}_n(t) = \frac{1}{\omega_n}\int_0^t A_n(\tau)\sin\omega_n(t-\tau)\,\mathrm{d}\tau, \tag{44}$$

where $\omega_n = \eta_n c$ is the nth modal circular frequency, and $c = (\bar{C}_{66}/\rho)^{1/2}$ is the shear wave speed. We have assumed that the initial conditions are zero. The combined modal excitation $A_n(t)$ is calculated via equations (32) and (33). Because only stress and potential are prescribed at the boundary, we merely have to evaluate $T_n(t)$ and $\psi_n(t)$ in equations (32) and (33). Although the boundary surfaces involved in equations (32) are infinite in both x_1 and x_3 directions, we need to only integrate over unit length in both directions. The result of $A_n(t)$ is

$$A_n(t) = N_n^{-1}[u^n(h)\sigma_1(t) + u^n(-h)\sigma_2(t) - V_1(t)D^n(h) + V_2(t)D^n(-h)], \tag{45}$$

where $D^n(x) = e_{26}u^n_{,2} - \varepsilon_{22}\varphi^n_{,2}$, and the generalized mass N_n (per unit area) is

$$N_n = \rho\int_{-h}^{h}[u^n(x)]^2\,\mathrm{d}x. \tag{46}$$

Equations (41)–(46) constitute the general solution for arbitrary $V_1(t)$, $V_2(t)$, $\sigma_1(t)$ and $\sigma_2(t)$. To give an explicit numerical example, we assume $V_1(t) = V_2(t) = 0$ and apply the triangular transient loads in the x_1 direction on both sides:

$$\sigma_1(t) = \sigma_2(t) = \sigma(t) = \begin{cases} \sigma_0 t/T, & 0 \leqslant t < T/2, \\ \sigma_0(1 - t/T), & T/2 \leqslant t < T, \\ 0, & T \leqslant t \text{ or } t < 0, \end{cases} \tag{47}$$

where σ_0 is the peak total applied force per unit area, and T is the duration of excitation. It is obvious that only even modes will be excited. For even modes, the generalized masses are

$N_0 = 2\rho h$, $N_n = \rho h$, $n = 2, 4, 6, \ldots$, and the combined modal excitation $A_n(t)$ will be

$$A_n(t) = \begin{cases} \sigma(t)/\rho h, & n = 0, \\ (-1)^{n/2} 2\sigma(t)/\rho h, & n = 2, 4, 6 \ldots . \end{cases} \tag{48}$$

To express the results in dimensionless forms, we define the references for various physical quantities:

$$\begin{aligned} &\delta = \frac{2\sigma_0 h}{\bar{C}_{66}} = \frac{\sigma_0 T_0^2}{2\rho h} = \text{the reference for deflection,} \\ &\bar{\varphi} = e_{26}\delta/\varepsilon_{22} = \text{the reference for potential,} \\ &\bar{\sigma} = \sigma_0/2 = \text{the reference for stress,} \end{aligned} \tag{49}$$

where $T_0 = 2h/c$ is the one-way travel time for shear stress wave propagating through the plate thickness. The generalized co-ordinates can be expressed in the dimensionless forms:

$$\tilde{q}_0(t) = \frac{\bar{q}_0(t)}{\delta} = \begin{cases} \dfrac{T_0}{3T}\left(\dfrac{t}{T_0}\right)^3, & 0 \leqslant t < T/2 \\ \dfrac{1}{12}\left(\dfrac{T}{T_0}\right)^2 - \dfrac{T}{2T_0}\dfrac{t}{T_0} + \left(\dfrac{t}{T_0}\right)^2 - \dfrac{T_0}{3T}\left(\dfrac{t}{T_0}\right)^3, & T/2 \leqslant t < T, \\ \dfrac{T}{T_0}\left(\dfrac{t}{2T_0} - \dfrac{T}{4T_0}\right), & T \leqslant t, \end{cases}$$

$$\tilde{q}_n(t) = \frac{\bar{q}_n(t)}{\delta}$$

$$= \begin{cases} \dfrac{(-1)^{n/2}4}{n^2\pi^2}\left[\dfrac{t}{T} - \dfrac{1}{n\pi}\dfrac{T_0}{T}\sin n\pi\dfrac{t}{T_0}\right], & 0 \leqslant t < T/2, \\ \dfrac{(-1)^{n/2}4}{n^2\pi^2}\left[1 - \dfrac{t}{T} + \dfrac{2}{n\pi}\dfrac{T_0}{T}\sin n\pi\left(\dfrac{t}{T_0} - \dfrac{T}{2T_0}\right) - \dfrac{1}{n\pi}\dfrac{T_0}{T}\sin n\pi\dfrac{t}{T_0}\right], & T/2 \leqslant t < T, \\ \dfrac{(-1)^{n/2}4}{n^3\pi^3}\dfrac{T_0}{T}\left[2\sin n\pi\left(\dfrac{t}{T_0} - \dfrac{T}{2T_0}\right) - \sin n\pi\dfrac{t}{T_0} - \sin n\pi\left(\dfrac{t}{T_0} - \dfrac{T}{T_0}\right)\right], & T \leqslant t. \end{cases} \tag{50}$$

$n = 2, 4, 6, \ldots$

The scaled displacement, potential and stress can then be expressed as

$$\tilde{u}(x, t) = \frac{u(x, t)}{\delta} = \sum_{n=0, 2, 4, \ldots} \tilde{q}_n(t)\cos\frac{n\pi x}{2h}, \tag{51}$$

$$\tilde{\varphi}(x, t) = \frac{\varphi(x, t)}{\bar{\varphi}} = \tilde{u}(x, t) - \tilde{u}(h, t), \tag{52}$$

$$\tilde{\sigma}(x, t) = \frac{\sigma_{21}(x, t)}{\bar{\sigma}} = -2\pi \cdot \sum_{n=2, 4, 6, \ldots} n\tilde{q}_n(t)\sin\frac{n\pi x}{2h}. \tag{53}$$

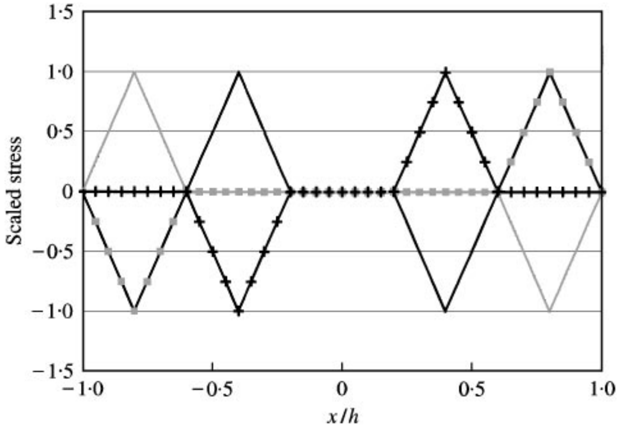

Figure 2. Scaled stresses for different t. $t = 0{\cdot}2T_0$, $t = 0{\cdot}4T_0$; ----- $t = 0{\cdot}6T_0$; $t = 0{\cdot}8T_0$; $t = 1{\cdot}0T_0$.

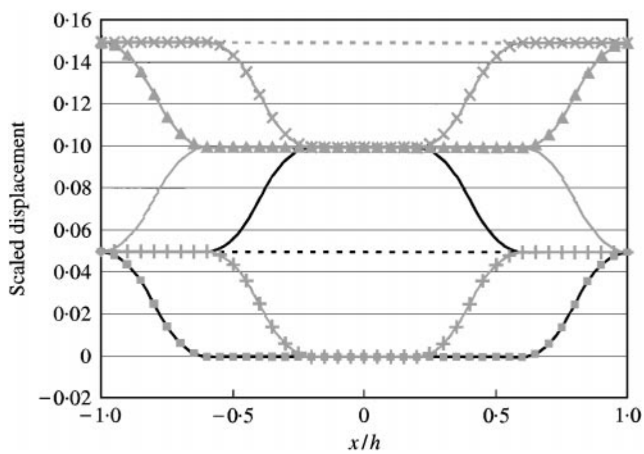

Figure 3. Scaled displacement for different t. $t = 0{\cdot}2T_0$, $t = 0{\cdot}4T_0$; ----- $t = 0{\cdot}6T_0$; $t = 0{\cdot}8T_0$; $t = 1{\cdot}0T_0$; $t = 1{\cdot}2T_0$; $t = 1{\cdot}4T_0$; $t = 1{\cdot}6T_0$.

In Figure 2, we show the stress distributions at various times t for the case in which duration T is set to be $0{\cdot}2T_0$. Note that $x = h$ means the upper surface, and $x = -h$ means the lower surface in Figure 1. In the beginning, as the transient shear forces are applied at the boundaries, the stress waves travel from two sides into the solid. The amplitude of the stress wave is exactly equal to that of excitations. At time $t = 0{\cdot}2T_0$, when the excitations stop, the wave fronts reach the depth $0{\cdot}4h$, which is equal to the product of wave speed c and time $0{\cdot}2T_0$. At time $t = 0{\cdot}6T_0$, when the upward and downward travelling waves coincide in space, the stress waves cancel out, and this solid is temporarily in a stress-free state. For $t \geqslant T = 0{\cdot}2T_0$, the stress distribution will repeat itself with the interval of T_0, which can be easily checked by equations (49) and (52). When the stress waves impinge on the boundary, the reflected stresses are reversed.

In Figures 3 and 4, the corresponding displacement and potential distributions at various times t are shown. As the stress waves travel back and forth, the elastic deformation also travels back and forth with a steady rigid-body motion. It is easily checked that the scaled

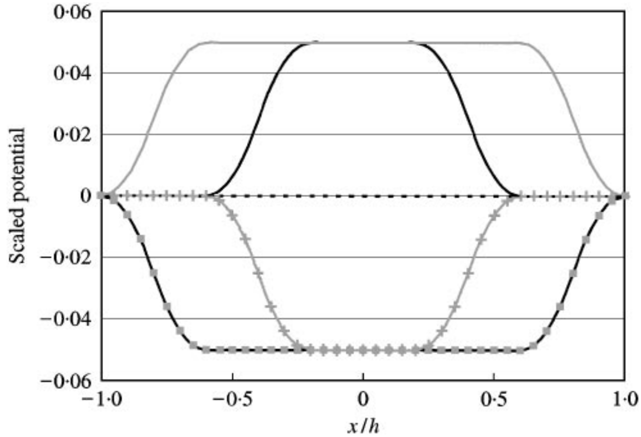

Figure 4. Scaled potential for different t. —■— $t = 0{\cdot}2T_0$, —+— $t = 0{\cdot}4T_0$; ----- $t = 0{\cdot}6T_0$; ___ $t = 0{\cdot}8T_0$; ——— $t = 1{\cdot}0T_0$.

amplitude of elastic deflection is 0·05. Note that the propagation speed of electric disturbance is also the equivalent shear wave speed c. In fact, in the piezoelectric equations (5) and (6), the speed of light is assumed to be infinity.

5. CONCLUSION

The normal mode expansion formulation for non-homogeneous linear piezoelectric solids is developed. It can be shown that, with the given normal modes, the displacement field can be expanded as the linear superposition of normal modes with the modal coefficients represented as surface and volume integrals over the exciting source functions. Six types of excitations are considered. These results can be used as a foundation for modal testing methods and for general dynamic analysis for linear piezoelectric materials. Because the theory covers generally non-homogeneous materials, it can be applied directly to arbitrary configuration of elastic-piezoelectric composites, which are widely used in the vibrational and acoustic sensor/actuator/transmitter applications. A simple example is given to exemplify the application of the present formulation.

REFERENCES

1. E. Dieulesaint and D. Roger 1980 *Elastic Waves in Solid, Applications to Signal Processing*. UK: John Wiley & Sons.
2. D. Stansfield 1990 *Underwater Electroacoustic Transducers*. UK: Bath University Press and Institute of Acoustics.
3. C. K. Lee, W. W. Chiang and T. C. O'Sullivan 1991 *Journal of the Acoustical Society of America* **90**, 374–384. Piezoelectric modal sensors/actuators pair for critical active damping vibration control.
4. J. Zelenka 1986 *Piezoelectric Resonators and their Applications*. Amsterdam: Elsevier Science Publishers.
5. P. Heyliger and D. A. Saravanos 1995 *Journal of the Acoustical Society of America* **98**, 1547–1557. Exact free-vibration analysis of laminated plates with embedded piezoelectric layers.
6. D. A. Saravanos, P. Heyliger and D. A. Hopkins 1997 *International Journal of Solids and Structures* **34**, 359–378. Layerwise mechanics and finite element for the dynamic analysis of piezoelectric composite plates.

7. Y. KAGAWA, T. TSUCHIYA and T. KATAOKA 1996 *Journal of Sound and Vibration* **191,** 519–538. Finite element simulation of dynamic responses of piezoelectric actuators.
8. R. D. MINDLIN 1961 *Problems of Continuum Mechanics*, 282–290. Philadelphia: Society of Industrial and Applied Mathematics. On the equations of motion of piezoelectric crystals.
9. H. F. TIERSTEIN 1969 *Linear Piezoelectric Plate Vibrations*. New York: Plenum Press.
10. A. C. ERINGEN and E. S. SUHUBI 1975 *Elastodynamics*, Vol. 2. London: Academic Press.
11. D. C. LIU 1999 *International Journal of Applied Mathematics* **1**, 281–293. On the dynamic solution for the forced vibration of piezoelectric solids.