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A METHOD FOR THE EFFICIENT CONSTRUCTION OF ACOUSTIC PRESSURE CROSS-SPECTRAL MATRICES

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A method is presented for constructing, with the minimum number of physical measurements, the full cross-spectral matrix of acoustic pressures associated with a number of measurement positions. It is necessary to evaluate the elements of the matrix in question when using inverse methods for the reconstruction of acoustic source strength spectra. The method presented uses the concept of "reference microphones". The relation between the rank of the cross-spectral matrix of acoustic pressures and the number of uncorrelated acoustic sources is discussed and used to determine the required number of reference microphones. A method is proposed for selecting this number in the inverse problem in which information regarding acoustic sources is unknown. The results of computer simulations are presented which explore the main features of the technique under various conditions. Experimental results are also presented which validate the technique.

1. INTRODUCTION

In order to reconstruct the cross-spectral matrix of acoustic source strengths by inverse techniques, it is necessary to first measure the cross-spectral matrix of acoustic pressures in the radiated sound field [1, 2]. This matrix can be constructed by measuring directly all auto-spectra at the field points considered and the cross-spectra between all pairs of field points. However, this often leads to a tedious and expensive task, especially when the number of field points is large. For example, when we wish to measure acoustic pressure auto- and cross-spectra at 100 field points by using dual channel acquisition equipment, then the dimension of the matrix of acoustic pressure auto- and cross-spectra becomes 100×100 and thus we need 5050 measurements ($= 100 \times 101/2$, since this matrix is an Hermitian matrix). In this case, it is natural to attempt to develop an alternative technique which constructs the full auto- and cross-spectral matrix of acoustic pressures with a minimum number of measurements. This paper proposes such a technique using a small number of reference microphones.

Hald [3], in developing the application of nearfield acoustic holography (NAH), adopted the concept of the reference microphone, with a view to obtaining the full cross-spectral matrix with a minimum number of measurements of acoustic pressure cross-spectra on the "hologram plane". We also employ the concept of the reference microphone to construct the full cross-spectral matrix of acoustic pressures with a minimum number of measurements on the "measurement plane". Although our work adopts the concept of the reference microphone as used as Hald, there are important differences. First of all, here we divide "conceptually" the entire number of measurement microphones (or number of measurement positions) into reference microphones (or reference positions) and moving microphones (or moving positions) on the measurement plane. However, Hald made use of reference microphones that were independent of the measurement array and which were located between acoustic sources and scanning microphones on the hologram plane. In our work, the full cross-spectral matrix of acoustic pressures on the measurement plane comprises contributions from both the reference and moving positions (or microphones). On the contrary, in Hald's work, the full cross-spectral matrix of acoustic pressures consists of only the auto- and cross-spectral matrix of acoustic pressures measured on the hologram plane, not including those sensed by reference microphones. Furthermore, our mathematical development differs from that presented by Hald. For example, Hald constructed the cross-spectral matrix of acoustic pressures on the hologram plane using a number of reference microphones based on the number of uncorrelated sources. In what follows, however, we construct this matrix using reference microphones based on the number of uncorrelated sources and the contaminating noise.

In this paper, the theoretical development of this technique is presented by employing the concept of the rank of matrix. The heart of this technique is the proof of the rank equality between the matrix of acoustic pressure cross-spectra measured at the entire number of field points and a certain sub-matrix of acoustic pressure cross-spectra. To verify the rank equality, it is first necessary to understand the relation between the rank of the acoustic pressure cross-spectral matrix and the number of uncorrelated acoustic sources, and this is therefore described. Also, some methods for the estimation of the ranks of these matrices are discussed. They are eigenvalue decomposition, singular value decomposition, principal component analysis and virtual coherence. It is crucial, in securing the rank equality referred to above, to select properly the number of reference microphones. Accordingly, a method is proposed for selecting this number in an inverse problem in which information regarding acoustic sources is unknown. In order to clarify the main features of the theory developed, the results of computer simulations are presented for the problems in which acoustic sources are either mutually uncorrelated or correlated and in which the effect of output noise is also included. Finally, this technique is validated from experiments which use the acoustic pressures radiated from two volume velocity sources and a simply supported plate mounted in a finite baffle.

2. THEORETICAL DEVELOPMENT

2.1. USE OF REFERENCE MICROPHONES

When we assume that there is no measurement noise, the m-dimensional complex vector $\mathbf{p}$ of acoustic pressures is related to the n-dimensional complex vector $\mathbf{q}$ of acoustic source strengths by using the $m \times n$ complex matrix $\mathbf{H}$ of transfer functions such that

$$\mathbf{p} = \mathbf{Hq}. \tag{1}$$

As described in references [1, 2], when seeking to reconstruct the strengths of a number of stationary random sources of sound from the measured pressure fluctuations we use the expression

$$\mathbf{S}_{qq} = \mathbf{H}^{+}\mathbf{S}_{pp}\mathbf{H}^{+H}, \tag{2}$$

where it is assumed that $m \geqslant n$ and $\mathbf{H}^{+} = [\mathbf{H}^{\mathrm{H}}\mathbf{H}]^{-1}\,\mathbf{H}^{\mathrm{H}}$ is the pseudoinverse of $\mathbf{H}$. The superscript H denotes Hermitian transpose. Note that the matrices of source strength cross-spectra and acoustic pressure cross-spectra are, respectively, defined by

$$\mathbf{S}_{qq} = E[\mathbf{q}\,\mathbf{q}^{\mathrm{H}}], \qquad \mathbf{S}_{pp} = E[\,\mathbf{p}\mathbf{p}^{\mathrm{H}}], \tag{3, 4}$$

where the expectation operator $E[\ ]$ implies that the elements of the vectors $\mathbf{p}$ and $\mathbf{q}$ are given by the Fourier transforms of time histories of finite duration T which are subsequently averaged in the limit $T \to \infty$. It is clear from equation (2) therefore that a measurement of the cross-spectral matrix $\mathbf{S}_{pp}$ is required in order to reconstruct $\mathbf{S}_{qq}$. References [1, 2] describe methods for undertaking this inversion procedure, particularly in circumstances where $\mathbf{H}$ is ill-conditioned.

In order to proceed, as can be seen from Figure 1, we conceptually partition the complex vector $\mathbf{p}$ consisting of acoustic pressures sensed at the entire number of measurement positions into a complex vector $\mathbf{p}_R$ which contains acoustic pressures measured at u reference positions and a complex vector $\mathbf{p}_M$ which consists of acoustic pressures measured at v moving positions.

It is now assumed that the transfer function matrices $\mathbf{H}_R$ and $\mathbf{H}_M$ relate the acoustic source strengths $\mathbf{q}$ to the acoustic pressures $\mathbf{p}_R$ and $\mathbf{p}_M$ at the reference and moving positions respectively. Thus, we write

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}_R \\ \mathbf{p}_M \end{bmatrix} = \begin{bmatrix} p_1(\omega) \\ p_2(\omega) \\ \vdots \\ p_u(\omega) \\ p_{u+1}(\omega) \\ p_{u+2}(\omega) \\ \vdots \\ p_{u+v}(\omega) \end{bmatrix} = \begin{bmatrix} \mathbf{H}_R \\ \mathbf{H}_M \end{bmatrix} \mathbf{q} = \mathbf{H}\mathbf{q}, \tag{5}$$

where the entire number of measurement positions $m = u + v$ which is the sum of the number u of reference positions and the number v of moving positions. With the vector

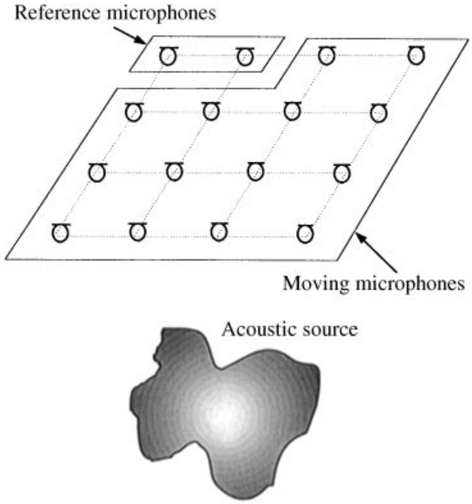

Figure 1. The partition of the entire number of measurement positions (or microphones) into the reference and moving positions (or microphones).

$\mathbf{p}$ partitioned as in equation (5), the acoustic pressure cross-spectral matrix $\mathbf{S}_{pp}$ is given by

$$\mathbf{S}_{pp} = \begin{bmatrix} \mathbf{S}_{RR} & \mathbf{S}_{RM} \\ \mathbf{S}_{RM}^{\mathrm{H}} & \mathbf{S}_{MM} \end{bmatrix}, \tag{6}$$

where $\mathbf{S}_{RR} = E\,[\mathbf{p}_R\,\mathbf{p}_R^{\mathrm{H}}]$, $\mathbf{S}_{RM} = E\,[\mathbf{p}_R\,\mathbf{p}_M^{\mathrm{H}}]$ and $\mathbf{S}_{MM} = E\,[\mathbf{p}_M\,\mathbf{p}_M^{\mathrm{H}}]$.

To construct all the components of $\mathbf{S}_{pp}$ given by equation (6), we undertake the following procedure: we first *measure* the reference position cross-spectral matrix $\mathbf{S}_{RR}$ and the reference-moving position cross-spectral matrix $\mathbf{S}_{RM}$, then *calculate* the moving position auto- and cross-spectral matrix $\mathbf{S}_{MM}$ from the measured $\mathbf{S}_{RR}$ and $\mathbf{S}_{RM}$. Hence, the measurement of only $\mathbf{S}_{RR}$ and $\mathbf{S}_{RM}$ is required in constructing the full matrix of $\mathbf{S}_{pp}$. In the next section, we describe how to calculate the matrix $\mathbf{S}_{MM}$.

2.2. THE MOVING POSITION CROSS-SPECTRAL MATRIX $\mathbf{S}_{MM}$

Let the first u columns of $\mathbf{S}_{pp}$ be represented by the $m \times u$ (or $(u+v) \times u$) matrix $\mathbf{S}_1$ and the remaining columns $m \times v$ (or $(u+v) \times v$) matrix, $\mathbf{S}_2$ respectively. Thus,

$$\mathbf{S}_1 = \begin{bmatrix} \mathbf{S}_{RR} \\ \mathbf{S}_{RM}^{\mathrm{H}} \end{bmatrix}, \qquad \mathbf{S}_2 = \begin{bmatrix} \mathbf{S}_{RM} \\ \mathbf{S}_{MM} \end{bmatrix}. \tag{7, 8}$$

Therefore, the matrix $\mathbf{S}_{pp}$ can be expressed as

$$\mathbf{S}_{pp} = [\mathbf{S}_1 \;\; \mathbf{S}_2]. \tag{9}$$

To calculate the moving position auto- and cross-spectral matrix $\mathbf{S}_{MM}$ from the measured matrices $\mathbf{S}_{RR}$ and $\mathbf{S}_{RM}$, we have to *assume* the rank equality of $\mathbf{S}_1$ and $\mathbf{S}_{pp}$, i.e.,

$$\operatorname{rank}(\mathbf{S}_{pp}) = \operatorname{rank}(\mathbf{S}_1). \tag{10}$$

If this is the case, since the matrix $\mathbf{S}_2$ does not contribute to the rank of the matrix $\mathbf{S}_{pp}$, the columns of $\mathbf{S}_2$ in equation (9) can be expressed by linear combinations of the columns of $\mathbf{S}_1$. Accordingly (see reference [4]), there exists a $u \times v$ matrix $\mathbf{T}$ enabling $\mathbf{S}_2$ to be written as

$$\mathbf{S}_2 = \mathbf{S}_1\mathbf{T}. \tag{11}$$

Using equations (7) and (8), equation (11) can also be expressed as

$$\mathbf{S}_{RM} = \mathbf{S}_{RR}\,\mathbf{T}, \qquad \mathbf{S}_{MM} = \mathbf{S}_{RM}^{\mathrm{H}}\,\mathbf{T}. \tag{12, 13}$$

Arranging equation (12) with respect to $\mathbf{T}$ and substituting this into equation (13), we can find the moving position auto- and cross-spectral matrix $\mathbf{S}_{MM}$. Thus,

$$\mathbf{S}_{MM} = \mathbf{S}_{RM}^{\mathrm{H}}\,\mathbf{S}_{RR}^{-1}\,\mathbf{S}_{RM}. \tag{14}$$

Note that equation (14) is valid only when $\mathbf{S}_{RR}$ is of full rank, otherwise the generalized inverse has to be employed instead of the direct inverse.

This procedure results in a great reduction in the number of measurements required to construct $\mathbf{S}_{pp}$. Thus, while the number of required direct measurements is $(u+v)(u+v+1)/2$ (since the $\mathbf{S}_{pp}$ is $(u+v) \times (u+v)$ and Hermitian), the technique proposed here needs only $u(u+1)/2$ (since $\mathbf{S}_{RR}$ is $u \times u$ and Hermitian) plus uv (since $\mathbf{S}_{RM}$ is $u \times v$). When

$u = 5$ and $v = 95$, for example, the number of required direct measurements is 5050, but this technique reduces that to 490. These numbers are computed for dual channel acquisition, and therefore when multi-channel equipment is used, the number of required measurements will be reduced still further. A simple example of the application of this method is described in reference [5]. It should however, be emphasized that, in using equation (14), it is a pre-requisite to meet the requirement of the rank equality of the matrices $\mathbf{S}_1$ and $\mathbf{S}_{pp}$.

3. ESTIMATION OF THE RANK OF THE CROSS-SPECTRAL MATRIX OF ACOUSTIC PRESSURES

3.1. INTRODUCTION

In order to determine the rank of the cross-spectral matrix $\mathbf{S}_{pp}$, there are a number of tools available. First, note that $\mathbf{S}_{pp}$ is a positive semi-definite Hermitian matrix whose rank is equal to the number of non-zero eigenvalues (including repetitions). Thus, one can express the eigenvalue decomposition (EVD) of the matrix by

$$\mathbf{S}_{pp} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{\mathrm{H}}. \tag{15}$$

where $\mathbf{Q}$ is a unitary matrix containing all the eigenvectors of $\mathbf{S}_{pp}$ and $\boldsymbol{\Lambda}$ is the diagonal matrix of eigenvalues of $\mathbf{S}_{pp}$. Also note that in this case $\boldsymbol{\Lambda} = \boldsymbol{\Sigma}$ which is the diagonal matrix of singular values of $\mathbf{S}_{pp}$ deduced from the singular-value decomposition (SVD) of $\mathbf{S}_{pp}$. Note that $\mathbf{S}_{pp}$ is said to be unitarily similar to $\boldsymbol{\Lambda}$ and thus the rank of $\mathbf{S}_{pp}$ is equal to the rank of $\boldsymbol{\Lambda}$. This expression can also be interpreted in terms of principal component analysis (PCA) [6, 7] or, in the context of this work, in terms of principle spectral analysis (PSA). Thus, one assumes that the measured vector of pressures results from a number of uncorrelated "virtual" acoustic pressures (or "principal components") where

$$\mathbf{p} = \mathbf{Q}\mathbf{v} \tag{16}$$

and $\mathbf{v}$ is the vector of Fourier spectra of the virtual acoustic pressures. Thus, one can write

$$\mathbf{S}_{pp} = \mathbf{Q}\,\mathbf{S}_{vv}\,\mathbf{Q}^{\mathrm{H}}, \tag{17}$$

where $\mathbf{S}_{vv}$ is the diagonal matrix of auto-spectra of the virtual acoustic pressures which comprise the principle spectral components. It is evident that $\boldsymbol{\Lambda} = \boldsymbol{\Sigma} = \mathbf{S}_{vv}$. Also note that the matrix of cross-spectra between the real and virtual acoustic pressures is given by

$$\mathbf{S}_{vp} = E\,[\mathbf{v}\mathbf{p}^{\mathrm{H}}] = \mathbf{S}_{vv}\,\mathbf{Q}^{\mathrm{H}} \tag{18}$$

and the "virtual coherence" [8] between the real and virtual acoustic pressures is derived from the elements of this matrix and is given by

$$\gamma^2_{v_i p_j}(\omega) = \frac{|S_{v_i p_j}(\omega)|^2}{S_{v_i v_i}(\omega)\;\; S_{p_j p_j}(\omega)}, \tag{19}$$

where $S_{v_i v_i}(\omega)$ corresponds to the iith component of the singular value matrix $\boldsymbol{\Sigma}$ or the eigenvalue matrix $\boldsymbol{\Lambda}$ of $\mathbf{S}_{pp}$. Thus, $\gamma^2_{v_i p_j}(\omega)$ indicates the degree to which $S_{p_j p_j}(\omega)$ results from the virtual acoustic pressure $v_i(\omega)$. Note that $\gamma^2_{v_i p_j}(\omega)$ takes values between 0 and 1, and the summation of the contributions of all virtual signals to the jth physical signal is 1. That is, $\sum_{i=1}^{V} \gamma^2_{v_i p_j}(\omega) = 1$. Note also that if the sum of parts (say, K) of V virtual coherences with respect to a physical signal approaches unity, it indicates that there are K dominant uncorrelated signals.

3.2. RELATION BETWEEN THE RANK OF ACOUSTIC PRESSURE CROSS- SPECTRAL MATRIX AND THE NUMBER OF UNCORRELATED ACOUSTIC SOURCES

It has been seen from these considerations that the rank of the pressure cross-spectral matrix indicates the number of uncorrelated signals. As described above, in a system with n acoustic source strengths $\mathbf{q}$ which may be mutually uncorrelated or correlated and m acoustic pressures $\mathbf{p}$, the relationship between the cross-spectral matrices of $\mathbf{q}$ and $\mathbf{p}$ is expressed as $\mathbf{S}_{pp} = \mathbf{H}\mathbf{S}_{qq}\mathbf{H}^{\mathrm{H}}$. In this equation, the rank of $\mathbf{S}_{qq}$ indicates the number of uncorrelated acoustic source strengths and the rank of $\mathbf{S}_{pp}$ is then determined by the rank of the matrix $\mathbf{H}$. If $\mathbf{H}$ is of full rank, then the rank of $\mathbf{S}_{pp}$ is equal to that of $\mathbf{S}_{qq}$. Note that pre-multiplication or post-multiplication of any matrix by any non-singular (i.e., full rank) matrix does not alter its rank [9]. Thus, if m microphones are geometrically arranged for $\mathbf{H}$ to be of full rank, the rank of $\mathbf{S}_{pp}$ will correspond to the number of uncorrelated acoustic source strengths. Finally, note the similarity in the relationships $\mathbf{S}_{pp} = \mathbf{H}\mathbf{S}_{qq}\mathbf{H}^{\mathrm{H}}$ and $\mathbf{S}_{pp} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathrm{H}}$. Thus, the rank of $\mathbf{S}_{pp}$ is equal to the rank of $\mathbf{\Lambda}$, without any constraint, because of the fact that $\mathbf{Q}$ is a unitary matrix and so is of full rank. Based on similar reasoning the rank of $\mathbf{S}_{pp}$ corresponds to the rank of $\mathbf{S}_{qq}$ if and only if $\mathbf{H}$ is of full rank.

3.3. RANK EQUALITY AND THE CHOICE OF THE NUMBER OF REFERENCE MICROPHONES

This section is devoted to a discussion of which submatrix of $\mathbf{S}_{RR}$ and $\mathbf{S}_{RM}^{\mathrm{H}}$ determines the rank of $\mathbf{S}_1$. Also, a method is proposed for the selection of the optimal number of reference positions (or microphones) which ensures the rank equality between the matrices $\mathbf{S}_{pp}$ and $\mathbf{S}_1$. Let us first consider the definition of $\mathbf{S}_1$ given by equation (7). The dimension of $\mathbf{S}_1$ is $m \times u$ with $u < m$ (recall that m and u represent the entire number of microphones and the number of reference microphones, respectively), and thus $\operatorname{rank}(\mathbf{S}_1) \leqslant u$. In addition, $\mathbf{S}_1$ consists of two submatrices, $\mathbf{S}_{RR}$ $(u \times u)$ and $\mathbf{S}_{RM}^{\mathrm{H}}$ $(v \times u)$ (equation (7)). Therefore, when $u < v$, $\operatorname{rank}(\mathbf{S}_{RR}) \leqslant u$, $\operatorname{rank}(\mathbf{S}_{RM}^{\mathrm{H}}) \leqslant u$, when $u > v$, $\operatorname{rank}(\mathbf{S}_{RR}) \leqslant u$, $\operatorname{rank}(\mathbf{S}_{RM}^{\mathrm{H}}) \leqslant v$, and when $u = v$, $\operatorname{rank}(\mathbf{S}_{RR}) \leqslant u$, $\operatorname{ranks}(\mathbf{S}_{RM}^{\mathrm{H}}) \leqslant u$. (Recall that if a matrix $\mathbf{A}$ is of $m \times n$, $\operatorname{rank}(\mathbf{A}) \leqslant \min(m, n)$).

Now, we wish to check which of two submatrices, $\mathbf{S}_{RR}$ and $\mathbf{S}_{RM}$, determines the rank of $\mathbf{S}_1$. Note that the rank of $\mathbf{S}_{RM}^{\mathrm{H}}$ corresponds to that of $\mathbf{S}_{RM}$ because an elementary operation such as conjugate transpose does not alter the rank of a matrix. Firstly, it should be pointed out that the case of $u > v$ is unusual. In such a case, the number of reference microphones is larger than that of the moving microphones and is of little practical relevance, although we have considered this case here for the sake of completeness. Denoting the number of uncorrelated sources as w, we can consider five possible combinations of u, v and w: $w > u > v$, $u > v > w$, $u > w > v$, $w = u > v$, $u > v = w$. For these, ranks of the matrices of $\mathbf{S}_{RR}$, $\mathbf{S}_{RM}$, $\mathbf{S}_1$, and $\mathbf{S}_{pp}$ are given by cases 1–5 of Table 1. In computing these ranks, we have to recall two things. One is that rank of the cross-spectral matrix is the same as the number of the uncorrelated sources. The other is that in such a case as $u > v$, $\operatorname{rank}(\mathbf{S}_{RR}) \leqslant u$ and $\operatorname{rank}(\mathbf{S}_{RM}) \leqslant v$. Secondly, for the case of $u < v$ (this is the usual case), $\operatorname{rank}(\mathbf{S}_{RR}) \leqslant u$ and $\operatorname{rank}(\mathbf{S}_{RM}) \leqslant u$. In this case, there are also five possible combinations of u, v and w: $u < v < w$, $w < u < v$, $u < w < v$, $w = u < v$, and $u < v = w$. The ranks of the matrices of

TABLE 1

Rank of $\mathbf{S}_{RR}$, $\mathbf{S}_{RM}$, and $\mathbf{S}_{pp}$, and $\mathbf{S}_1$, where u, v and w are the number of reference microphones, moving microphones, and uncorrelated acoustic sources respectively

	Case no.	No. of microphones		Ranks of matrices			
		u	v	$\mathbf{S}_{RR}$	$\mathbf{S}_{RM}$	$\mathbf{S}_{pp}$	$\mathbf{S}_1$
$u > v$	1	$w > u$	$w > v$	u	v	w	u
	2	$w < u$	$w < v$	w	w	w	w
	3	$w < u$	$w > v$	w	v	w	w
	4	$w = u$	$w > v$	w	v	w	w
	5	$w < u$	$w = v$	w	w	w	w
$u < v$	6	$w > u$	$w > v$	u	u	w	u
	7	$w < u$	$w < v$	w	w	w	w
	8	$w > u$	$w < v$	u	u	w	u
	9	$w = u$	$w < v$	w	u	w	w
	10	$w > u$	$w = v$	u	u	w	u
$u = v$	11	$w = u$	$w = u$	w	w	w	w
	12	$w > u$	$w > u$	u	u	w	u
	13	$w < u$	$w < u$	w	w	w	w

$\mathbf{S}_{RR}$, $\mathbf{S}_{RM}$, $\mathbf{S}_1$, and $\mathbf{S}_{pp}$ are given by cases 6–10 of Table 1. Finally, for the case of $u = v$, we can consider three possible combinations of u, v and w: $u = v = w$, $u = v < w$, and $u = v > w$. The ranks of the matrices $\mathbf{S}_{RR}$, $\mathbf{S}_{RM}$, $\mathbf{S}_1$, and $\mathbf{S}_{pp}$ are given by cases 11–13 of Table 1. According to Table 1, it is evident that rank$(\mathbf{S}_1)$ is always the same as rank$(\mathbf{S}_{RR})$.

An example of the determination of the ranks of $\mathbf{S}_{RR}$, $\mathbf{S}_{RM}$, $\mathbf{S}_1$ and $\mathbf{S}_{pp}$ with different number of reference and moving microphones (i.e., u and v) is shown in Table 2. These results have been obtained from numerical simulations using the model of Figure 2. In this model, it is assumed that there are four mutually uncorrelated acoustic point monopoles radiating sound in a free field. For the cases 2, 3, 5, 7–9, we use this model with 16 microphones. For the cases 1, 4, 6, 10, five microphones (numbered 1–5) are used. Also, we use this model with eight microphones (numbered 1–8), six microphones (numbered 1–6), and 10 microphones (numbered 1–10) for the cases 11, 12, 13 respectively. In this model, the four sources are made "mutually uncorrelated" (i.e., $w = 4$) by assigning four different normally distributed random signals having variances $\sigma_1^2 = 1$, $\sigma_2^2 = 4$, $\sigma_3^2 = 9$, and $\sigma_4^2 = 16$ as four acoustic source strengths. It is also assumed in this model that there is no output noise. As an example, consider the case of $u = 5$ and $v = 11$ (case 7). In this case the ranks of $\mathbf{S}_{RR}$ and $\mathbf{S}_{RM}$ will be less than or equal to 5. However, since the number of uncorrelated acoustic source strengths is 4, the ranks of these two matrices cannot exceed 4. Also, since the rank of the cross-spectral matrix equals the number of uncorrelated sources, the ranks of these two matrices both become 4. The other cases can be explained similarly.

Therefore, it is clear that the rank of $\mathbf{S}_1$ is determined by the rank of $\mathbf{S}_{RR}$. Thus, in order to prove the rank equality of $\mathbf{S}_1$ and $\mathbf{S}_{pp}$, it is necessary to choose the number of reference microphones appropriately. As a consequence, since the rank of $\mathbf{S}_{pp}$ is equal to the number of uncorrelated sources, w, the number u of reference microphones must be equal to or more than w. This can be checked from cases 2–5, 7, 9, 11, and 13 of Tables 1 and 2. In other words, since the rank of $\mathbf{S}_{pp}$ equals the number of significant singular values, we have to choose the number of reference microphones to be equal to or greater than the number of the significant singular values of $\mathbf{S}_{pp}$.

TABLE 2

Rank of $\mathbf{S}_{RR}$, $\mathbf{S}_{RM}$, $\mathbf{S}_{pp}$, *and* $\mathbf{S}_1$, *obtained using the models of Figure* 2, *where the number of uncorrelated source strengths is* 4 (*i. e.*, $w = 4$)

	Case no.	No. of microphones		Ranks of matrices			
		u	v	$\mathbf{S}_{RR}$	$\mathbf{S}_{RM}$	$\mathbf{S}_{pp}$	$\mathbf{S}_1$
$u > v$	1	3	2	3	2	4	3
	2	11	5	4	4	4	4
	3	13	3	4	3	4	4
	4	4	1	4	1	4	4
	5	12	4	4	4	4	4
$u < v$	6	2	3	2	2	4	2
	7	5	11	4	4	4	4
	8	3	13	3	3	4	3
	9	4	12	4	4	4	4
	10	1	4	1	1	4	1
$u = v$	11	4	4	4	4	4	4
	12	3	3	3	3	4	3
	13	5	5	4	4	4	4

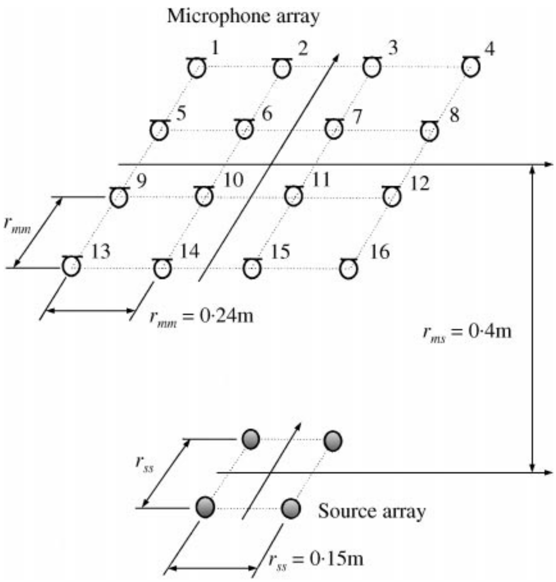

Figure 2. A simulation model for computing the ranks of $\mathbf{S}_{RR}$, $\mathbf{S}_{RM}$, $\mathbf{S}_1$ and $\mathbf{S}_{pp}$.

It should now be noted that the application of this conclusion regarding the number of required reference microphones is not difficult in a “forward problem” in which we have prior knowledge of the number of uncorrelated acoustic sources. However, it is in general problematic in an “inverse problem” because the number of uncorrelated sources is unknown. Accordingly, in order to properly select the number of reference microphones in an inverse problem, we propose the following approach. At first, select P reference microphones and then calculate the rank of $\mathbf{S}_{RR}$ obtained from measurements. For convenience denote this rank as K_P. After that, decrease the number of reference mircophones by 1, i.e., P-1 and calculate again the rank of $\mathbf{S}_{RR}$ and denote this rank as K_{P-1}. Then check the correspondence of K_P and K_{P-1} as follows:

(1) If K_P and K_{P-1} are equal, repeat a further decrease in the number of reference microphones and calculation of rank of $\mathbf{S}_{RR}$ until K_{P-i} and K_{P-i-1} become different. If this occurs, then K_{P-i} is the number of significant singular values of $\mathbf{S}_{pp}$ and thus is the required number of reference microphones which provides the rank equality of $\mathbf{S}_1$ and $\mathbf{S}_{pp}$.
(2) If K_P and K_{P-1} differ, increase P by 1, and calculate again the rank of $\mathbf{S}_{RR}$, which is denoted by K_{P+1}. Check whether K_P and K_{P+1} are the same. If so, K_P is the number of significant singular values of $\mathbf{S}_{pp}$ and is thus equal to the required number of reference microphones. If this is not so, repeat a further increase in the number of reference microphones and calculation of rank of $\mathbf{S}_{RR}$ until K_{P+i} and K_{P+i+1} become equal. If this is the case, then K_{P+i} is the number of significant singular values of $\mathbf{S}_{pp}$ and therefore the optimal number of reference microphones which we wish to find.

A simple example of this procedure is described in detail in reference [5].

4. SIMULATION RESULTS I: NO MEASUREMENT NOISE

4.1. UNCORRELATED ACOUSTIC SOURCE STRENGTHS

To clarify some features of the theory developed above, we conduct a set of computer simulations. The first simulation is carried out using the model illustrated in Figure 2. Recall that in this model four acoustic sources are made "mutually uncorrelated" by assinging four different normally distributed random signals having variances, $\sigma_1^2 = 1$, $\sigma_2^2 = 4$, $\sigma_3^2 = 9$ and $\sigma_4^2 = 16$ as the four acoustic source strengths. It is also assumed in this model that there is no measurement noise. Before we calculate the matrix $\mathbf{S}_{MM}$ from the measured matrices $\mathbf{S}_{RR}$ and $\mathbf{S}_{RM}$, we first have to check the rank equality of $\mathbf{S}_1$ and $\mathbf{S}_{pp}$. The rank of $\mathbf{S}_1$ is determined by the rank of $\mathbf{S}_{RR}$ and the number of necessary reference microphones has to be at least equal to the number of uncorrelated acoustic sources or the number of significant singular values of $\mathbf{S}_{pp}$ (which is 4 in this case). Comparison of the ranks of $\mathbf{S}_{pp}$ and $\mathbf{S}_1$ plotted in the upper part of Figure 3 supports this statement.

The effect of the number of reference microphones on the accuracy of the calculation of $\mathbf{S}_{MM}$ is now investigated. As a measure of accuracy of calculation, we use the normalized difference R_1 between the directly measured and calculated $\mathbf{S}_{MM}$ defined as

$$\mathrm{R}_1 = \frac{\| (\mathbf{S}_{MM})_{mea} - (\mathbf{S}_{MM})_{cal} \|_e}{\| (\mathbf{S}_{MM})_{mea} \|_e}, \tag{20}$$

where subscripts *mea* and *cal* denote "measured" and "calculated" and $\| \; \|_e$ denotes the euclidean norm of the matrix. The results of the lower part of Figure 3 reveal that the accurate calculation of $\mathbf{S}_{MM}$ can be achieved only when the rank equality between $\mathbf{S}_{pp}$ and $\mathbf{S}_1$ is satisfied. That is, when using three reference microphones, the normalized difference R_1 is over 10^{-3}, whilst for the cases of using four or five reference microphones it is below 10^{-11}. On the other hand, the fact that the rank of $\mathbf{S}_{pp}$ is 4 also indicates that there are four principal auto-spectra of virtual acoustic pressures. The results of Figure 4 emphasize this point. The rank of $\mathbf{S}_{pp}$ can also be identified from the virtual coherence (VC). The results of Figure 5 illustrate the VC for the first, second, third and fourth virtual acoustic pressure with respect to the first physical acoustic pressure. Their sum is unity and this signifies that

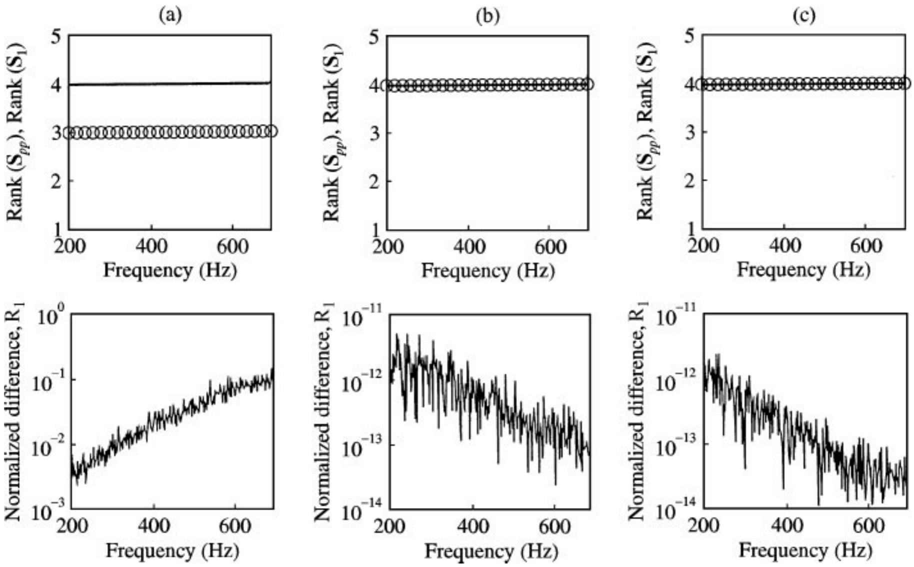

Figure 3. Effect of the number of reference microphones ((a) 3, (b) 4, (c) 5) on variations of rank of $\mathbf{S}_{pp}$ (solid), rank of $\mathbf{S}_1$ (circle) and the normalized difference R_1. There are four uncorrelated acoustic sources.

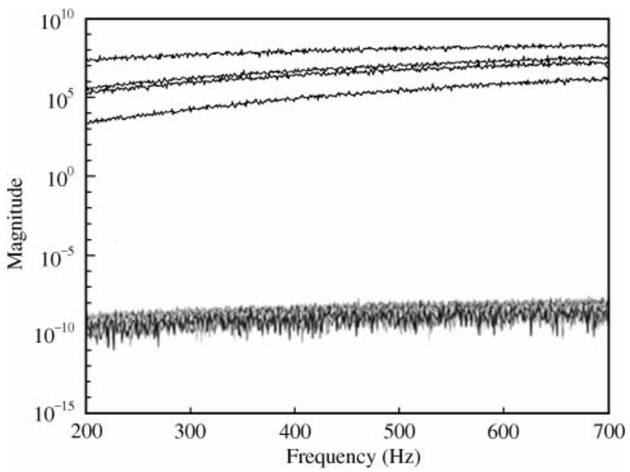

Figure 4. Principal auto-spectra (or singular values) of $\mathbf{S}_{pp}$ for the model (Figure 2) comprising four uncorrelated sources under the assumption of no output noise.

there are no other virtual acoustic pressures. It is therefore concluded that the rank of this model is 4.

From the results of Figure 3, we select four microphones as reference microphones (or reference positions) for the model of Figure 2. In this case, the dimensions of the matrices $\mathbf{S}_{RR}$, $\mathbf{S}_{MM}$ and $\mathbf{S}_{pp}$ are 4×4, 12×12 and 16×16 respectively. Based on the rank equality of $\mathbf{S}_{pp}$ and $\mathbf{S}_1$, we can now estimate $\mathbf{S}_{MM}$ using equation (14). The results of Figure 6 illustrate that the acoustic pressure auto- and cross-spectra estimated at the fifth and eighth positions, which correspond, respectively, to the first and fourth moving positions, are in a good agreement with those calculated directly. The normalized difference R_1 reveals that there is a successful prediction for all moving positions. Also, Figure 6 shows the normalized

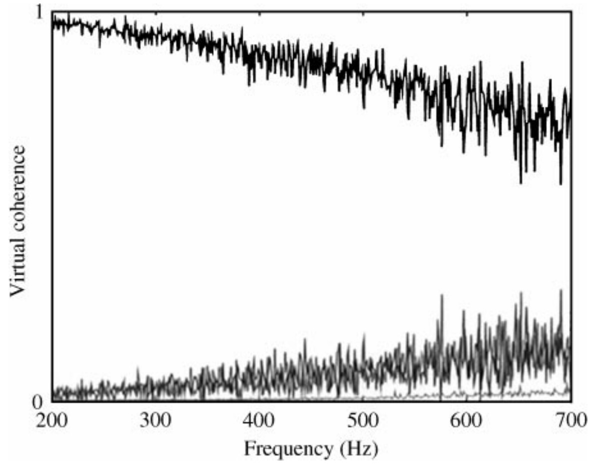

Figure 5. Virtual coherences of the first (black thick), the second (grey thick), the third (black thin), and the fourth (grey thin) virtual acoustic pressure with respect to the physical acoustic pressure sensed at the microphone 1 for the model in Figure 2 comprising four uncorrelated sources and the assumption of no output noise.

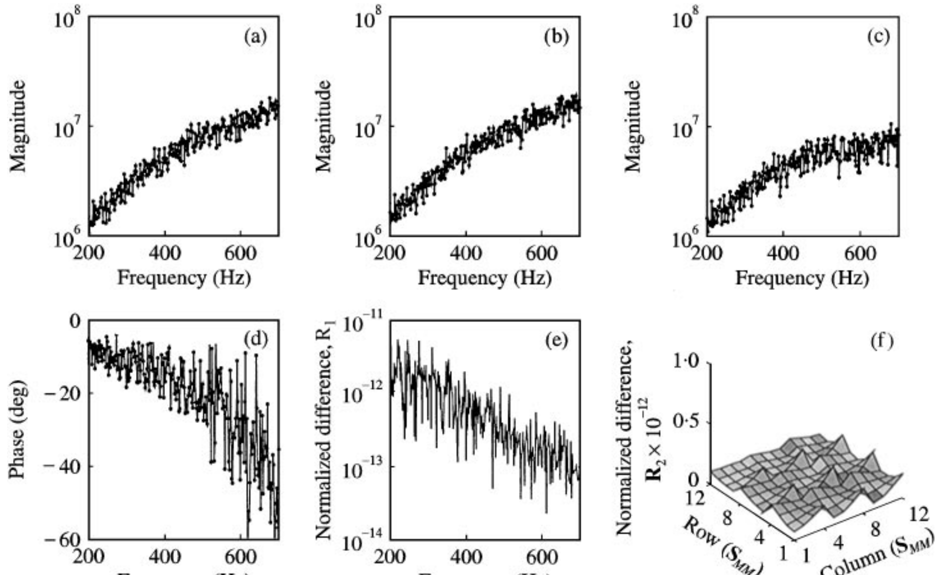

Figure 6. A comparison of the directly calculated (solid) and estimated (dotted) $\mathbf{S}_{MM}$: (a) autospectra at the first moving position, (b) autospectra at the fourth moving position, (c) and (d) magnitude and phase of cross-spectra between the first and fourth moving positions, (e) normalized difference R_1, (f) normalized difference matrix $\mathbf{R}_2$ at $kr_{ss} = 1{\cdot}65$ ($= 600$ Hz, $r_{ss} = 0{\cdot}15$ m). These results are for the model of Figure 2 comprising four uncorrelated sources and the assumption of no output noise.

difference matrix $\mathbf{R}_2$ defined by

$$\mathbf{R}_2 = \frac{|(\mathbf{S}_{MM})_{mea} - (\mathbf{S}_{MM})_{cal}|}{|(\mathbf{S}_{MM})_{mea}|}, \tag{21}$$

where $|\ |$ denotes the absolute value. The value presented in Figure 6 are for $kr_{ss} = 1{\cdot}65$ ($= 600$ Hz, $r_{ss} = 0{\cdot}15$ m). According to the above results, it is evident that the technique using reference microphones to construct $\mathbf{S}_{pp}$ is reliable in accuracy.

4.2. CORRELATED ACOUSTIC SOURCE STRENGTHS

Now consider the case of a model having correlated sources. As an example, we use a simply supported plate mounted in an infinite baffle as shown in Figure 7. The plate is excited at a point by a normally distributed random force. As a result, the surface velocities of the plate generate the acoustic field. The plate is discretized into 16 contiguous small rectangular elements each of which is regarded as a point monopole source. It is also assumed that there is no output noise.

Since a single force excites this plate, it is anticipated that there may exist only one uncorrelated acoustic source among 16 discretized acoustic sources. So we use only one microphone (numbered 1 in Figure 7) as the reference microphone. Thus, the rank of $\mathbf{S}_1$ is equal to that of $\mathbf{S}_{pp}$ and they are of rank 1. This means that there will be one principal auto-spectra of $\mathbf{S}_{pp}$ and this is ensured from Figure 8. Thus, there exists only 1 virtual acoustic pressure. Figure 9 illustrates the virtual coherence of the virtual acoustic pressure with respect to the physical acoustic pressure sensed at the microphone numbered by 1 in

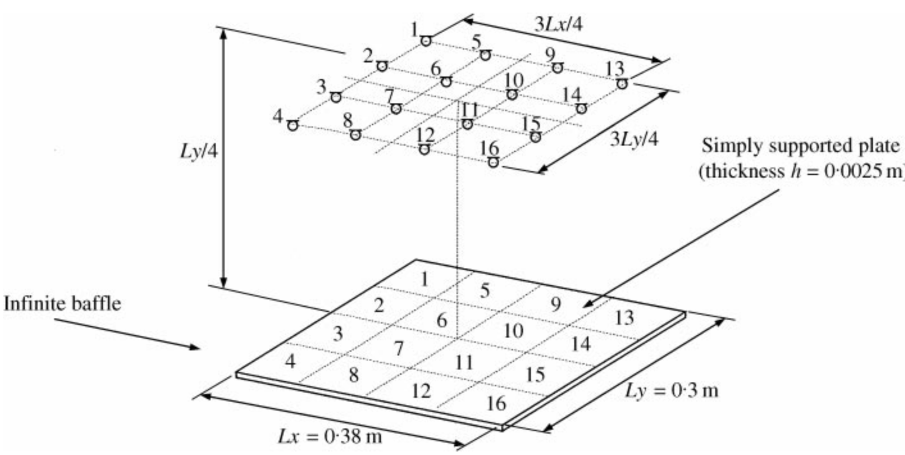

Figure 7. Geometry of a simply supported plate mounted in an infinite baffle used for the computer simulation.

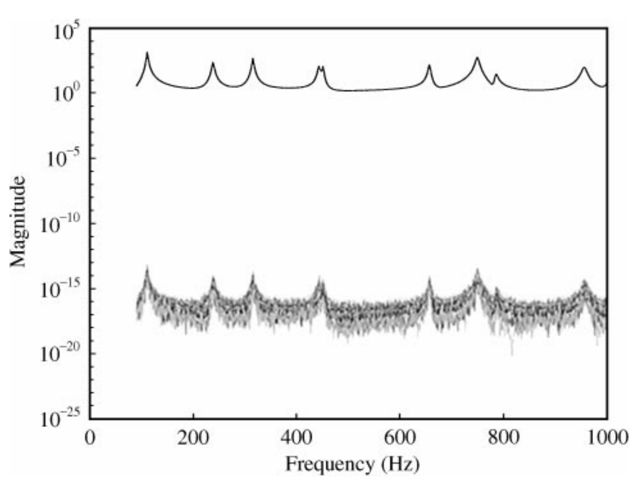

Figure 8. Principal auto-spectra (or singular values) of $\mathbf{S}_{pp}$ for the simply supported plate model (Figure 7) under the assumption of no output noise.

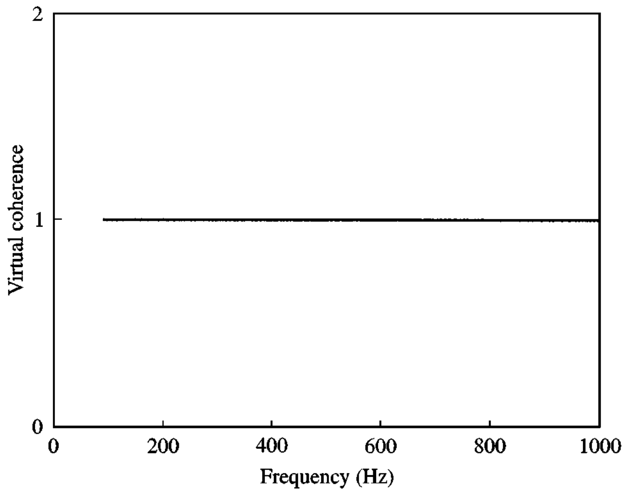

Figure 9. Virtual coherence of the virtual acoustic pressure with respect to the physical acoustic pressure sensed at the microphone 1 for the simply supported plate model (Figure 7) under the assumption of no output noise.

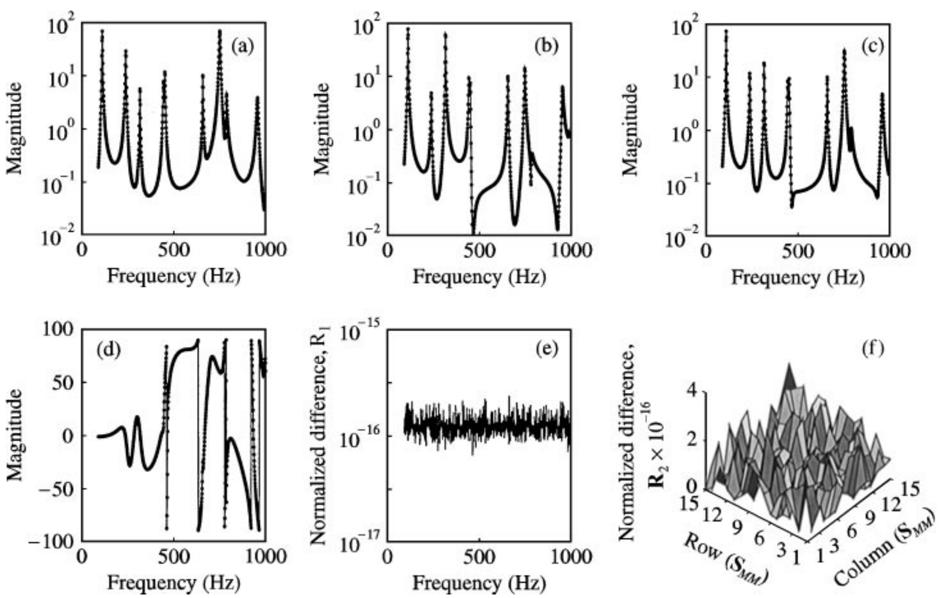

Figure 10. A comparison of the directly calculated (solid) and estimated (dotted) $\mathbf{S}_{MM}$: (a) auto-spectra at the first moving position, (b) autospectra at the fourth moving position, (c) and (d) magnitude and phase of cross-spectra between the first and fourth moving positions, (e) normalized difference R_1, (f) normalized difference matrix $\mathbf{R}_2$ at $kr_{ss} = 1{\cdot}37$ (= 786 Hz, $r_{ss} = 0{\cdot}095$ m) which is the (2,3) resonant frequency. These results are for the simply supported plate model of Figure 7 under the assumption of no output noise.

Figure 7. Only one virtual acoustic pressure has unity as the value of virtual coherence. Note that the virtual coherence with respect to the physical acoustic pressures sensed at the other microphones showed the same results, although they are not presented here. With the achievement of rank equality, the estimated magnitudes and phases of the components of $\mathbf{S}_{MM}$ exhibit excellent agreement with the directly calculated values, as plotted in Figure 10. Note also that auto- and cross-spectra of Figure 10 were obtained from the acoustic pressures normalized by unit force and so the shape is smooth despite the excitation by a normally distributed random force.

5. SIMULATION RESULTS II: WITH OUTPUT NOISE

5.1. SENSOR NOISE MODEL

We now investigate the performance of the technique using reference mircophones for a more realistic model where output noise contaminates the acoustic pressures. In measuring the cross-spectral matrices, we make an assumption that the noise will be equally distributed over all the components of $\mathbf{S}_{pp}$. Accordingly, the matrix $\mathbf{S}_{pp}$ and its submatrices $\mathbf{S}_{RR}$, $\mathbf{S}_{RM}$, $\mathbf{S}_{MM}$, $\mathbf{S}_1$, and $\mathbf{S}_2$ will be denoted by $\mathbf{S}_{\hat{p}\hat{p}}$, $\mathbf{S}_{\hat{R}\hat{R}}$, $\mathbf{S}_{\hat{R}\hat{M}}$, $\mathbf{S}_{\hat{M}\hat{M}}$, $\mathbf{S}_{\hat{1}}$, and $\mathbf{S}_{\hat{2}}$ respectively where ^ signifies data contaminated by the output noise. From these measured matrices $\mathbf{S}_{\hat{R}\hat{R}}$ and $\mathbf{S}_{\hat{R}\hat{M}}$, the moving position auto- and cross-spectral matrix $\mathbf{S}_{\hat{M}\hat{M}}$ is obtained by following the procedure used in reaching equation (14). Thus, when the matrix $\mathbf{S}_{\hat{R}\hat{R}}$ is of full rank

$$\mathbf{S}_{\hat{M}\hat{M}} = \mathbf{S}^{\mathrm{H}}_{\hat{R}\hat{M}}\, \mathbf{S}^{-1}_{\hat{R}\hat{R}}\, \mathbf{S}_{\hat{R}\hat{M}} \tag{22}$$

or when the matrix $\mathbf{S}_{\hat{R}\hat{R}}$ is rank-deficient $\mathbf{S}^{-1}_{\hat{R}\hat{R}}$, is replaced by the psuedo-inverse matrix.

5.2. UNCORRELATED ACOUSTIC SOURCE STRENGTHS

In order to observe the effect of output noise the model shown in Figure 2 is again used. Output noise (15%) is added into all components of acoustic pressure auto- and cross-spectra $\mathbf{S}_{pp}$. Like the previous case which was not concerned with the effect of noise, we select four reference microphones as a first trial. (Recall that in the absence of output noise, the use of four reference microphones provided a good estimation of $\mathbf{S}_{MM}$, as illustrated in Figure 6.) In this case, the rank of $\mathbf{S}_{\hat{p}\hat{p}}$ is 5 (note that previously the rank of $\mathbf{S}_{pp}$ was 4) whilst the rank of $\mathbf{S}_{\hat{1}}$ still is kept unchanged at 4 because four reference microphones are used. The number of significant singular values (Figure 11) of the matrix $\mathbf{S}_{\hat{p}\hat{p}}$ increases to 5 from the value of 4 associated with the matrix $\mathbf{S}_{pp}$ (Figure 4). The failure to verify the assumption of rank equality between $\mathbf{S}_{\hat{1}}$ and $\mathbf{S}_{\hat{p}\hat{p}}$ by using only four reference microphones makes the estimated magnitudes and phases of the components of $\mathbf{S}_{\hat{M}\hat{M}}$ deviate from the directly calculated values, as illustrated in Figure 12. Also, the measure of deviation of all components of $\mathbf{S}_{\hat{M}M}$ with frequency can be observed from the normalized difference R_1. The

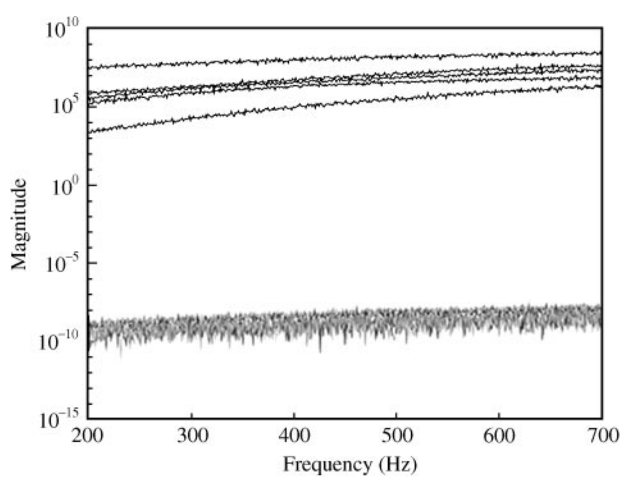

Figure 11. Principal auto-spectra (or singular values) of $\mathbf{S}_{\hat{p}\hat{p}}$ for the model of Figure 2.

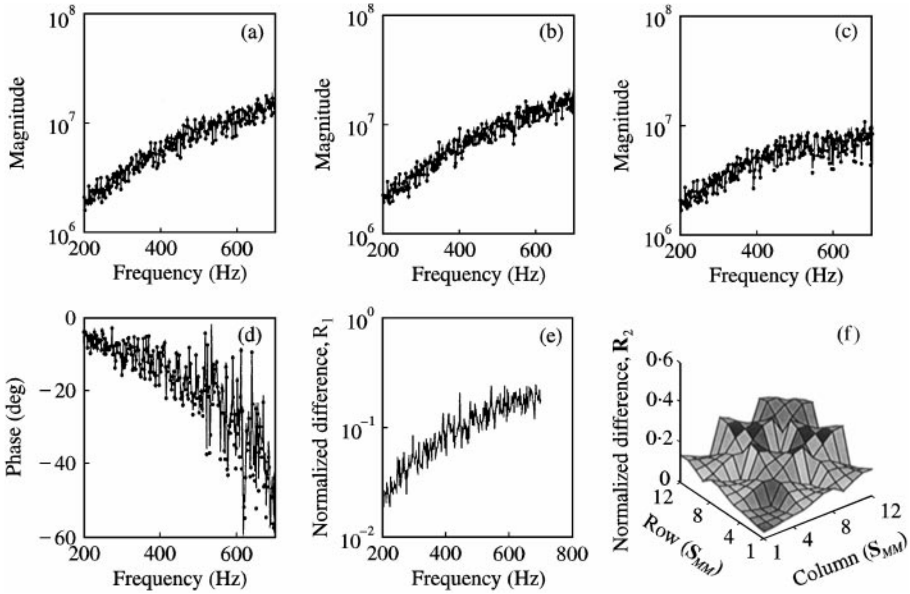

Figure 12. A comparison of the directly calculated (solid) and estimated (dotted) $\mathbf{S}_{\hat{M}\hat{M}}$: (a) auto-spectra at the first moving position, (b) auto-spectra at the fourth moving position, (c) and (d) magnitude and phase of cross-spectra between the first and fourth moving positions, (e) normalized difference R_1, (f) normalized difference matrix $\mathbf{R}_2$ at $kr_{ss} = 1{\cdot}65$ ($= 600$ Hz, $r_{ss} = 0{\cdot}15$ m) for the model of Figure 2 with output noise. Four reference microphones are used.

quality of R_1 (i.e., 10^{-2}–1) plotted in Figure 12 is much larger than that of R_1 (i.e., $10^{-14} \sim 10^{-11}$) shown in Figure 6 in which the rank equality between $\mathbf{S}_{\hat{1}}$ and $\mathbf{S}_{\hat{p}\hat{p}}$ was verified. Figure 12 also shows the normalized difference $\mathbf{R}_2$($\sim 10^{-1}$) is much larger than that ($\sim 10^{-12}$) of Figure 6.

Since these unsatisfactory results come from the discrepancy between the ranks of $\mathbf{S}_1$ and $\mathbf{S}_{\hat{p}\hat{p}}$, we have to alter the number of reference microphones and now five reference microphones are selected instead of 4. As pointed out earlier, since rank of $\mathbf{S}_{\hat{1}}$ is the same as that of $\mathbf{S}_{\hat{R}\hat{R}}$, the use of five reference microphones yields a submatrix $\mathbf{S}_{\hat{1}}$ of rank 5. Therefore, ranks of $\mathbf{S}_{\hat{1}}$ and $\mathbf{S}_{\hat{p}\hat{p}}$ become equal. Subsequently, the estimated magnitudes and phases of components of $\mathbf{S}_{\hat{M}\hat{M}}$ are now in very good agreement with those directly calculated values (Figure 13).

5.3. CORRELATED ACOUSTIC SOURCE STRENGTHS

Finally, we consider the simply supported plate model depicted in Figure 7 which has correlated sources. For this model, measurement noise (15%) is also added. When the number of reference microphones is chosen to be 1 as before, a discrepancy appears in the ranks of $\mathbf{S}_{\hat{1}}$ and $\mathbf{S}_{\hat{p}\hat{p}}$. That is, the addition of output noise forces the rank of $\mathbf{S}_{\hat{p}\hat{p}}$ to increase to 2. The reason for this can easily be understood from viewing the singular values of $\mathbf{S}_{\hat{p}\hat{p}}$ shown in Figure 14. Hence, the estimation of the magnitudes and phases are inaccurate, as illustrated in Figure 15. Thus, in order to make the ranks of $\mathbf{S}_{\hat{1}}$ and $\mathbf{S}_{\hat{p}\hat{p}}$ equal, two reference microphones are used. As a result, rank equality is achieved and a satisfactory estimation of $\mathbf{S}_{\hat{M}\hat{M}}$ is made as illustrated in Figure 16. It is therefore apparent from the above results that the proper choice of the number of reference microphones is at the heart of this technique. If this is achieved, the construction of full auto- and cross-spectral matrix of acoustic pressures can be made faster whilst maintaining accuracy.

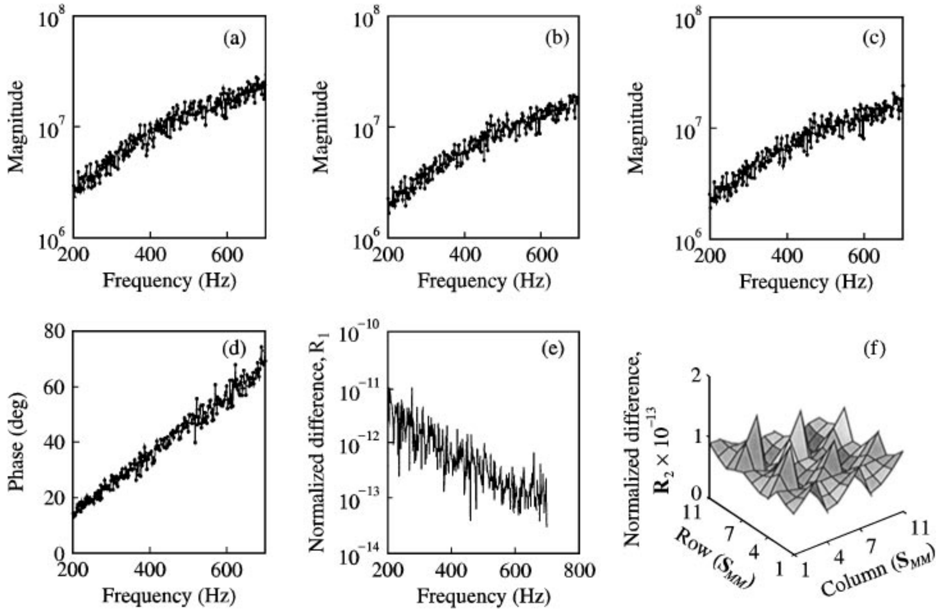

Figure 13. A comparison of the directly calculated (solid) and estimated (dotted) $\mathbf{S}_{\hat{M}\hat{M}}$: (a) auto-spectra at the first moving position, (b) auto-spectra at the fourth moving position, (c) and (d) magnitude and phase of cross-spectra between the first and fourth moving positions, (e) normalized difference R_1, (f) normalized difference matrix $\mathbf{R}_2$ at $kr_{ss} = 1{\cdot}65$ ($= 600$ Hz, $r_{ss} = 0{\cdot}15$ m). These results are for the model of Figure 2 with output noise when five reference microphones are used.

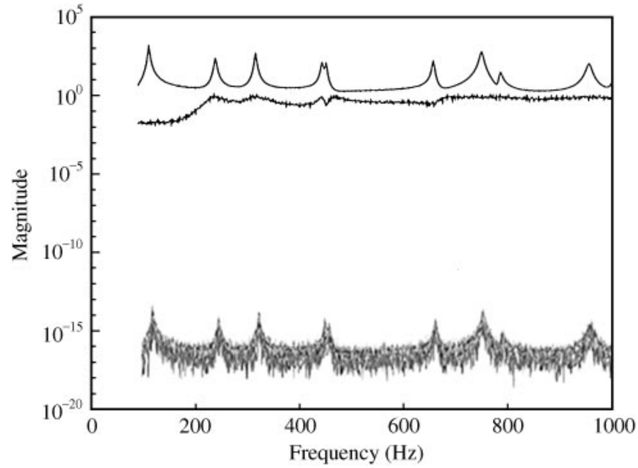

Figure 14. Principal autospectra (or singular values) of $\mathbf{S}_{\hat{p}\hat{p}}$ for the model Figure 7.

6. EXPERIMENTAL VERIFICATION

6.1. PRACTICAL RANK ESTIMATION

We now verify the reference microphones technique through experiments and describe some practical considerations. We use two experimental systems which were explained in detail in references [5, 10]. One is the system used for the reconstruction of the strengths of two volume velocity sources (Figure 17) and the other for the reconstruction of

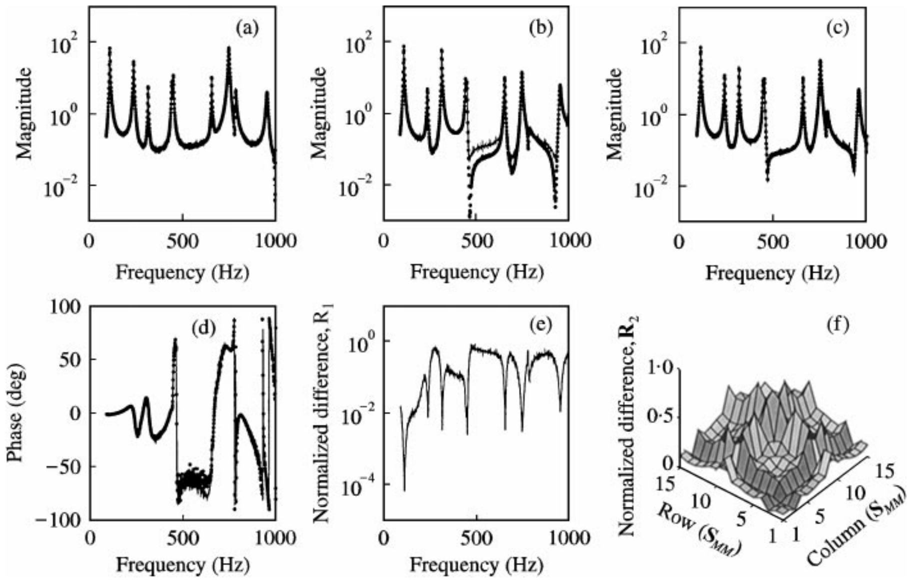

Figure 15. A comparison of the directly calculated (solid) and estimated (dotted) $\mathbf{S}_{\hat{M}\hat{M}}$: (a) auto-spectra at the first moving position, (b) auto-spectra at the fourth moving position, (c) and (d) magnitude and phase of cross-spectra between the first and fourth moving positions, (e) normalized difference R_1, (f) normalized difference matrix $\mathbf{R}_2$ at $kr_{ss} = 1{\cdot}37$ (= 786 Hz, $r_{ss} = 0{\cdot}095$ m) which is the (2,3) resonant frequency for the simply supported plate model of Figure 7 with output noise. One reference microphone is used.

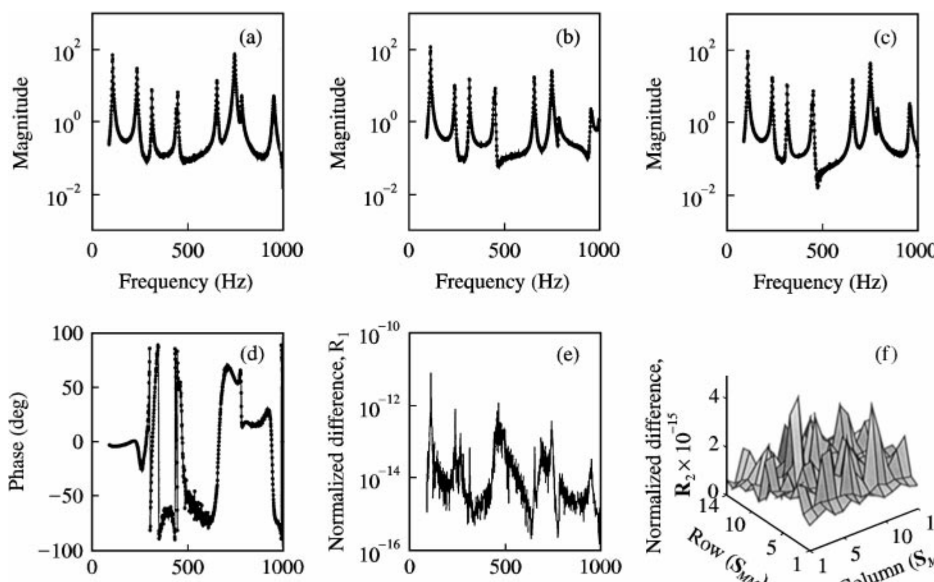

Figure 16. A comparison of the directly calculated (solid) and estimated (dotted) $\mathbf{S}_{\hat{M}\hat{M}}$: (a) auto-spectra at the first moving position, (b) auto-spectra at the fourth moving position, (c) and (d) magnitude and phase of cross-spectra between the first and fourth moving positions, (e) normalized difference R_1, (f) normalized difference matrix $\mathbf{R}_2$ at $kr_{ss} = 1{\cdot}37$ (= 786 Hz, $r_{ss} = 0{\cdot}095$ m) which is the (2,3) resonant frequency. These results are for the simply supported plate model of Figure 7 with output noise. Two reference microphones are used.

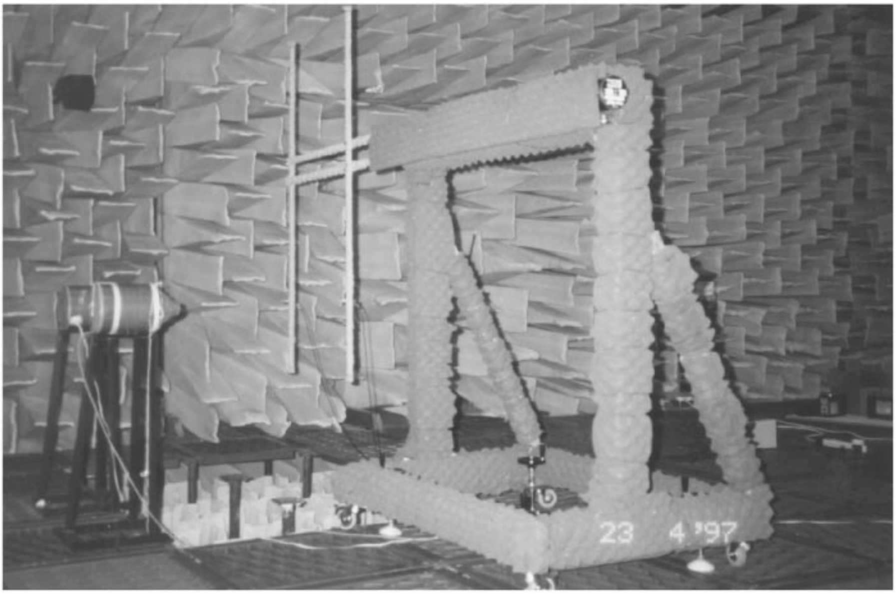

Figure 17. Experimental arrangement for the reconstruction of strengths of two volume velocity sources.

the volume velocities of a randomly vibrating simply supported plate mounted in a finite baffle (Figure 18).

Before conducting the experimental verification, we here consider carefully the estimate of the rank of matrix $\mathbf{S}_{\hat{p}\hat{p}}$ which consists of experimental data. As pointed out earlier, a reliable rank estimator of a matrix is the SVD. That is, the rank of a matrix is the number of singular values larger than a threshold level. Accordingly, since this threshold level plays a role of distinguishing the significant singular values from the negligible singular values, the determination of this level is important in rank estimation. In general, in the rank calculation using the data in connection with the numerical simulations, the threshold level is chosen based on the machine epsilon of the computer used (see reference [11]).

However, as far as measured data are concerned, such a threshold level is not suitable, because, for example, the quantization error related to the analogue-to-digital converter used in the data acquisition process is, in general, larger than this machine epsilon (see reference [12], for example). Thus, it is reasonable to choose the threshold level, considering this kind of uncontrollable error from the practical point of view. Another parameter that makes difficult the determination of the rank of matrix $\mathbf{S}_{\hat{R}\hat{R}}$ from the SVD is the signal processing technique usually used to obtain the acoustic pressure auto- and cross-spectra. As is well known, when auto- and cross-spectra are estimated by the segment averaging method, a large number of data segments are necessary to reduce the random error and a long data segment is required to reduce the bias error. Thus, we cannot help but identify incorrectly the significant singular values from the cross-spectral matrix $\mathbf{S}_{\hat{R}\hat{R}}$ having random error and bias error. Kompella *et al.* [12] studied the effect of the number of data segments and segment length on the singular values, in connection with the determination of the number of incoherent sources contributing to the response to a system. They reached similar conclusions regarding the number of data segments and the segment length. In addition, attention has to be paid to acquiring acoustic pressures with good signal-to-noise ratio. Otherwise, as described earlier, it becomes problematic to identify correctly the

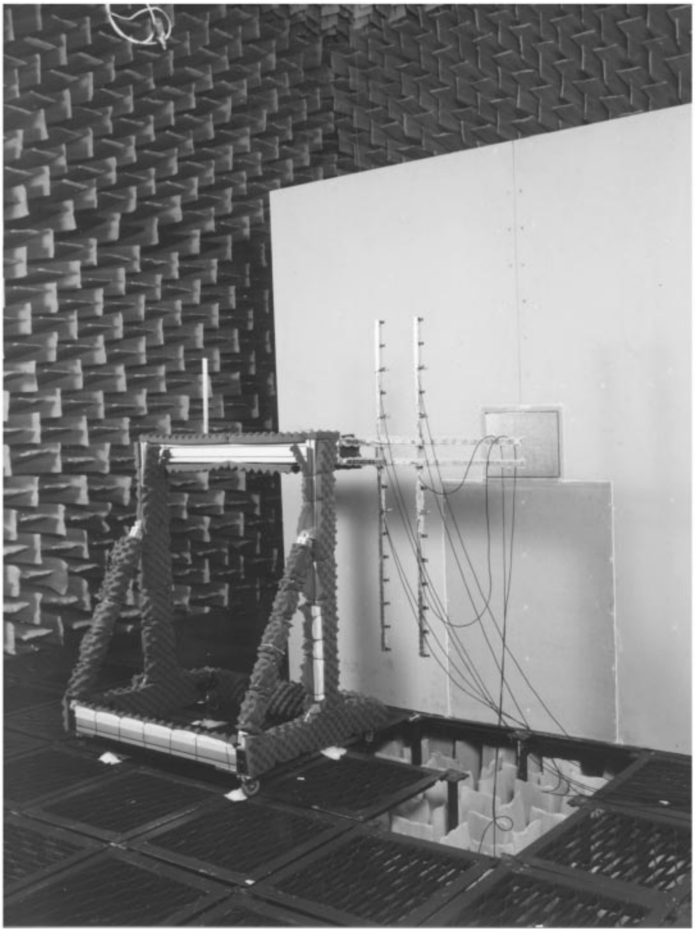

Figure 18. Experimental arrangement for the reconstruction of volume velocities of a randomly vibrating plate mounted in a finite baffle.

number of significant singular values due purely to uncorrelated acoustic sources since background noise also modifies the magnitudes of the singular values of $\mathbf{S}_{\hat{R}\hat{R}}$.

However, our purpose is the identification of the required number of reference microphones necessary to secure the rank equality between the matrices $\mathbf{S}_{\hat{1}}$ and $\mathbf{S}_{\hat{p}\hat{p}}$ by inspecting the significant singular values of the matrix $\mathbf{S}_{\hat{R}\hat{R}}$. Thus, even if there exist spurious significant singular values which result from imperfect measurement, we have to choose the reference microphones to be equal to or greater than the number of (true and spurious) significant singular values. This is borne out by the computer simulations presented previously.

6.2. EXPERIMENTS

With these considerations in mind, we describe the first experimental system shown in Figure 17. Six microphones are placed to sense the acoustic pressures radiated by two volume velocity sources which are driven by one random noise generator. Since we use only

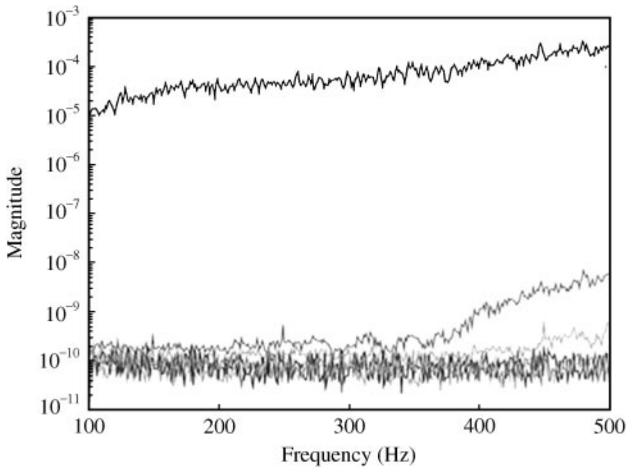

Figure 19. Six principal auto-spectra (or singular values) of $\mathbf{S}_{\hat{p}\hat{p}}$ for the model (Figure 17) consisting of the two volume velocity sources driven by one random noise generator and six microphones.

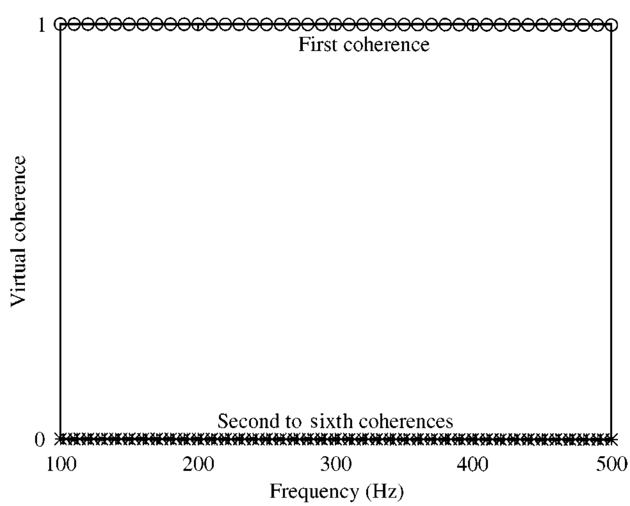

Figure 20. Virtual coherences of the first to sixth virtual acoustic pressure with respect to the physical acoustic pressure sensed at the microphone 1 in Figure 17 consisting of the two volume velocity sources driven by one random noise generator and six microphones.

one random noise generator, it is expected that this model has one significant singular value (or one significant principal autospectrum of virtual acoustic pressure). This is ensured from the singular value distributions of $\mathbf{S}_{\hat{p}\hat{p}}$ shown in Figure 19. Also observing the virtual coherence (Figure 20) of the virtual acoustic pressure with respect to the physical pressure measured at a microphone (Figure 17) reveals that there exists only one uncorrelated acoustic source. That is the first of six virtual coherences is very close to unity. Accordingly, we use one reference microphone (so that ranks of $\mathbf{S}_{\hat{R}\hat{R}}$ and $\mathbf{S}_{\hat{1}}$ are one) and estimate the moving microphone auto- and cross-spectra $\mathbf{S}_{\hat{M}\hat{M}}$. As can be seen from Figure 21, the estimated magnitudes and phases of the components of $\mathbf{S}_{\hat{M}\hat{M}}$ show good agreement with the directly measured values.

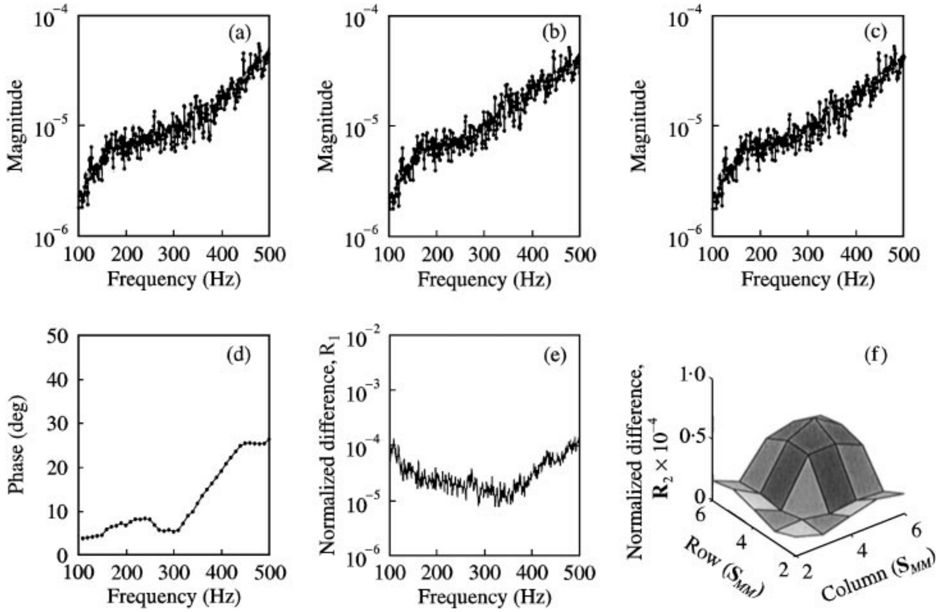

Figure 21. A comparison of the directly measured (solid) and estimated (dotted) $\mathbf{S}_{\hat{M}\hat{M}}$: (a) auto-spectra at the first moving position, (b) auto-spectra at the second moving position, (c) and (d) magnitude and phase of cross-spectra between the first and second moving positions, (e) normalized difference R_1, (f) normalized difference matrix $\mathbf{R}_2$ at $ka = 0{\cdot}1$ (= 400 Hz, $a = 0{\cdot}014$ m) for the model of Figure 17 consisting of the two volume velocity sources driven by one random noise generator and six microphones. One reference microphone is used.

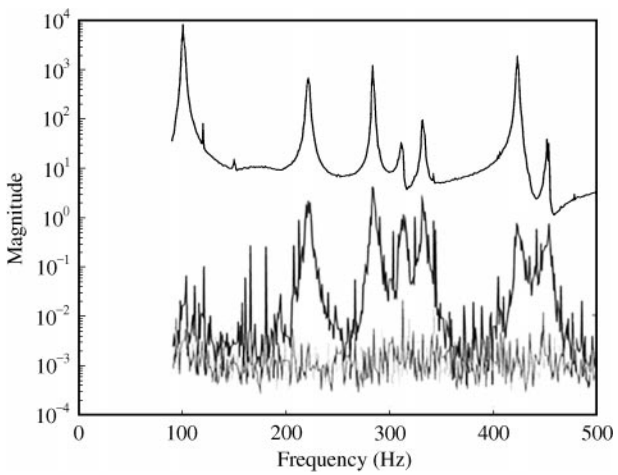

Figure 22. Principal auto-spectra (or singular values) of $\mathbf{S}_{\hat{p}\hat{p}}$ for the model (Fig. 18) consisting of the simply supported plate excited by one electromagnetic driver and four microphones.

Now consider the experimental model of the randomly vibrating simply supported plate mounted in a finite baffle shown in Figure 18. (See reference [10] for details.) We use four microphones to measure the acoustic pressures radiated by the plate excited by one electromagnetic driver. Thus, one would assume that this system has one uncorrelated acoustic source and thus the rank of matrix $\mathbf{S}_{\hat{p}\hat{p}}$ will be one. However, whereas the number of uncorrelated acoustic sources is obviously one, judging the rank of this matrix from this singular value distribution (Figure 22) is not straightforward. Among the four singular values of $\mathbf{S}_{\hat{p}\hat{p}}$, whilst the third and fourth singular values are relatively small, the second singular value seems to be significant at some frequencies, for example, at about 310–350 and 420–480 Hz. Although the first singular value is associated with one uncorrelated acoustic source, the second singular value is not. The latter spurious singular value is possibly due to "measurement noise" caused by acoustic reflections. This can be readily understood by viewing the geometrical placement of the microphones shown in Figure 18 which are 0·166 m away from the plate. The fact that there are two significant singular values (true and spurious) at those frequencies is also observed from the virtual coherences shown in Figure 23. The sum of the first and second virtual coherences with respect to the acoustic pressure sensed at microphone 1 (Figure 18) results in nearly unity at those frequencies (the virtual coherences with respect to other acoustic pressure showed the similar results). This indicates that the rank of $\mathbf{S}_{\hat{p}\hat{p}}$ is two at those frequencies. At the other frequencies, the value of the first virtual coherence approaches unity, suggesting that the rank of $\mathbf{S}_{\hat{p}\hat{p}}$ is unity.

The effect of the use of one reference microphone on the estimation of $\mathbf{S}_{\hat{M}\hat{M}}$ is now investigated. The results of Figure 24 reveal that the estimated values of acoustic pressure autospectra at the first and second moving microphones and magnitudes and phases of acoustic pressure cross-spectra between the first and second moving microphones follow well the directly measured values at most frequencies. However, a discrepancy is observed at frequencies about 310–350 and 420–480 Hz, because at those frequencies we cannot meet the requirement of the rank equality between two matrices $\mathbf{S}_{\hat{p}\hat{p}}$ and $\mathbf{S}_{\hat{\imath}}$ with only one reference microphone. That is to say, as could be seen from the singular-value distributions (Figure 22) and the virtual coherences (Figure 23), the rank of $\mathbf{S}_{\hat{p}\hat{p}}$ is two at those frequencies. When the number of reference microphones is increased to two, the estimated auto- and cross-spectra become closer to the directly measured values at those frequencies by the virtue of the rank equality. This point can also be observed clearly by comparing the normalized differences R_1 and $\mathbf{R}_2$ of Figure 25 with those of Figure 24.

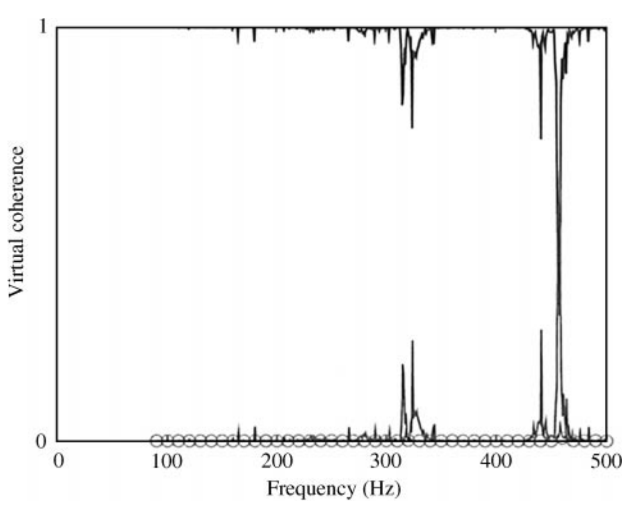

Figure 23. Virtual coherences of the first (black thick), the second (grey thick), the third (black thin), and the fourth virtual acoustic pressure (grey thin circle) with respect to the physical acoustic pressure sensed at the microphone 1 in Figure 18 consisting of the simply supported plate excited by one electromagnetic driver and four microphones.

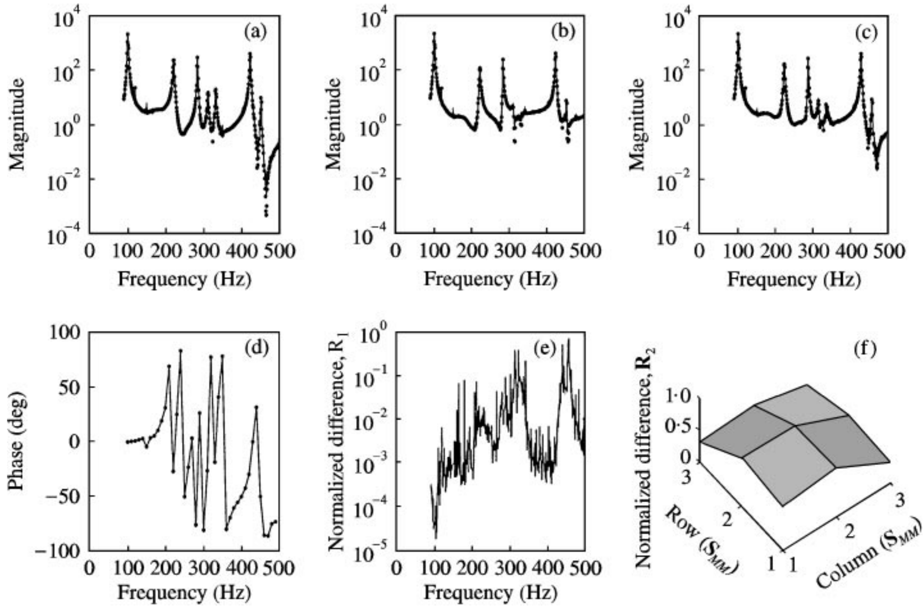

Figure 24. A comparison of the directly measured (solid) and estimated (dotted) $\mathbf{S}_{\hat{M}\hat{M}}$: (a) auto-spectra at the first moving position, (b) auto-spectra at the second moving position, (c) and (d) magnitude and phase of cross-spectra between the first and second moving positions, (e) normalized difference R_1, (f) normalized difference matrix $\mathbf{R}_2$ at $ka = 0{\cdot}1$ (= 458 Hz, $a = 0{\cdot}014$ m) for the model of Figure 18 consisting of the simply supported plate excited by one electromagnetic driver. One reference microphone is used.

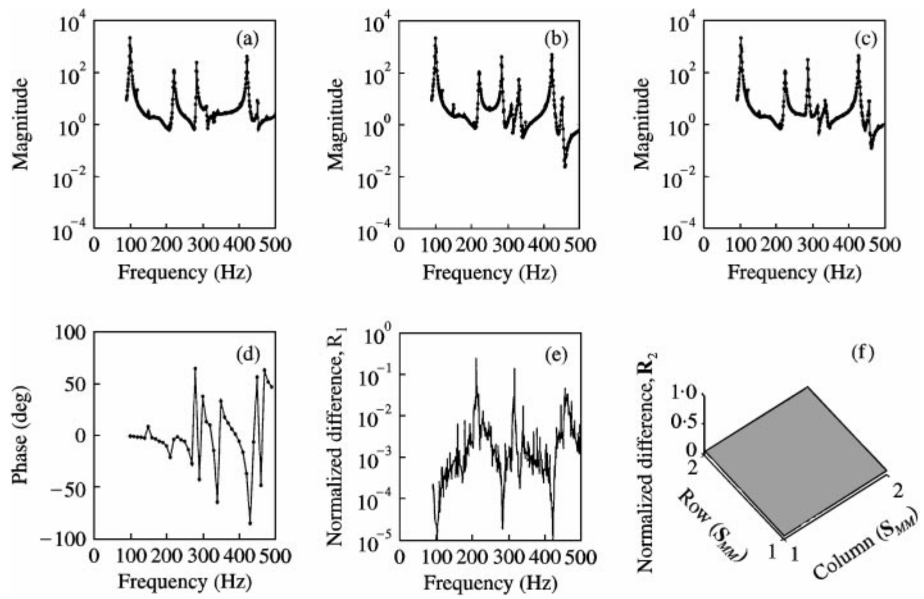

Figure 25. A comparison of the directly measured (solid) and estimated (dotted) $\mathbf{S}_{\hat{M}\hat{M}}$: (a) auto-spectra at the first moving position, (b) auto-spectra at the second moving position, (c) and (d) magnitude and phase of cross-spectra between the first and second moving positions, (e) normalized difference R_1, (f) normalized difference matrix $\mathbf{R}_2$ at $ka = 0{\cdot}1$ (= 458 Hz, $a = 0{\cdot}014$ m) for the model of Figure 18 consisting of the simply supported plate excited by one electromagnetic driver. Two reference microphones are used.

7. CONCLUSIONS

For a rapid construction of the full matrix of acoustic pressure cross-spectra, we have suggested a technique using the concept of reference microphones. This has been seen as a useful tool, because this technique enables the construction of this matrix with good precision, saving measurement effort. The use of u reference microphones and v moving microphones reduces the total number of measurements to $(u(u+1)/2)+uv$ from $(u+v)(u+v+1)/2$ (which is the number required for direct measurement, when dual channel experimental equipment is employed). This method has constructed satisfactorily the full matrix regardless of the nature of acoustic sources (correlated or uncorrelated) and also when acoustic pressure data are corrupted by noise. The prerequisite of using this technique is to validate the assumption of the rank equality between the full matrix $\mathbf{S}_{pp}$ (or $\mathbf{S}_{\hat{p}\hat{p}}$) and its submatrix $\mathbf{S}_1$ (or $\mathbf{S}_{\hat{1}}$). To do this, it is necessary to select properly the number of reference microphones. This is determined by knowing how many significant singular values of the matrix $\mathbf{S}_{pp}$ (or $\mathbf{S}_{\hat{p}\hat{p}}$) are present. That is to say, the number of reference microphones should be at least equal to the number of significant singular values of the matrix $\mathbf{S}_{pp}$ (or $\mathbf{S}_{\hat{p}\hat{p}}$). However, since we do not have information regarding the acoustic sources in the inverse problem, we have proposed a method of choosing the number of reference microphones. Regarding the choice of reference microphones, attention should be drawn to the case in which the output noise corrupts acoustic pressures. Since the output noise increases the number of significant singular values of the matrix $\mathbf{S}_{\hat{p}\hat{p}}$, compared with that of $\mathbf{S}_{pp}$, in this case the number of reference microphones has to be chosen by examining carefully the singular value distribution of the matrix $\mathbf{S}_{\hat{R}\hat{R}}$ of reference position auto- and cross-spectra.

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