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# A POWER-SERIES SOLUTION FOR A STRONGLY NON-LINEAR TWO-DEGREE-OF-FREEDOM SYSTEM

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A power-series method is presented for the analysis of a conservative strongly non-linear two-degree-of-freedom (d.o.f.) system with cubic non-linearity. The method is based on transforming the time variable into an harmonically oscillating time whereby the governing differential equations become well conditioned for power-series analysis. The oscillating time frequency is obtained by enforcing Rayleigh's energy principle. The results show good agreement with those obtained using the Lindstedt–Poincaré method.



## 1. INTRODUCTION

In recent years, attempts have been made by researchers to extend the use of classical perturbation techniques to strongly non-linear oscillators. Geer and Anderson [1] used a hybrid perturbation-Galerkin technique which employs a perturbation expansion to give an approximate solution which is then used for a subsequent Galerkin analysis. Burton [2, 3] used the classical Lindstedt–Poincaré (LP) method and defined an expansion parameter to enable accurate low order solutions to be obtained for oscillators with odd non-linearity. More recently, Cheung *et al.* [4] proposed a modified LP method (MLP) by defining a new expansion parameter which remains small even if the original parameter grows without bound. The MLP method is suitable for a system with even or odd non-linearity and has been used to analyse a strongly non-linear single-d.o.f. system [4] and later generalized to multiple-d.o.f. systems [5].

A major drawback of the perturbation methods is the excessive labour needed for algebraic manipulations of successive perturbation steps. This promoted the use of recently developed symbolic softwares to reduce the effort involved in such perturbation problems. In this paper, a power-series method [6] is presented for an undamped strongly non-linear two-d.o.f. system previously analyzed by Cheung *et al.* [5]. The power-series analysis of the undamped oscillators is facilitated by transforming the time variable into an harmonically oscillating time. The method yields results that compare well with those of existing techniques and has the advantage of requiring minimum computational effort and simple computer code.

## 2. FORMULATION

Consider the free vibrations of a two-d.o.f. system governed by the equations.

$$\ddot{x} + x + Ax^3 + Bx^2y + Cxy^2 + Dy^3 = 0, \tag{1}$$

$$\ddot{y} + 9y + Ex^3 + Fx^2y + Gxy^2 + Hy^3 = 0 \tag{2}$$

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subject to the initial conditions $x(0) = x_0$, $\dot{x}(0) = 0$, $y(0) = y_0$, $\dot{y}(0) = 0$. The overdot denotes differentiation with respect to time $t$. This system has been analyzed by Cheung *et al.* [5] and describes the transverse free vibration of an undamped clamped–hinged beam using a two-mode approximation. The constants in equations (1) and (2) have the following values; $A = 0{\cdot}2788$, $B = -0{\cdot}3111$, $C = 1{\cdot}116$, $D = -0{\cdot}3864$, $E = B/3$, $F = C$, $G = 3\mathrm{D}$, $H = 3{\cdot}8703$. A direct power-series expansion for the displacements $x$, $y$ in terms of $t$, results in a convergent solution over a small time interval only in the neighbourhood of $t = 0$. However, power-series expansions for conservative systems that are convergent for all times are facilitated upon transforming the time variable $t$ into an harmonically oscillating time $\tau$ as follows:

$$\tau = \sin \omega t, \tag{3}$$

whereby the infinite extent of time $t$ $(0 \leqslant t < \infty)$ is reduced to a finite time scale $(-1 \leqslant \tau \leqslant 1)$, and the new time $\tau$ oscillates at a frequency $\omega$ to be determined. Upon introducing equation (3) into equations (1) and (2), the transformed equations of motion become

$$\omega^2(1-\tau^2)x'' - \omega^2 \tau x' + x + Ax^3 + Bx^2 y + Cxy^2 + Dy^3 = 0, \tag{4}$$

$$\omega^2(1-\tau^2)y'' - \omega^2 \tau y' + 9y + Ex^3 + Fx^2 y + Gxy^2 + Hy^3 = 0 \tag{5}$$

subject to the initial conditions $x(0) = x_0$, $x'(0) = 0$, $y(0) = y_0$, $y'(0) = 0$. The prime denotes differentiation with respect to time $\tau$. This transformation permits power-series representation of $x$ and $y$ in terms of $\tau$. According to the theory of ordinary differential equations [7], equations (4) and (5) have one ordinary point at $\tau = 0$ and two regular singular points at $\tau = \pm 1$. It is convenient to write power-series expansion for $x$ and $y$ about the ordinary point as

$$x(\tau) = a_1 + a_2\tau + a_3\tau^2 + \cdots = \sum_{k=1}^{\infty} a_k \tau^{k-1}, \tag{6}$$

$$y(\tau) = b_1 + b_2\tau + b_3\tau^2 + \cdots = \sum_{k=1}^{\infty} b_k \tau^{k-1}, \tag{7}$$

where $a_i$, $b_i$ are constant coefficients to be determined. Since $\tau$ is periodic, equations (6) and (7) are capable of capturing periodic motion which is appropriately assumed to start from the maximum displacement position. Under this condition, all the terms having odd powers of $\tau$ in equations (6) and (7) vanish and the same motion is repeated every half-cycle (positive or negative) of the oscillating time. This requires the oscillating time frequency to be equal to one-half the vibration frequency.

$$\omega = \frac{\Omega}{2}. \tag{8}$$

Introducing equations (6) and (7) into equation (4), one obtains

$$\omega^2(1-\tau^2)\sum_{k=1}^{\infty} a_k(k-1)(k-2)\tau^{k-3} - \omega^2\tau \sum_{k=1}^{\infty} a_k(k-1)\tau^{k-2} + \sum_{k=1}^{\infty}(a_k + Ac_k + Bd_k + Ce_k + Df_k)\tau^{k-1} = 0 \ldots, \tag{9}$$

in which the non-linear terms are expanded as

$$x^3 = \sum_{k=1}^{\infty} c_k \tau^{k-1}, \quad x^2 y = \sum_{k=1}^{\infty} d_k \tau^{k-1}, \quad xy^2 = \sum_{k=1}^{\infty} e_k \tau^{k-1}, \quad y^3 = \sum_{k=1}^{\infty} f_k \tau^{k-1}, \tag{10}$$

which result from different multiplications of equations (6) and (7). It follows that the constant coefficients $c_k$, $d_k$, $e_k$ and $f_k$ can be computed once the constants $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ are known. An appropriate shifting of indices in the first two terms in equation (9) is now introduced so that all terms have the same power, thus

$$\sum_{k=1}^{\infty} \{\omega^2 [k(k+1) a_{k+2} - (k-1)^2 a_k] + a_k + Ac_k + Bd_k + Ce_k + Df_k\} \tau^{k-1} = 0. \tag{11}$$

Equation (11), representing the first equation of motion, is satisfied exactly by making the coefficient of each power vanish identically. This condition introduces the recurrence relation

$$a_{k+2} = \frac{[(k-1)^2 \omega^2 - 1] a_k - Ac_k - Bd_k - Ce_k - Df_k}{k(k+1)\omega^2}, \quad k = 1, 2, 3 \ldots \tag{12}$$

between the series coefficients. Similarly, introducing equations (6) and (7) into equation (5) and following the same steps described above, the following recurrence relation is established:

$$b_{k+2} = \frac{[(k-1)^2 \omega^2 - 1] b_k - Ec_k - Fd_k - Ge_k - Hf_k}{k(k+1)\omega^2}, \quad k = 1, 2, \ldots . \tag{13}$$

By introducing the initial conditions associated with equations (4) and (5) into equations (6) and (7), one obtains the following coefficients:

$$a_1 = x_0, \quad a_2 = 0, \quad b_1 = y_0, \quad b_2 = 0. \tag{14}$$

The remaining coefficients depend recursively on these four fundamental coefficients and on the oscillating time frequency in accordance with equations (12) and (13). It follows that a solution, as expressed in equations (5) and (6), is obtained once the oscillating time frequency is determined. For that purpose, Rayleigh's energy principle is invoked. This principle states that, for a conservative system, the maximum potential and kinetic energies are equal. For the system under consideration, the potential energy $V$ is obtained from its relation with the non-linear elastic forces

$$F_x = \frac{\partial V}{\partial x} \quad \text{and} \quad F_y = \frac{\partial V}{\partial y}$$

in equations (1) and (2), respectively, as

$$V = \frac{1}{2}(x^2 + 9y^2) + \frac{1}{4}(Ax^4 + Hy^4) + \frac{B}{3} x^3 y + \frac{C}{2} x^2 y^2 + Dxy^3. \tag{15}$$

The maximum potential energy $V_{max}$ is associated with the maximum displacement position assumed to occur at the start of the motion $t = \tau = 0$, and is determined by setting $x = x_0$, $y = y_0$ in equation (15). The kinetic energy of the system is given by

$$T = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\dot{y}^2 = \tfrac{1}{2}\omega^2(1 - \tau^2)(x'^2 + y'^2). \tag{16}$$

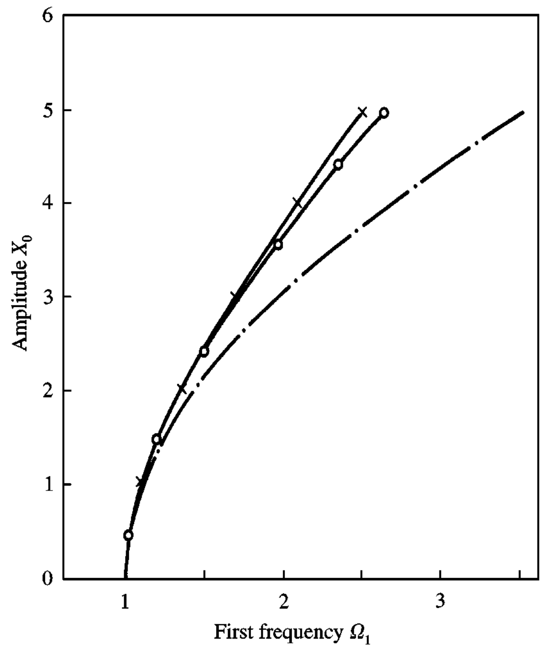


Figure 1. Amplitude–frequency relation ($x_0 - \Omega_1$). (—·—·—), LP method: (×——×), MLP method (○—○), present.

The maximum kinetic energy $T_{max}$ occurs at the equilibrium position for which $\omega t = \pi/4, 3\pi/4, 5\pi/4, \ldots$, etc. From equation (3), this position is reached at $\tau = \pm 1/\sqrt{2}$. By using this result in equation (16), one obtains

$$T_{max} = \tfrac{1}{4}\,\omega^2 (x'^2 + y'^2)|_{\tau = 1/\sqrt{2}}\,. \tag{17}$$

## 3. RESULTS AND DISCUSSION

The free vibration solution to equations (1) and (2) begins by assuming a set of initial conditions for the displacements $x$ and $y$. For the first non-linear frequency of vibration, it is assumed that $x(0) = x_0$, $\dot{x}(0) = 0$, $y(0) = 0$, $\dot{y}(0) = 0$. This determines the first two fundamental constants in equations (6) and (7) from equation (14). The remaining coefficients are determined recursively from equations (12) and (13) for an assumed value of the oscillating time frequency. A search for the correct oscillating time frequency is made by computing the error function $\varepsilon = V_{max} - T_{max}$ for each $\omega$ and the actual frequency is obtained when $\varepsilon = 0$ which ensures that Rayleigh's energy principle is satisfied. It was observed that for small amplitude oscillations, the error function $\varepsilon$ had a stationary minimum value at the correct solution whereas it changed sign for large amplitudes. In Figure 1, the first non-linear vibration frequency $\Omega_1$ obtained by the LP and MLP method [5] is compared with the present method. Good agreement is seen between the power-series solution and the MLP method. The error in the classical LP method is significant at large amplitudes.

A convergence test was made for the power-series solution. Figure 2 shows the convergence of the first frequency for amplitude $x_0 = 3$ as the number of terms are increased. For smaller amplitudes, fewer terms are required to obtain accurate solutions.

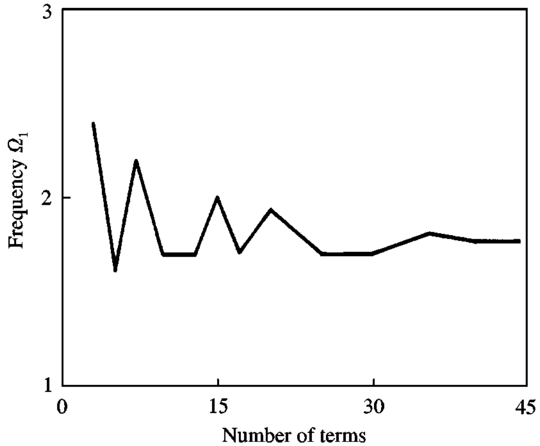


Figure 2. Convergence of the first vibration frequency for amplitude $x_0 = 3$.

TABLE 1

*Odd power-series coefficients for amplitude $x_0 = 3$, number of terms = 25*

| $a$ | $i = 1$ | $i = 3$ | $i = 5$ | $i = 7$ | $i = 9$ |
|---|---|---|---|---|---|
| $a_i$ | 0·3000E1 | − 0·7117E1 | 0·5063E1 | − 0·7364E1 | 0·9661E1 |
| $a_{i+10}$ | − 0·1477E2 | 0·2241E2 | − 0·3496E2 | 0·5489E2 | − 0·8713E2 |
| $a_{i+20}$ | 0·1392E3 | − 0·2236E3 | 0·3608E3 | | |
| $b_i$ | 0·0000Eo | 0·1893E1 | − 0·5676E1 | 0·9241E1 | − 0·1515E2 |
| $b_{i+10}$ | 0·2505E2 | − 0·4196E2 | 0·7041E2 | − 0·1181E3 | 0·1980E3 |
| $b_{i+20}$ | − 0·3718E3 | 0·5660E3 | − 0·9317E3 | | |

The solution generated had zero even power-series coefficients because of the vanishing of initial velocities. Table 1 shows the odd coefficients for amplitude $x_0 = 3$ with 25 terms. A progressive increase in the absolute value of the coefficients is evident. This characterizes the solution coefficients at large amplitudes and may be explained by a ratio test. By noting that only even powers of $\tau$ exist in equations (6) and (7), the convergence of the solution for $x$ is assured providing the ratio between two consecutive terms $|a_{n+2}\tau^2/a_n| < 1$, so that $|a_{n+2}/a_n| < q$, where $q = |1/\tau^2|$. For small amplitude vibrations, convergent power-series solutions are obtained over the entire time domain corresponding to $|\tau| = 1$ for which $q = 1$ and the series coefficients therefore decrease in absolute value with an increase of the index. For large amplitude vibrations, convergent solutions are obtained [8] over one-quarter cycle corresponding to $|\tau| = 1/\sqrt{2}$ for which $q = 2$ and the series coefficients may therefore increase such that $|a_{n+2}/a_n| < 2$ as can be verified from Table 1. The same argument is extended to the coefficients of $y$.

Figure 3 shows the results for the amplitude–frequency relation of the second frequency $(y_0 - \Omega_2)$.

## 4. CONCLUSION

A power-series solution has been presented for a strongly non-linear two-d.o.f. system with cubic non-linearity. The results show good agreement with the modified

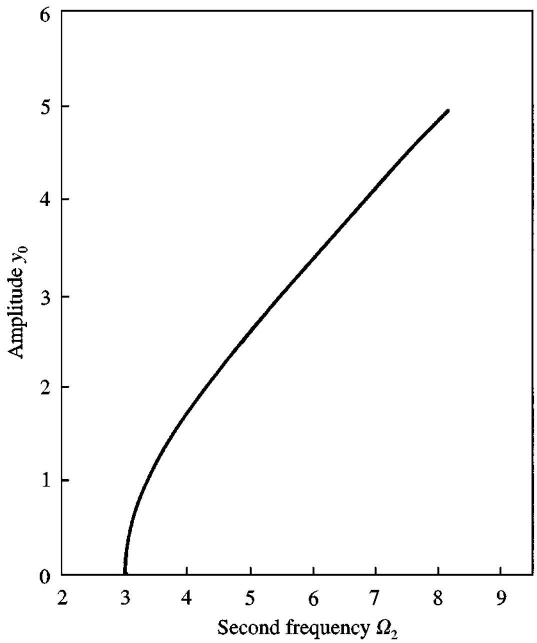


Figure 3. Amplitude–frequency relation ($y_0 - \Omega_2$).

Lindstedt–Poincare' method for the first vibration frequency. The method can be applied to strongly non-linear conservative oscillators and avoids tedious algebraic manipulations inherent in perturbation techniques.

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