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LETTERS TO THE EDITOR

STRUCTURED VIBRATION CHARACTERISTICS OF PLANETARY GEARS WITH UNEQUALLY SPACED PLANETS

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1. INTRODUCTION

The free vibration of planetary gears with equally spaced planets has been extensively studied [1–5]. In a recent study, the highly structured free-vibration properties were rigorously characterized in reference [4]. These special properties result from the cyclic symmetry of planetary gears. Because of certain design purposes and assembly limitations, however, the planets are sometimes unequally spaced and the cyclic symmetry is lost. The free vibration of systems with unequally spaced planets has not been previously investigated, and the question remains of how the highly structured free-vibration properties of equally spaced planet systems change due to unequal planet spacing. This technical note analytically investigates this question. A case of particular interest is diametrically opposed planets with each pair of planets on the ends of a carrier diameter. A lumped-parameter model of planetary gears with N planets is shown in Figure 1. Details of the dynamic model are given in reference [4]. x_h, y_h, $h = c, r, s$ denote the translations of the carrier, ring, and sun, and ζ_n, η_n, $n = 1, \ldots, N$ are the radial and tangential translations of planet n. u_h, $h = c$, r, s, $1, \ldots, N$ are rotational co-ordinates. Circumferential planet locations are specified by the fixed angles ψ_n, where ψ_n is measured relative to the rotating, carrier-fixed basis vector $\mathbf{i}$ so that $\psi_1 = 0$. In this note, all planets are identical, all sun–planet mesh stiffnesses k_{sp} are equal, and all ring–planet mesh stiffnesses k_{rp} are equal. The free vibration eigenvalue problem is

$$(\mathbf{K} - \omega_i^2 \mathbf{M})\phi_i = \mathbf{0}, \qquad \phi_i = [\underbrace{x_c, y_c, u_c,}_{\mathbf{p}_c} \underbrace{x_r, y_r, u_r,}_{\mathbf{p}_r} \underbrace{x_s, y_s, u_s,}_{\mathbf{p}_s} \underbrace{\zeta_1, \eta_1, u_1}_{\mathbf{p}_1}, \ldots, \underbrace{\zeta_N, \eta_N, u_N}_{\mathbf{p}_N}]^{\mathrm{T}}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{K}$ are inertia and stiffness matrices, and ω_i are natural frequencies. $\mathbf{p}_h$, $h = c$, r, s, $1, \ldots, N$, are modal deflections of the carrier, ring, sun, and planets.

2. EQUALLY SPACED PLANETS

When all planets are equally spaced, the system has cyclic symmetry and its natural frequencies and vibration modes have a well-defined structure [4] as summarized below.

(1) Three degenerate natural frequencies have multiplicity $N - 3$. Their associated modes are *planet modes* (Figure 2(a)) in which only the planets deflect; the carrier, ring,

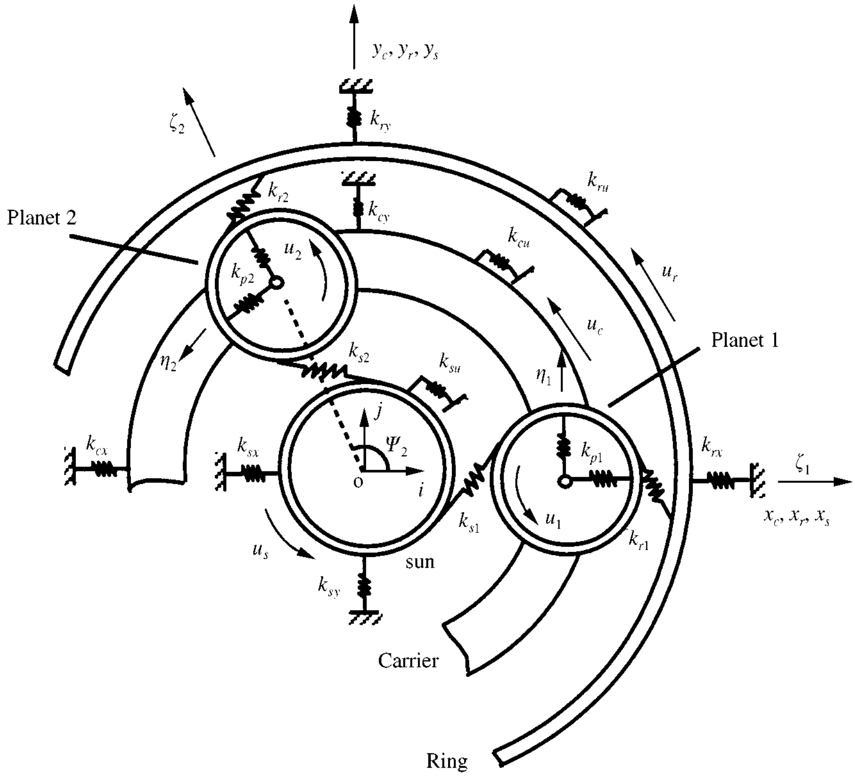

Figure 1. Lumped parameter model of planetary gears and system co-ordinates. All translational co-ordinates x_h, y_h, $h = c, r, s$ and ζ_n, η_n, $n = 1, \ldots, N$ are with respect to the frame $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ rotating at constant carrier speed Ω_c.

and sun have no motion. The component modal deflections have the form

$$\mathbf{p}_h = [0, 0, 0]^{\mathrm{T}}, \quad h = c, r, s, \qquad \mathbf{p}_n = w_n \mathbf{p}_1, \tag{2}$$

where w_n are scalars to be determined ($w_1 = 1$).

(2) Six distinct natural frequencies have multiplicity 1. Their associated modes are *rotational modes* (Figure 2(b)) which have pure rotation (no translation) of the carrier, ring, and sun. All planets have identical motion. The component modal deflections have the form

$$\mathbf{p}_h = [0, 0, u_h]^{\mathrm{T}}, \quad h = c, r, s, \qquad \mathbf{p}_n = \mathbf{p}_1. \tag{3}$$

(3) Six degenerate natural frequencies have multiplicity 2. Their associated modes are *translational modes* (Figure 2(c, d)) which have pure translation (no rotation) of the carrier, ring, and sun. For a pair of degenerate orthonormal ($\phi_i^{\mathrm{T}} \mathbf{M} \hat{\phi}_i = 0$) translational modes ϕ_i and $\hat{\phi}_i = [\hat{\mathbf{p}}_c, \hat{\mathbf{p}}_r, \hat{\mathbf{p}}_s, \hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_N]^{\mathrm{T}}$, the component modal deflections have the form

$$\mathbf{p}_h = [x_h, y_h, 0]^{\mathrm{T}}, \qquad \hat{\mathbf{p}}_h = [-y_h, x_h, 0]^{\mathrm{T}}, \qquad h = c, r, s,$$

$$\mathbf{p}_n = \cos\psi_n \mathbf{p}_1 + \sin\psi_n \hat{\mathbf{p}}_1, \qquad \hat{\mathbf{p}}_n = -\sin\psi_n \mathbf{p}_1 + \cos\psi_n \hat{\mathbf{p}}_1. \tag{4}$$

3. ARBITRARILY SPACED PLANETS

In general, much of the above well-defined structure of the natural frequency spectra and vibration modes is lost when the planets are arbitrarily spaced. A notable exception is the planet modes. Additionally, for the practically important case of diametrically opposed planets, the free vibration retains its unique properties. The analytical procedure employed here is to substitute candidate vibration modes directly into equation (1) to verify that they are true modes. Expansion of equation (1) into $N + 3$ groups of equations associated with the individual components gives

$$(\mathbf{K}_{cb} + \sum \mathbf{K}_{c1}^{n} - \omega_i^2 \mathbf{M}_c)\mathbf{p}_c + \sum \mathbf{K}_{c2}^{n} \mathbf{p}_n = \mathbf{0}, \tag{5}$$

$$(\mathbf{K}_{rb} + \sum \mathbf{K}_{r1}^{n} - \omega_i^2 \mathbf{M}_r)\mathbf{p}_r + \sum \mathbf{K}_{r2}^{n} \mathbf{p}_n = \mathbf{0}, \tag{6}$$

$$(\mathbf{K}_{sb} + \sum \mathbf{K}_{s1}^{n} - \omega_i^2 \mathbf{M}_s)\mathbf{p}_s + \sum \mathbf{K}_{s2}^{n} \mathbf{p}_n = \mathbf{0}, \tag{7}$$

$$(\mathbf{K}_{c2}^{n})^{\mathrm{T}}\mathbf{p}_c - (\mathbf{K}_{r2}^{n})^{\mathrm{T}}\mathbf{p}_r + (\mathbf{K}_{s2}^{n})^{\mathrm{T}}\mathbf{p}_s + (\mathbf{K}_{pp} - \omega_i^2 \mathbf{M}_p)\mathbf{p}_n = \mathbf{0}, \quad n = 1, 2, \ldots, N. \tag{8}$$

where the summation index n ranges from 1 to N throughout this paper. Components of the 3×3 submatrices in equations (5)–(8) are given in reference [4] and used in the subsequent analysis.

3.1. PLANET MODES

Insertion of equation (2) into equations (5)–(8) and algebraic manipulation yield the reduced eigenvalue problem for planet modes,

$$\sum w_n \sin \psi_n = 0, \qquad \sum w_n \cos \psi_n = 0, \qquad \sum w_n = 0, \tag{9}$$

$$(\mathbf{K}_{pp} - \omega_i^2 \mathbf{M}_p)\mathbf{p}_1 = \mathbf{0}. \tag{10}$$

Equations (9) and (10) are decoupled. The under-determined equations (9) have $N - 3$ independent sets of non-trivial solutions for the w_n. For each of the three eigensolutions $(\omega_i, \mathbf{p}_1)$ solved from equation (10), $N - 3$ independent planet modes can be constructed according to equation (2). Note that equation (10) is independent of the planet spacing ψ_n, so only the coefficients w_n obtained from equation (9) are affected by ψ_n. As for planetary gears with equally spaced planets, systems with arbitrary planet spacing always have three sets of planet modes of the form (2) with multiplicity $N - 3$.

3.2. ROTATIONAL MODES

In general, the rotational and translational modes couple together for arbitrary planet spacing and no special modal structure or natural frequency multiplicity can be identified. For certain planet spacing, however, they still have distinguishing properties. A case of particular interest is that of diametrically opposed planets, which is common in industrial applications. Consider a system with each of $N/2$ pairs of planets located along arbitrarily oriented diameters. A pair of opposing planets have the position relation $\psi_{n+N/2} = \psi_n + \pi$. In this case,

$$\sum \sin \psi_n = 0, \qquad \sum \cos \psi_n = 0. \tag{11}$$

To check the validity of equation (3), it is substituted into equation (5). The resulting component equations are

$$-k_p \sum \sin\psi_n u_c - k_p \sum (\cos\psi_n \zeta_1 - \sin\psi_n \eta_1) = 0, \tag{12}$$

$$k_p \sum \cos\psi_n u_c - k_p \sum (\sin\psi_n \zeta_1 + \cos\psi_n \eta_1) = 0, \tag{13}$$

$$(k_{cu} + Nk_p - \omega_i^2 I_c/r_c^2)u_c - Nk_p \eta_1 = 0. \tag{14}$$

Equations (12) and (13) vanish as a result of equation (11) and only equation (14) remains. Similarly each of equations (6) and (7) reduce to one equation

$$(k_{ru} + Nk_{rp} - \omega_i^2 I_r/r_r^2)u_r - Nk_{rp}(u_1 - \zeta_1 \sin\alpha_r + \eta_1 \cos\alpha_r) = 0, \tag{15}$$

$$(k_{su} + Nk_{sp} - \omega_i^2 I_s/r_s^2)u_s + Nk_{sp}(u_1 - \zeta_1 \sin\alpha_s - \eta_1 \cos\alpha_s) = 0. \tag{16}$$

Using equations (3) and (11), all the equations in equation (8) are equivalent and can be represented by any one of them, say $n = 1$,

$$(\mathbf{K}_{c2}^1)^{\mathrm{T}}\mathbf{p}_c - (\mathbf{K}_{r2}^1)^{\mathrm{T}}\mathbf{p}_r + (\mathbf{K}_{s2}^1)^{\mathrm{T}}\mathbf{p}_s + (\mathbf{K}_{pp} - \omega_i^2 \mathbf{M}_p)\mathbf{p}_1 = \mathbf{0}. \tag{17}$$

Equations (14)–(17) comprise a reduced eigenvalue problem that yields six eigensolution pairs $(\omega_i, [u_c, u_r, u_s, \zeta_1, \eta_1, u_1]^{\mathrm{T}})$. From these eigensolutions, six rotational modes ϕ_i are obtained according to equation (3). Equations (14)–(17) are independent of the planet spacing angles ψ_n and are identical to those for equally spaced planets. Accordingly, these natural frequencies and vibration modes are the same as for equally spaced planets. The critical condition in the above derivation is actually equation (11), not diametrically opposed planets. Thus, more generally, systems satisfying equation (11) have six rotational modes with property (3). For arbitrarily distributed planets not satisfying equation (11), rotational modes do not exist.

3.3. TRANSLATIONAL MODES

While the translational modes couple with the rotational modes for truly arbitrary planet spacing, they retain their structure for systems satisfying equation (11). The notable difference with equally spaced planet systems is that the natural frequencies are no longer degenerate because the cyclic symmetry is disturbed. To start with, the planet deflection relations in a translational mode are derived from equation (4). For any three planets i, j, k,

$$\mathbf{p}_i = \cos\psi_i \mathbf{p}_1 + \sin\psi_i \hat{\mathbf{p}}_1, \qquad \mathbf{p}_j = \cos\psi_j \mathbf{p}_1 + \sin\psi_j \hat{\mathbf{p}}_1, \qquad \mathbf{p}_k = \cos\psi_k \mathbf{p}_1 + \sin\psi_k \hat{\mathbf{p}}_1. \tag{18}$$

Eliminating $\mathbf{p}_1$ and $\hat{\mathbf{p}}_1$ from equation (18) yields

$$\sin(\psi_i - \psi_j)\mathbf{p}_k + \sin(\psi_j - \psi_k)\mathbf{p}_i + \sin(\psi_k - \psi_i)\mathbf{p}_j = \mathbf{0}, \quad i, j, k = 1, \ldots, N, \tag{19}$$

so the nth planet deflection can be expressed as a linear combination of $\mathbf{p}_1$ and $\mathbf{p}_2$. The component modal deflections for a translational mode become

$$\mathbf{p}_h = [x_h, y_h, 0]^{\mathrm{T}}, \qquad h = c, r, s, \qquad \mathbf{p}_n = [\sin(\psi_2 - \psi_n)\mathbf{p}_1 + \sin\psi_n \mathbf{p}_2]/\sin\psi_2. \tag{20}$$

Equations (20) and (11) give $\sum \mathbf{p}_n = \mathbf{0}$, a result that is necessary in the algebraic reductions that follow. Similar to the derivation of equations (12)–(16), the third equation in each of equations (5)–(7) vanishes and only six equations remain:

$$(k_c + Nk_p - \omega_i^2 m_c)x_c - k_p \sum (\cos\psi_n \zeta_n - \sin\psi_n \eta_n) = 0,$$

$$(k_c + Nk_p - \omega_i^2 m_c)y_c - k_p \sum (\sin\psi_n \zeta_n + \cos\psi_n \eta_n) = 0, \tag{21}$$

$$(k_r + k_{rp} \sum \sin^2\psi_{rn} - \omega_i^2 m_r)x_r$$

$$- k_{rp} \sum \sin\psi_{rn} \cos\psi_{rn} y_r - k_{rp} \sum \sin\psi_{rn}(\sin\alpha_r \zeta_n - \cos\alpha_r \eta_n - u_n) = 0,$$

$$(k_r + k_{rp} \sum \cos^2\psi_{rn} - \omega_i^2 m_r)y_r$$

$$- k_{rp} \sum \sin\psi_{rn} \cos\psi_{rn} x_r + k_{rp} \sum \cos\psi_{rn}(\sin\alpha_r \zeta_n - \cos\alpha_r \eta_n - u_n) = 0, \tag{22}$$

$$(k_s + k_{sp} \sum \sin^2\psi_{sn} - \omega_i^2 m_s)x_s$$

$$- k_{sp} \sum \sin\psi_{sn} \cos\psi_{sn} y_s + k_{sp} \sum \sin\psi_{sn}(\sin\alpha_s \zeta_n + \cos\alpha_s \eta_n - u_n) = 0,$$

$$(k_s + k_{sp} \sum \cos^2\psi_{sn} - \omega_i^2 m_s)y_s$$

$$- k_{sp} \sum \sin\psi_{sn} \cos\psi_{sn} x_s - k_{sp} \sum \cos\psi_{sn}(\sin\alpha_s \zeta_n + \cos\alpha_s \eta_n - u_n) = 0, \tag{23}$$

where $\psi_{rn} = \psi_n + \alpha_r$, $\psi_{sn} = \psi_n - \alpha_s$ and α_r, α_s are the pressure angles at the ring/planet and sun/planet meshes. Note that ζ_n, η_n, u_n, $n = 1, \ldots, N$, can be further reduced to six independent variables $\zeta_1, \eta_1, u_1, \zeta_2, \eta_2, u_2$ by equation (20). Use of equations (20) and (11) in equation (8) for different n results in only two independent equations [4]. Thus, equation (1) reduces to 12-degree-of-freedom eigenvalue problem consisting of equations (21)–(23) and (8) for $n = 1, 2$. Twelve natural frequencies ω_i and corresponding modes $[x_c, y_c, x_r, y_r, x_s, y_s, \zeta_1, \eta_1, u_1, \zeta_2, \eta_2, u_2]$ are obtained. The complete vibration modes are constructed according to equation (20). This reduced eigenvalue problem is different from those obtained for equally spaced planets and has no symmetry between x_h and y_h, $h = c, r, s$. Consequently, the 12 natural frequencies associated with the translational modes are distinct, in general. Therefore, planetary gears with planet positions satisfying equations (11) (for example, diametrically opposed planets) have 12 distinct vibration modes that have the special structure (20) of a translational mode.

4. EXAMPLE

As an example, the planetary gear used in a U.S. Army helicopter is studied. The system has four planets and a fixed ring; the parameters are given in reference [4]. The first case considers equally spaced planets and typical vibration modes are shown in Figure 2. In the second case, the planets are diametrically opposed with position angles 0, 80, 180, and 260°. The planet mode natural frequencies [Figures 2(a) and 3(a)] are the same as for equally spaced planets; the vibration modes change slightly because the coefficients w_n of the planet deflections are altered. The rotational modes and their associated natural frequencies Figures 2(b) and 3(b) are identical. Comparing Figures 2(c, d), 3(c, d), the translational modes still have well-defined structure with diametrically opposed planets. The degenerate

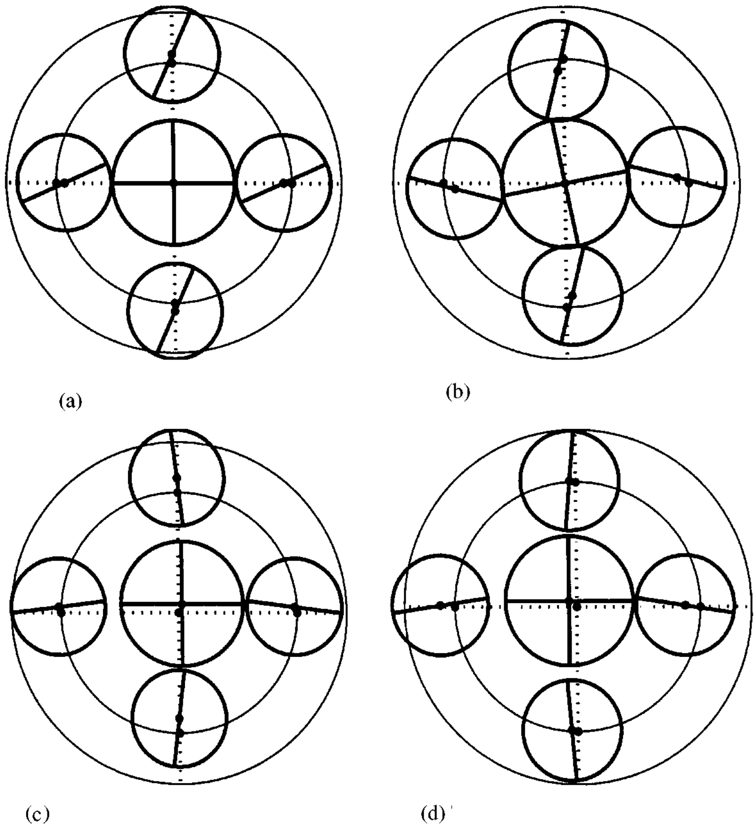

Figure 2. Typical vibration modes for equally spaced planets. The movements of the carrier and ring are not shown in order to clarify the figures. Dotted lines are the equilibrium positions and solid lines are the deflected positions. Large dots represents the component centers. (a) Planet mode (6981 Hz), (b) rotational mode (1661 Hz) and (c), (d) a pair of translational modes (8251 Hz).

translational mode natural frequencies split into distinct ones as a result of the unequal spacing.

5. DISCUSSION

The unique modal properties of planetary gears derived previously for equally spaced planet systems are preserved in certain unequally spaced planet systems. Planet modes of multiplicity $N-3$ are remarkably insensitive to planet location and retain their special properties for arbitrary planet spacing. Coupling between rotational and translational modes occurs for arbitrary planet spacing, and distinct properties cannot be identified. For systems satisfying equation (11), however, rotational and translational modes have structured properties. This includes the common case of diametrically opposed planet pairs.

An important implication of these results is on the use of planet phasing to suppress certain vibration modes in planetary gear response [6–9]. The most recent of these studies [9] considers diametrically opposed planet systems. Because the special properties of rotational and translational modes are preserved for such systems, conclusions regarding the effectiveness of planet phasing to eliminate excitation of these modes under operating conditions are possible.

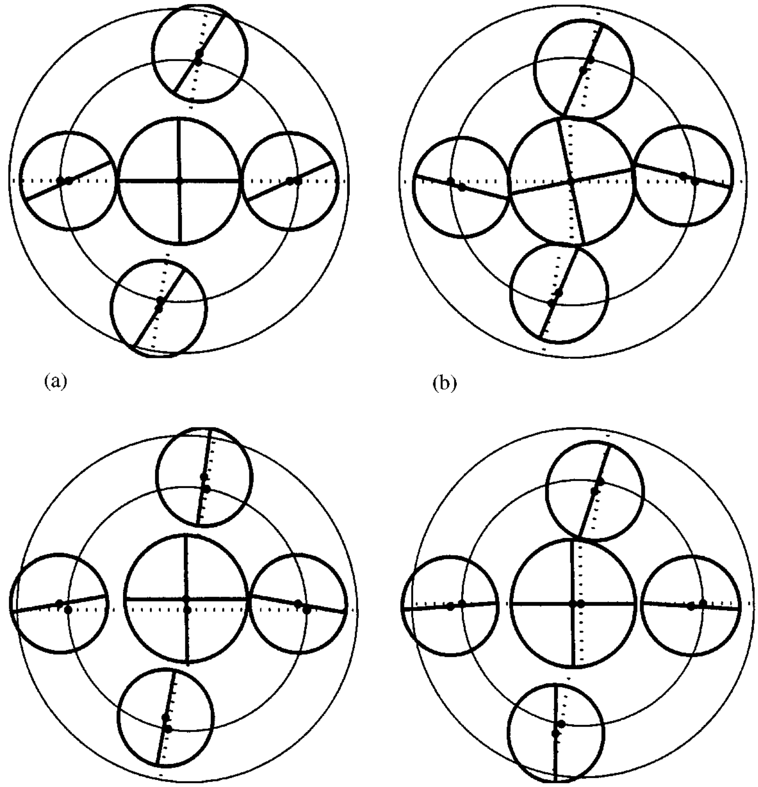

Figure 3. Typical vibration modes for diametrically opposed planets. The figure description is the same as in Figure 2. (a) Planet mode (6981 Hz), (b) rotational mode (1661 Hz), (c) translational mode (8529 Hz) and (d) translational mode (7929 Hz).

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