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EXACT SOLUTIONS FOR FREE LONGITUDINAL VIBRATIONS OF NON-UNIFORM RODS

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An exact approach for free longitudinal vibrations of one-step non-uniform rods with classical and non-classical boundary conditions is presented. In this paper, the expression for describing the distribution of mass is arbitrary, and the distribution of longitudinal stiffness is expressed as a functional relation with the mass distribution and *vice versa*. Using appropriate functional transformation, the governing differential equations for free vibrations of one-step non-uniform rods are reduced to analytically solvable differential equations for several functional relations between stiffness and mass. The fundamental solutions that satisfy the normalization conditions are derived and used to establish the frequency equations for one-step rods with classical and non-classical boundary conditions. Using the fundamental solutions of each step rod and a recurrence formula developed in this paper, a new exact approach for determining the longitudinal natural frequencies and mode shapes of multi-step non-uniform rods is proposed. Numerical examples demonstrate that the calculated longitudinal natural frequencies and mode shapes are in good agreement with the experimental data and those determined by the finite element method, and the proposed procedure is an efficient and exact method.

1. INTRODUCTION

The longitudinal vibration of non-uniform beams and rods is a subject of considerable scientific and practical interest that has been studied extensively, and is still receiving attention in the literature because of its relevance to structural, mechanical and aeronautical engineering. Investigations into earthquake-induced structural damages [1] showed that structural damage in the upper part of a high-rise structure is usually caused by the combined actions of horizontal and vertical ground motions. Thus, it is necessary to determine the natural frequencies and mode shapes in the vertical direction for high-rise structures at the design stage for certain cases. When analyzing the free vibrations of high-rise structures, it is possible to regard such structures as a cantilever rod with varying cross-sections [2–6]. However, in general, it is not possible or, at least, very difficult to get the exact analytical solutions of differential equations for free vibrations of rods with variably distributed mass and stiffness. These exact rod solutions are available only for certain rod shapes and boundary conditions. Wang [2] derived the closed-form solutions for the free longitudinal vibration of a rod with variably distributed stiffness and mass that were described by exponential functions. Eisenberger [7] found exact longitudinal natural frequencies of a variable cross-section rod with polynomial variation in the cross-sectional area and mass distribution along the member using the exact element method. By using the discrete Green function and transforming the fundamental equation governing the

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longitudinal free vibration of a non-uniform circular bar into a boundary integral equation, Matsuda *et al.* [8] obtained the eigenvalues of such a bar. Bapat [9] obtained exact solutions for the longitudinal vibration of exponential and catenoidal rods. Lau [10] and Abrate [11] derived closed-form solutions for the free longitudinal vibration of rods whose cross-section varies as $A(x) = A_0(x/L)^2$ and $A(x) = A_0[1 + a(x/L)]^2$ respectively. Kumar *et al.* [12] obtained exact solutions for the longitudinal vibration of non-uniform rods whose cross-section varies as $A(x) = (a + bx)^n$ and $A(x) = A_0 \sin^2(ax + b)$.

In the research work mentioned above, it was usually assumed that the mass per unit length of a rod is proportional to its longitudinal stiffness. This calculation model is suitable for a part of high-rise structures [2–4] and solid rods [13]. However, this assumption is not reasonable for tall buildings and many high-rise structures. This is due to the fact that the mass of floors is a significant part of the total mass of a tall building and the variation of mass at different floors is not significant. Hence, the distribution of mass with building height is not necessarily proportional to that of the stiffness. This is confirmed by a series of shaking tests on buildings of various types in which the mass and stiffness of individual buildings have been measured and reported [14].

In the Li *et al.* [5] study, the distribution of mass of a rod was assumed to be not necessarily proportional to that of stiffness, i.e.,

$$\bar{m}(x) = a\mathrm{e}^{-b(x/L)}, \quad K(x) = a\mathrm{e}^{-\beta(x/L)} \text{ and } \bar{m}(x) = a\left(1 + b\frac{x}{L}\right)^C, \quad K(x) = \alpha\left(1 + \beta\frac{x}{L}\right)^\gamma.$$

The exact solutions for the above cases were obtained.

A review of the technical literature dealing with this problem indicates that generally the authors of previous studies have directed their investigations to special functions for describing the distributions of mass and longitudinal stiffness to derive closed-form solutions. However, exact solutions for free longitudinal vibration of non-uniform rods with arbitrary distribution of mass or stiffness have not been obtained in the literature. In this paper, the expression for describing the distribution of mass of a rod is arbitrary, and the stiffness distribution is a functional expression of the mass distribution and *vice versa*. Using appropriate functional transformation, the governing differential equations for free vibrations of one-step non-uniform rods are reduced to Bessel equations or ordinary differential equations with constant coefficients for several functional relations between stiffness and mass. The fundamental solutions that satisfy the normalization condition are derived and used to establish the frequency equation for one-step non-uniform rods with classical or non-classical boundary conditions. A new exact approach that combines the recurrence formula and fundamental solutions of one-step rods developed in this paper lead to a single-frequency equation for multi-step non-uniform rods with any number of steps. With the proposed method, no recourse to a computer-based technique is required. It does not require a large amount of computer memory storage and CPU time.

The objective of this paper is to present the exact solutions for free vibration of non-uniform rods with arbitrary distribution of mass or stiffness. In the absence of the exact solutions, this problem can also be handled using the finite element method. However, the present exact solutions which can be easily implemented could provide adequate insight into the physics of the problem and supplement the existing database, and further serve as the benchmark for researchers and engineers to examine the merits of new numerical methods in this field. Therefore, it is always desirable to obtain exact solutions to such problems.

2. FUNCTIONAL SOLUTIONS OF NON-UNIFORM RODS

The governing differential equation for vibration mode shape function of a non-uniform rod is given by [5]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[K(x)\frac{\mathrm{d}X(x)}{\mathrm{d}x}\right]+\bar{m}(x)\omega^2 X(x)=0 \tag{1}$$

in which $K(x)$, $\bar{m}(x)$, ω and $X(x)$ are the longitudinal stiffness, mass per unit length, circular natural frequency and vibration mode shape function respectively.

In order to solve equation (1), we let

$$X(x)=X(\eta),\quad K(x)=\text{arbitrary},\quad \bar{m}(x)=K^{-1}(x)p(\eta),\quad \eta=\int K^{-1}(x)\,\mathrm{d}x \tag{2}$$

or

$$X(x)=X(\eta),\quad \bar{m}(x)=\text{arbitrary},\quad K(x)=\bar{m}^{-1}(x)p(\eta),\quad \eta=\int K^{-1}(x)\,\mathrm{d}x \tag{3}$$

In the above equations, functional transformation is introduced. Pouyet and Lataillade [15] applied a similar transformation to torsional vibrations of a shaft with non-uniform cross-section.

In equations (2) and (3), the expression for describing the distribution of longitudinal stiffness of a rod is arbitrary, and the distribution of mass is expressed as a functional relation with the distribution of stiffness and *vice versa.*

Substituting equation (2) or (3) into equation (1) leads to

$$\frac{\mathrm{d}^2X(\eta)}{\mathrm{d}t^2}+\omega^2p(\eta)X(\eta)=0. \tag{4}$$

Obviously, the solution of the above equation is dependent on the expression of $p(\eta)$. It is difficult to obtain the exact solution of equation (4) for general cases. The following two important cases that cover many cases of ordinary structural members are considered herein.

Case 1 $\quad p(\eta)=(a+b\eta)^C,$ (5)

where a, b, c are parameters that can be determined by the values of $\bar{m}(x)$ or $K(x)$ at critical sections based on equation (2) or (3).

In this case, equation (4) becomes

$$\frac{\mathrm{d}^2X(\eta)}{\mathrm{d}\eta^2}+\omega^2(a+b\eta)^C X(\eta)=0. \tag{6}$$

By using the following functional transformation,

$$\zeta=(a+b\eta)^{1/2v},\ X=\zeta^v Z,\ v=\frac{1}{C+\bar{\alpha}}. \tag{7}$$

Equation (6) is reduced to a Bessel equation

$$\frac{\mathrm{d}^2Z}{\mathrm{d}\zeta^2}+\frac{1}{\zeta}\frac{\mathrm{d}Z}{\mathrm{d}\zeta}+\left(\alpha^2-\frac{v^2}{\zeta^2}\right)Z=0 \tag{8}$$

where

$$\alpha = \frac{2v\omega}{|b|}, \tag{9}$$

and $|b|$ represents the absolute value of b.

The vibration mode shape function can be expressed as

$$X = \begin{cases} (a+b\eta)^{1/2}\{C_1 \mathrm{J}_v[\alpha(a+b\eta)^{1/2}] + C_2 \mathrm{J}_{-v}[\alpha(a+b\eta)^{1/2}]\}, & v = \text{a non-integer}, \\ (a+b\eta)^{1/2}\{C_1 \mathrm{J}_v[\alpha(a+b\eta)^{1/2}] + C_2 \mathrm{Y}_v[\alpha(a+b\eta)^{1/2}]\}, & v = \text{an integer}, \end{cases} \tag{10}$$

where J_v and Y_v are Bessel functions of the first and second kind of order v, respectively; C_1 and C_2 are integral constants that can be determined by the boundary conditions.

If $C = -2$, then it can be found from equation (7) that $v = \infty$. In this case, equation (6) becomes an Euler equation, the general solution of which can be written as

$$X = \begin{cases} (a+b\eta)^{1/2}(C_1 \sin[\bar{\alpha}\ln(a+b\eta)] + C_2 \cos[\alpha \ln(\mathrm{a}+\mathrm{b}\eta)]\} & \text{for } 4\omega^2 - b^2 > 0, \\ C_1(a+b\eta)^{(1/2)+\bar{\alpha}} + C_2(a+b\eta)^{(1/2)-\bar{\alpha}} & \text{for } 4\omega^2 - b^2 < 0, \\ C_1(a+b\eta)^{1/2} + C_2(a+b\eta)^{1/2}\ln(a+b\eta) & \text{for } 4\omega^2 - b^2 = 0, \end{cases} \tag{11}$$

where

$$\bar{\alpha} = \frac{|4\omega^2 - b^2|^{1/2}}{2|b|}. \tag{12}$$

Case 2 $\quad p(\eta) = a\mathrm{e}^{b\eta}, \qquad (13)$

where a and b are parameters that can be determined by the values of $\bar{m}(x)$ and $K(x)$ at critical sections based on equation (2) or (3).

Using the functional transformation

$$\zeta = \mathrm{e}^{b\eta/2}, \tag{14}$$

and substituting equation (14) into (4), one obtains

$$\frac{\mathrm{d}^2 X}{\mathrm{d}\zeta^2} + \frac{1}{\zeta}\frac{\mathrm{d}X}{\mathrm{d}\zeta} + \bar{\alpha}X = 0 \tag{15}$$

where

$$\bar{\alpha} = \frac{2\omega a^{1/2}}{|b|}, \quad a > 0. \tag{16}$$

Equation (15) is a Bessel equation of the zeroth order. The general solution of equation (15) can be written as

$$X = C_1 \mathrm{J}_0(\bar{\alpha}\mathrm{e}^{b\eta/2}) + C_2 Y_0(\bar{\alpha}\mathrm{e}^{b\eta/2}). \tag{17}$$

Case 3 $p(\eta) = a$. This case corresponds to a uniform rod. The general solution of a uniform rod can be written as

$$X = C_1 \sin\left(\sqrt{\frac{\bar{m}}{K}}\,\omega x\right) + C_2 \cos\left(\sqrt{\frac{\bar{m}}{K}}\,\omega x\right), \tag{18}$$

where $\bar{m}$ and K are the mass per unit length and longitudinal stiffness respectively.

It is evident that one solution of equation (4) represents a class of solutions, because $\bar{m}(x)$ is an arbitrary function, and $K(x)$ is a functional relation with $\bar{m}(x)$ and *vice versa*. This type of solution is called functional solution by the author in this paper.

3. FREQUENCY EQUATIONS OF ONE-STEP RODS

The general solutions of mode shape functions for all the cases mentioned above are expressed as the unified form

$$X(x) = C_1 S_1(x) + C_2 S_2(x), \tag{19}$$

where $S_1(x)$ and $S_2(x)$ are special solutions of mode shapes, which can be found from equations (10) or (11), (16) and (18) for the Cases 1, 2 and 3 respectively.

In order to simplify the analysis, the linearly independent fundamental solutions, $\overline{S_1}(x)$ and $\overline{S_2}(x)$, are chosen such that they satisfy the following normalization condition at the origin of the co-ordinate system:

$$\begin{bmatrix} \overline{S_1}(0) & \overline{S_1'}(0) \\ \overline{S_2}(0) & \overline{S_2'}(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{20}$$

In general, $\overline{S_1}(x)$ and $\overline{S_2}(x)$ can be given by

$$\begin{bmatrix} \overline{S_1}(x) \\ \overline{S_2}(x) \end{bmatrix} = \begin{bmatrix} S_1(0) & S_1'(0) \\ S_2(0) & S_2'(0) \end{bmatrix} \begin{bmatrix} S_1(x) \\ S_2(x) \end{bmatrix} \tag{21}$$

where

$$\overline{S_i'}(0) = \left.\frac{\mathrm{d}S_i(x)}{\mathrm{d}x}\right|_{x=0}, \quad i = 1, 2. \tag{22}$$

In general, the mode shape function of a one-step rod can be expressed in terms of the fundamental solutions as

$$X(x) = X_0 \overline{S_1}(x) + \frac{N_0}{K(0)} \overline{S_2}(x), \tag{23}$$

where $X_0 = X(0)$ is the displacement at $x = 0$, N_0 is the axial force at $x = 0$; they are called the initial parameters.

It is easy to obtain the mode shape functions and frequency equations for various boundary conditions by using the fundamental solutions and the initial parameters as follows:

(1) *Fixed–free rod* (Figure 1). The boundary conditions for this case are

$$x = 0, \quad X(0) = 0, \quad x = L, \quad X'(L) = 0. \tag{23}$$

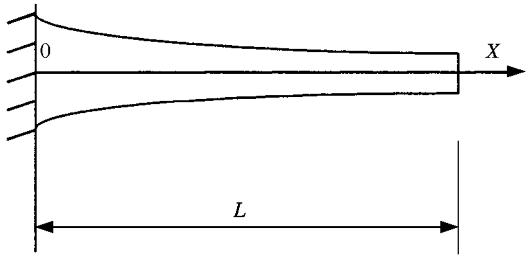

Figure 1. A fixed–free rod.

Using equation (23) and the boundary condition at $x = 0$, one gets the mode shape function of the rod shown in Figure 1 as follows:

$$X(x) = \frac{N_0}{K(0)}\,\overline{S_2}(x), \tag{24}$$

where $N_0/K(0)$ can be taken as any value, for example, as 1.

The frequency equation is established by using the boundary condition at $x = L$ as

$$\overline{S_1'}(L) = 0. \tag{25}$$

(2) *Fixed–fixed rod*: The boundary conditions for this case are

$$x = 0, \quad X(0) = 0, \quad x = L, \quad X(L) = 0. \tag{26}$$

The mode shape function has the same form as that given in equation (24), but the frequency equation is

$$\overline{S_2}(L) = 0. \tag{27}$$

(3) *Free–free rod*: The boundary conditions for this case are

$$x = 0, \quad X'(0) = 0, \quad x = L, \quad X'(L) = 0. \tag{28}$$

The mode shape function can be written as

$$X(x) = X_0\overline{S_1}(x), \tag{29}$$

where X_0 can be taken as any value, for example, as 1.

The frequency equation is obtained by using the boundary condition at $x = L$ as

$$\overline{S_1}(L) = 0. \tag{30}$$

(4) *Fixed–spring rod with a concentrated mass at the spring end* (Figure 2). The boundary conditions for this case are

$$x = 0, \quad X(0) = 0, \quad x = L, \quad X'(L) = \frac{X(L)}{K(L)}(M_L\omega^2 - K_L). \tag{31}$$

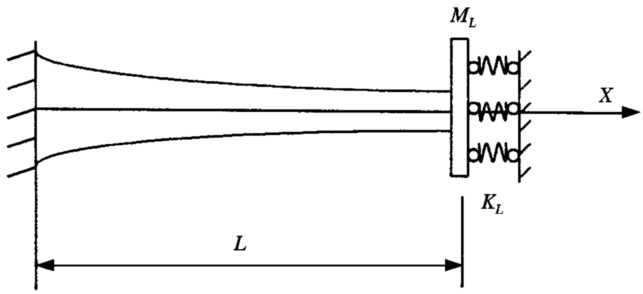

Figure 2. A fixed–spring rod with a concentrated mass at the spring end.

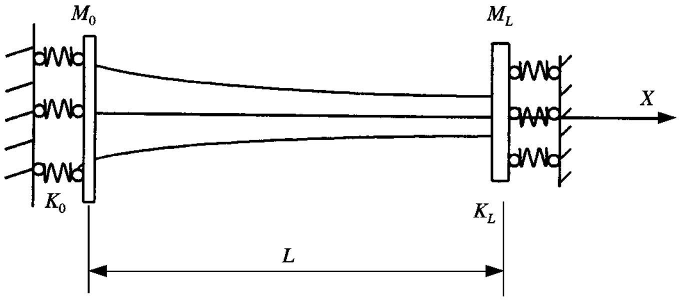

Figure 3. A spring–spring rod with concentrated masses at the ends.

The mode shape function has the same form as that given in equation (24), but the frequency equation is

$$K(L)\overline{S_2'}(L) = \overline{S_2}(L)(M_L\omega^2 - K_L). \tag{32}$$

If the right end of the rod is free with a concentrated mass, then set $K_L = 0$ in the above equation.

(5) *Spring–spring rod with concentrated masses at the ends* (Figure 3). The boundary conditions for this case are

$$\begin{aligned} x = 0, &\quad K(0)X'(0) = -\,X_0(M_0\omega^2 - K_0), \\ x = L, &\quad K(L)X'(L) = X(L)(M_L\omega^2 - K_L), \end{aligned} \tag{33}$$

The mode shape function can be written as

$$X(x) = X_0\left\{\overline{S_1}(x) - \frac{\overline{S_2}(x)}{K(0)}(M_0\omega^2 - K_0)\right\}, \tag{34}$$

Using the boundary condition at $x = L$ and setting $X_0 = 1$, one obtains the frequency equation as

$$K(L)\overline{S_1'}(L) - \frac{K'(L)}{K(0)}(M_0\omega^2 - K_0)\overline{S_2'}(L) = (M_L\omega^2 - K_L)\left[\overline{S_1}(L) - \frac{\overline{S_2}(L)}{K(0)}(M_0\omega^2 - K_0)\right]. \tag{35}$$

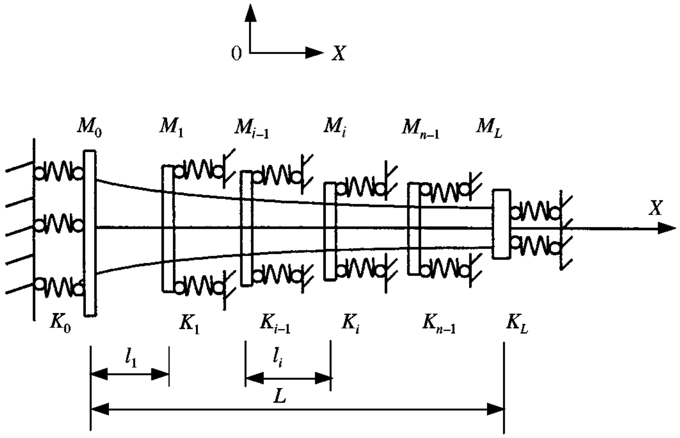

Figure 4. A spring–spring rod with concentrated masses at the ends and at the other $(n-1)$ points.

(6) *Spring–spring rod with concentrated masses at the ends and other* $(n-1)$ *points* (Figure 4). The boundary conditions for this case are the same as those described in equation (33), and the mode shape function can be written as

$$X(x) = X_1(x) - \sum_{i=1}^{n} \frac{X_i(l_i)}{K(l_i)} [M_i\omega^2 - K_i]\overline{S_2}(x - l_i)H(x - l_i), \tag{36}$$

where l_i is the length of the ith step, the origin of the co-ordinate is set at the left end of this step, and $X_1(x)$ is given by

$$X_1(x) = X_0\left[\overline{S_1}(x) - \frac{\overline{S_2}(x)}{K(0)}(M_0\omega^2 - K_0)\right], \tag{37}$$

where $H(\cdot)$ is Heaviside function, and $X_1(x)$ is the mode shape function of the first step. $X_i(x)$ can be determined using the recurrence formula as

$$X_i(x) = X_{i-1}(x) - \frac{X_{i-1}(l_{i-1})}{K(l_{i-1})}[M_{i-1}\omega^2 - K_{i-1}]\overline{S_2}(x - l_{i-1})H(x - l_{i-1}). \tag{38}$$

Using the boundary condition at $x = L$ and setting $X_0 = 1$, one obtains the frequency equation as follows:

$$K(L)\left\{\overline{S_1'}(L) - \frac{\overline{S_2'}(L)}{K(0)}(M_0\omega^2 - K_0) - \sum_{i=1}^{n} \frac{X_i(l_i)}{K(l_i)}[M_i\omega^2 - K_i]\overline{S_2'}(L - l_i)\right\}$$

$$= (M_L\omega^2 - K_L)\left\{\overline{S_1}(L) - \frac{\overline{S_2}(L)}{K(0)}(M_0\omega^2 - K_0) - \sum_{i=1}^{n} \frac{X_i(l_i)}{K(l_i)}[M_i\omega^2 - K_i]\overline{S_2}(L - l_i)H(L - l_i)\right\}, \tag{39}$$

where $M_n = M_L$.

4. FREQUENCY EQUATIONS OF MULTI-STEP RODS

A multi-step rod, with each step being a non-uniform rod, is shown in Figure 5. In order to establish the frequency equation of such a rod, it is convenient to use the fundamental solutions and recurrence formula developed in this paper.

If $\overline{S_{i1}}(x)$ and $\overline{S_{i2}}(x)$ are the fundamental solutions of the ith step, then the mode shape function of the ith step can be written as

$$X_i(x) = X_i(0)\overline{S_{i1}}(x) + \frac{N_i(0)}{K_i(0)}\overline{S_{i2}}(x). \tag{40}$$

Since

$$X_i(0) = X_{i-1}(l_{i-1}),\ N_i(0) = K_{i-1}(l_{i-1})X'_{i-1}(l_{i-1}), \tag{41}$$

then equation (40) can be rewritten as

$$X_i(x) = X_{i-1}(l_{i-1})\overline{S_{i1}}(x) + \frac{K_{i-1}(l_{i-1})}{K_i(0)}X'_{i-1}(l_{i-1})\overline{S_{i2}}(x), \tag{42}$$

where l_{i-1} is the length of the $(i-1)$th step, and the origin of the co-ordinate is set at the left end of this step.

This is a recurrence formula. The frequency equation can be obtained by using the recurrence formula as follows.

1. *Fixed–free rod* (Figure 5). The boundary conditions are given in equation (23). Using the boundary conditions at $x = 0$, one obtains the mode shape function of the first step rod as

$$X_1(x) = \frac{N_0}{K_1(0)}\overline{S_{12}}(x). \tag{43}$$

Using $X_l(x)$ and the recurrence formula, the mode shape functions of the ith step rod $(i = 2, 3, \ldots, n)$ are obtained.

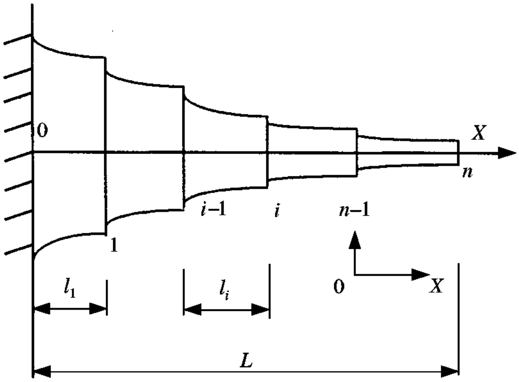

Figure 5. A multi-step rod.

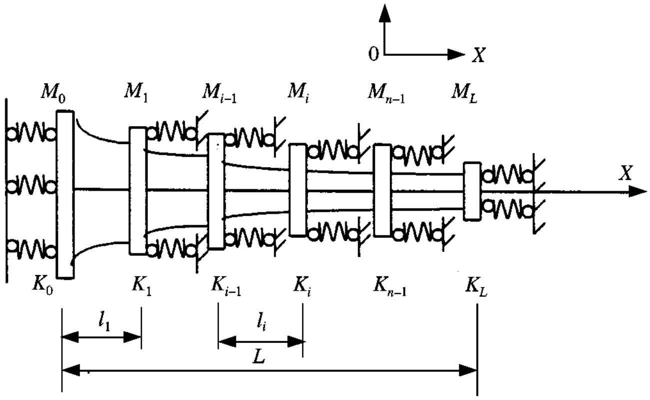

Figure 6. A multi-step rod with concentrated masses and spring supports.

The frequency equation can be established by using $X_n(x)$ and the boundary condition at $x = L$ as

$$X_n'(l_n) = 0. \tag{44}$$

2. *Fixed–fixed rod.* $X_1(x)$ has the same form as that given in equation (43), but the frequency equation is

$$X_n(l_n) = 0. \tag{45}$$

3. *Free–free rod.* $X_1(x)$ is given by

$$X_1(x) = X_0\overline{S_{11}}(x). \tag{46}$$

The frequency equation for this case is equation (44).

A multi-step rod with several concentrated masses and rectilinear springs as shown in Figure 6 is considered here.

Since

$$\begin{aligned} &X_i(0) = X_{i-1}(l_{i-1}), \\ &N_i(0) = K_{i-1}(l_{i-1})X_{i-1}'(l_{i-1}) - (M_{i-1}\omega^2 - K_{i-1})X_{i-1}(l_{i-1}) \end{aligned} \tag{47}$$

then the mode shape function of the ith step rod can be written as

$$X_i(x) = X_{i-1}(l_{i-1})\overline{S_{i1}}(x) + \frac{K_{i-1}(l_{i-1})}{K_i(0)}\left[X_{i-1}'(l_{i-1}) - \frac{X_{i-1}(l_{i-1})}{K_{i-1}(l_{i-1})}(M_{i-1}\omega^2 - K_{i-1})\right]\overline{S_{i2}}(x), \tag{48}$$

where l_{i-1} is the length of the $(i-1)$th step rod, and the origin of the co-ordinate is set at the left end of this step.

The boundary conditions for this case are

$$\begin{aligned} x = 0, \quad & X_1'(0) = -\frac{X(0)}{K_1(0)}(M_0\omega^2 - K_0), \\ x = L, \quad & X_n'(l_n) = \frac{X_n(l_n)}{K_n(l_n)}(M_n\omega^2 - K_n). \end{aligned} \tag{49}$$

Using the boundary condition at $x = 0$ leads to

$$X_1(x) = X(0)\left[\overline{S_{11}}(x) - \frac{1}{K_1(0)}(M_0\omega^2 - K_0)\overline{S_{12}}(x)\right], \tag{50}$$

where $X(0)$ can be taken as any value, for example, as 1.

$X_n(x)$ can be obtained by using $X_1(x)$ and the recurrence formula (48). The frequency equation can be established by using $X_n(x)$ and the boundary conditions at $x = L$ as follows:

$$X_n'(l_n) = \frac{X_n(l_n)}{K_n(l_n)}(M_n\omega^2 - K_n). \tag{51}$$

5. NUMERICAL EXAMPLE 1

In order to illustrate the proposed methods, a 20-story building with a height of 59·3 m which is located in Beijing is considered here for free vibration analysis. Based on the field dynamic measurement [3], this building can be simplified as a cantilever rod in the analysis for free longitudinal vibrations.

The procedure for determining the longitudinal natural frequencies and mode shapes of this building is as follows:

1. *Determination of the distributions of mass and longitudinal stiffness.* The stiffness distribution of this building is stepped variation, which is determined and shown in Figure 7(a).

(2) *Selection of expressions for describing the distributions of mass and longitudinal stiffness.* According to the real variations of mass and stiffness shown in Figure 7, in order to simplify the calculation, it is assumed that the distributions of mass and stiffness can be treated as continuous distributions as follows:

$$K(x) = K_0 e^{-\beta(x/L)}, \quad \bar{m}(x) = \bar{m}_0 e^{-\gamma(x/L)} \tag{52}$$

Based on the values of $\bar{m}(x)$ and $K(x)$ at $x = 0$ and $x = L$, we have

$$K(0) = K_0 = 3{\cdot}49 \times 10^{11}\ \text{N}, \quad \beta = \ln\frac{K(0)}{K(L)} = \ln\frac{3{\cdot}46}{2{\cdot}74} = 0{\cdot}23,$$

$$\bar{m}(0) = \bar{m}_0 = 3{\cdot}49 \times 10^{5}\ \text{kg/m}, \quad \gamma = \ln\frac{\bar{m}(0)}{\bar{m}(L)} = \ln\frac{3{\cdot}46}{2{\cdot}91} = 0{\cdot}18.$$

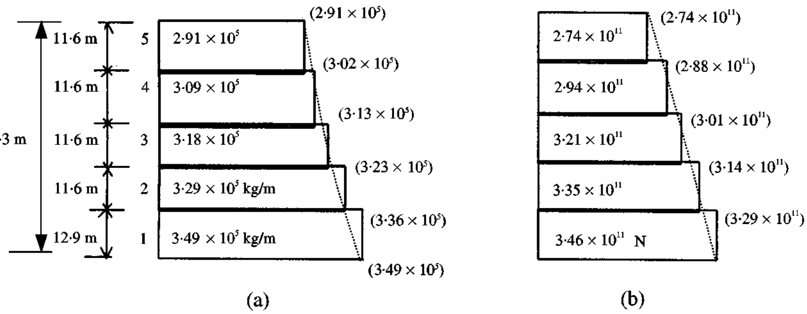

Figure 7. The distributions of mass and stiffness of a 20-story building: (a) mass; (b) stiffness.

The distributions of mass and stiffness given by equation (52) are also shown in Figure 7 (dotted line and values in parentheses) for comparison purposes.

(3) *Determination of* $p(\eta)$. Using equations (52) and (2), one obtains

$$p(\eta) = a\mathrm{e}^{b\eta}, \quad \eta = \int K^{-1}(x)\,\mathrm{d}x = \frac{L}{K_0\beta}\mathrm{e}^{\beta(x/L)} \tag{53}$$

where

$$a = \bar{m}_0 K_0, \qquad b = -(\beta + \gamma). \tag{54}$$

4. *Determination of the natural frequencies and mode shape functions*: Because the expression of $p(\eta)$ is the same as that in Case 2, the mode shape function can be found from equation (16) and expressed as

$$X(x) = C_1 J_0(\bar{\alpha}\mathrm{e}^{-(bL/2K_0\beta)\mathrm{e}^{\beta(x/L)}}) + C_2 Y_0(\bar{\alpha}\mathrm{e}^{-(bL/2K_0\beta)\mathrm{e}^{\beta(x/L)}}), \tag{55}$$

where

$$\bar{\alpha} = \frac{2\omega(\bar{m}_0 K_0)^{1/2}}{\beta + \gamma}. \tag{56}$$

In order to simplify the expression, we let

$$S_1(x) = J_0(\bar{\alpha}\mathrm{e}^{(bL/2K_0\beta)\mathrm{e}^{\beta(x/L)}}), \quad S_2(x) = Y_0(\bar{\alpha}\mathrm{e}^{(bL/2K_0\beta)\mathrm{e}^{\beta(x/L)}}). \tag{57}$$

Using equations (21) and (57) gives the fundamental solutions as

$$\begin{aligned} \overline{S_1}(x) &= a_{11}S_1(x) + a_{12}S_2(x), \\ \overline{S_2}(x) &= a_{21}S_1(x) + a_{22}S_2(x), \end{aligned} \tag{58}$$

where

$$
\begin{aligned}
a_{11} &= \frac{S_2'(0)}{S_1(0)S_2'(0) - S_2(0)S_1'(0)}, \\
a_{12} &= -\frac{S_1'(0)}{S_1(0)S_2'(0) - S_2(0)S_1'(0)}, \\
a_{21} &= -\frac{S_2(0)}{S_1(0)S_2'(0) - S_2(0)S_1'(0)}, \\
a_{22} &= \frac{S_1(0)}{S_1(0)S_2'(0) - S_2(0)S_1'(0)},
\end{aligned} \tag{59}
$$

$$S_1'(0) = \left.\frac{dS_1(x)}{dx}\right|_{x=0}, \quad S_2'(0) = \left.\frac{dS_2(x)}{dx}\right|_{x=0}$$

$$
\begin{aligned}
\frac{dS_1(x)}{dx} &= -J_1(\bar{\alpha}e^{(bL/2K_0\beta)e^{\beta(x/L)}})\frac{\bar{\alpha}b}{2K_0}e^{(bL/2K_0\beta)e^{\beta(x/L)}+\beta(x/L)} \\
\frac{dS_2(x)}{dx} &= -Y_1(\bar{\alpha}e^{(bL/2K_0\beta)e^{\beta(x/L)}})\frac{\bar{\alpha}b}{2K_0}e^{(bL/2K_0\beta)e^{\beta(x/L)}+\beta(x/L)}.
\end{aligned} \tag{60}
$$

The boundary conditions of this buildings are given in equation (23). Using equation (23), one obtains

$$X(x) = \frac{N_0}{K(0)}\bar{S}_2(x). \tag{61}$$

The frequency equation is equation (25), i.e.,

$$J_0(\bar{\alpha}e^{(bL/2K_0\beta)})Y_1(\bar{\alpha}e^{(bL/2K_0\beta)e^{\beta}}) = Y_0(\bar{\alpha}e^{(bL/2K_0\beta)})J_1(\bar{\alpha}e^{(bL/2K_0\beta)e^{\beta}}) \tag{62}$$

Solving equation (62) gives a set of $\bar{\alpha}_i$ ($i = 1, 2, \dots$), and the minimum value of $\bar{\alpha}_i$ is found as

$$\bar{\alpha}_1 = 4{\cdot}4534 \times 10^{10}.$$

Using equation (56) yields $\omega_1 = 26{\cdot}2723$ rad/s and $T_1 = 0{\cdot}2329$ s.

The field measured value of the fundamental period is 0·23 s [3].

It is evident that the computed value of the fundamental period shows good agreement with the measured one.

Substituting $\bar{\alpha}_1$ into equation (61) yeilds the fundamental mode shape, $X_1(x)$. The values of $X_1(x)$ are listed in Table 1.

The longitudinal natural frequencies and mode shapes of this building and also calculated by FEM (the non-uniform rod is divided into 40 uniform segments). It was found that $\omega_1 = 26{\cdot}2731$ rad/s^{-1}.

The free longitudinal vibrations of this building were measured on site by using the ambient method [3]. A sensor was fixed at the top floor of this building as a reference point, and the other four sensors were moved along the selected stories to make the measurements. The field measured values of the fundamental mode shape are also listed in Table 1 for comparison purposes. It is necessary to point out that all the data listed in Table 1 are relative values (normalized by the maximum amplitude). It can be seen from Table 1 that the values of the fundamental mode shape determined by the present method are very close to the measured data and those calculated by FEM.

TABLE 1

The fundamental mode shape of the 20-story building

X/L	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
Calculated values	0 (0) [0]	0·156 (0·156) [0·158]	0·309 0·309) [0·311]	0·454 (0·454) [0·456]	0·588 (0·588) [0·591]	0·707 (0·707) [0·709]	0·809 (0·809) [0·811]	0·891 (0·891) [0·892]	0·951 (0·950) [0·953]	0·987 (0·987) [0·989]	1·0 (1·0) [1·0]
Measured values	0	1·160	0·311	0·456	0·589	0·712	0·810	0·889	0·951	0·988	1·0

Note: (1) Values in brackets are calculated by FEM, (2) values in parentheses are determined by use of the model of a multi-step rod to be described in Numerical Example (3). These notes are also applicable to Table 2 and 4.

Using the aforementioned procedure, the second and third natural frequencies are found as $\omega_2 = 70{\cdot}4097$ rad/s and $\omega_3 = 110{\cdot}3437$ rad/s, and the corresponding mode shapes are determined and listed in Table 2. The FEM is also applied to compute the higher natural frequencies and mode shapes. It was found that $\omega_2 = 70{\cdot}4106$ rad/s and $\omega_3 = 110{\cdot}4179$ rad/s, and the mode shapes are presented in Table 2.

6. NUMERICAL EXAMPLE 2

This numerical example presents the determination of the longitudinal natural frequencies and mode shapes of the building introduced in Numerical Example 1 by using the model of a multi-step rod.

Because the mass and stiffness distributions are stepped variations, this building can be treated as a five-step cantilever rod as shown in Figure 7. The procedure for determining the natural frequencies and mode shapes of this building by using the model of five-step rod shown in Figure 7 is as follows.

1. *Determination of the distributions of mass and longitudinal stiffness.* The distributions of mass and stiffness of the five steps are determined and shown in Figure 7.

2. *Determination of the fundamental solutions for each step.* Because the mass and stiffness of each step are constants, the special solutions can be found from equation (18), and the fundamental solutions of the ith step can be determined from equation (20) as

$$\overline{S_{i1}}(x) = \cos\left(\sqrt{\frac{\bar{m}_i}{K_i}}\,\omega x\right), \qquad \overline{S_{i2}}(x) = \sin\left(\sqrt{\frac{\bar{m}_i}{K_i}}\,\omega x\right), \tag{63}$$

where $\bar{m}_i$ and K_i can be found from Figure 7 as follows:

$$\bar{m}_1 = 3{\cdot}49 \times 10^5\ \text{kg/m}, \qquad K_1 = 3{\cdot}46 \times 10^{11}\ \text{N},$$

$$\bar{m}_2 = 3{\cdot}29 \times 10^5\ \text{kg/m}, \qquad K_2 = 3{\cdot}35 \times 10^{11}\ \text{N},$$

$$\bar{m}_3 = 3{\cdot}18 \times 10^5\ \text{kg/m}, \qquad K_3 = 3{\cdot}21 \times 10^{11}\ \text{N},$$

$$\bar{m}_4 = 3{\cdot}09 \times 10^5\ \text{kg/m}, \qquad K_4 = 2{\cdot}94 \times 10^{11}\ \text{N},$$

$$\bar{m}_5 = 2{\cdot}91 \times 10^5\ \text{kg/m}, \qquad K_5 = 2{\cdot}74 \times 10^{11}\ \text{N}.$$

TABLE 2

The second and third mode shapes

X/L	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
$X_2(x/L)$	0	0·369	0·673	0·841	0·830	0·630	0·282	−0.146	−0·561	−0·871	−1·0
	(0)	(0·369)	(0·674)	(0·842)	(0·831)	(0·630)	(0·280)	(−0·146)	(−0·563)	(−0·872)	(−1·0)
	[0]	[0·370]	[0·674]	[0·841]	[0·832]	[0·632]	[0·281]	[−0·147]	[−0·562]	[−0·873]	[−1·0]
$X_3(x/L)$	0	0·575	0·832	0·602	0·001	−0·631	−0·912	−0·660	0·001	0·870	1·0
	(0)	(0·576)	(0·833)	(0·603)	(0·001)	(−0·632)	(−0·913)	(−0·663)	(0·001)	(0·869)	(1·0)
	[0]	[0·576]	[0·834]	[0·603]	[0·001]	[−0·632]	[−0·912]	[−0·663]	[0·001]	[0·872]	[1·0]

Note: (1) Values in brackets are calculated by FEM, (2) values in parentheses are determined by use of the model of a multi-step rod to be described in Numerical Example 2.

Obviously, the fundamental functions, $\overline{S_{i1}}(x)$ and $\overline{S_{i2}}(x)$, and their derivatives of the first order satisfy the normalization condition given in equation (20).

(3) *Determination of the natural frequencies and mode shape functions.* The mode shape function of the first step is give by equation (43). If we take $N_0/K_1(0)$ as 1, then

$$X_1(x) = \sin\left(\sqrt{\frac{\bar{m}_i}{K_i}}\omega x\right). \tag{64}$$

Using equations (42) and (63) yields the mode shape functions as follows:

$$X_2(x) = X_1(l_1)\cos\left(\sqrt{\frac{\bar{m}_2}{K_2}}\omega x\right) + \frac{K_1}{K_2}X_1'(l_1)\sin\left(\sqrt{\frac{\bar{m}_2}{K_2}}\omega x\right).$$

Since $X_1(l_1)$ and $X_1'(l_1)$ can be determined from equation (64), $X_2(x)$ is known. $X_3(x)$, $X_4(x)$ and $X_5(x)$ can also be determined by the same procedure.

The frequency equation is equation (44), i.e.,

$$K_5X_4(l_4)\sin\left(\sqrt{\frac{\bar{m}_5}{K_5}}\omega l_5\right) = K_4X_4'(l_4)\cos\left(\sqrt{\frac{\bar{m}_5}{K_5}}\omega l_5\right). \tag{65}$$

l_i can be found from Figure 7 as follows:

$$l_1 = 12{\cdot}9\ \text{m}, \quad l_2 = l_3 = l_4 = l_5 = 11{\cdot}6\ \text{m}.$$

Solving equation (65) yields a set of values, ω_j ($j = 1, 2, \ldots$). The minimum value of ω_j is found to be $\omega_1 = 26{\cdot}2602$ rad/s and $T_1 = 0{\cdot}2393$ s.

Substituting the value of ω_1 into $X_1(x)$, one obtains the fundamental mode shape which is presented in Table 1. Using the aforementioned procedure, the second and third longitudinal natural frequencies are found to be $\omega_2 = 70{\cdot}4093$ rad/s and $\omega_3 = 110{\cdot}3296$ rad/s, and the corresponding mode shapes are also determined and presented in Table 2.

It can be seen from the above results that the natural frequencies and mode shapes determined by using the model of continuously varying cross-section (Example 1) are almost the same as those determined by using the model of multi-step rod (Example 2) and the results calculated by FEM, suggesting that a multi-step rod can be treated as a one-step rod with continuously varying cross-section for free vibration analysis and vice versa.

7. NUMERICAL EXAMPLE 3

A 16-story building with a height of 49·0 m located in Wuhan, China, is considered here as a numerical example. The values of mass per unit length and longitudinal stiffness are calculated and shown in Figure 8.

Because the variation of the mass per unit length is relatively small (see Figure 8), it is reasonable to assume $\bar{m}(x)$ as a constant, i.e.,

$$\bar{m} = \tfrac{1}{4}(2{\cdot}91 + 3{\cdot}02 + 2{\cdot}90 + 2{\cdot}92) \times 10^5 = 2{\cdot}93 \times 10^5\ \text{kg/m}.$$

The expression for describing the distribution of longitudinal stiffness is selected as an exponential function

$$K(x) = K_0 e^{-\beta(x/H)}. \tag{66}$$

K_0 and β are found as

$$K_0 = 256 \times 10^{11}\ \mathrm{N}$$

$$\beta = \ln \frac{K(0)}{K(H)} = \ln \frac{2{\cdot}56}{2{\cdot}10} = 0{\cdot}198.$$

The distribution of stiffness given by equation (66) is also shown in Figure 8 (dotted line and the values in parentheses) for comparison purposes.

Based on the model of a one-step rod and that of a four-step rod as well as the FEM model (the non-uniform rod is divided into 40 uniform segments), the longitudinal natural frequencies of this building are determined and listed in Table 3. The field measured value of the fundamental natural frequency of this building is 29·92 rad/s [3]. It is evident that the computed value of the fundamental natural frequency shows good agreement with the measured one.

The calculated first three mode shapes of this building are listed in Table 4. The field measured values of the fundamental mode shape are also tabulated in Table 4 for comparison purposes. It can be seen from Table 4 that the values of the fundamental mode shape calculated by the present method are very close to the field measured ones.

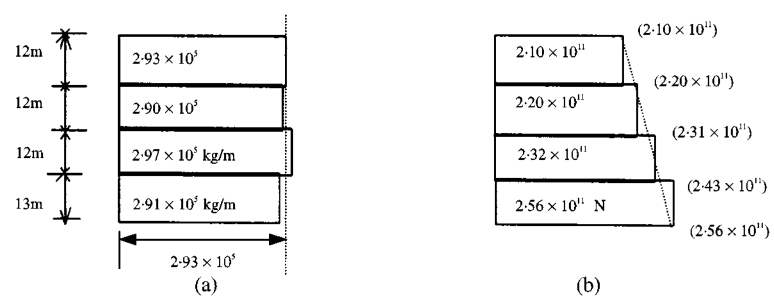

Figure 8. The distributions of mass and stiffness of a 16-story building: (a) mass; (b) stiffness.

Table 3

The longitudinal natural frequencies of the 16-*story building*

Natural frequencies (rad/s)	One-step rod model	Four-step model	FEM model
ω_1	29·7325	29·7307	29·7386
ω_2	77·7963	77·7921	77·7984
ω_3	124·4984	124·4761	124·4998

TABLE 4

The mode shapes of the 16-story building

x/L	0·0	0·125	0·25	0·375	0·5	0·625	0·75	0·875	1·0
$X_1(x/L)$	0	0·1910	0·3801	0·5496	0·7009	0·8297	0·9201	0·9762	1·0
	(0)	(0·1918)	(0.3807)	(0·5493)	(0·7006)	(0·8294)	(0·9200)	(0·9761)	(1·0)
	[0]	[0·1917]	[0·3805]	[0·5492]	[0·7007]	[0·8292]	[0·9202]	[0·9762]	[1·0]
	{0}	{0·192}	{0·382}	{0·550}	{0·706}	{0·827}	{0·918}	{0·975}	{1·0}
$X_2(x/L)$	0	0·4672	0·7964	0·8730	0·6404	0·1811	−0·3642	−0·8111	1·0
	(0)	(0·4671)	(0·7962)	(0·8731)	(0·6403)	(0·1810)	(−0·3640)	(−0·8110)	(1·0)
	[0]	[0·4670]	[0·7963]	[0·8733]	[0·6401]	[0·1814]	[−0·3645]	[−0·8110]	[1·0]
$X_3(x/L)$	0	0·6992	0·7964	0·1724	−0·6404	−0·9173	−0·3642	0·5420	1·0
	(0)	(0·6994)	(0·7963)	(0·1723)	(−0·6402)	(−0·9172)	(−0·3640)	(0·5421)	(1·0)
	[0]	[0·6995]	[0·7965]	[0·1720]	[−0·6400]	[−0·9170]	[−0·3640]	[0·5422]	[1·0]

Note: (1) Values in brackets are calculated by FEM, (2) values in parentheses are determined by use of the model of a multi-step rod and (3) the values in {} are the measured data.

8. CONCLUSIONS

In this paper, the expression for describing the distribution of mass is arbitrary, and the distribution of longitudinal stiffness is expressed as a functional relation with the mass distribution and *vice versa*. Using appropriate functional transformation, the governing differential equations for free vibrations of one-step rods are reduced to Bessel equations or ordinary differential equations with constant coefficients for several functional relations between stiffness and mass. Simple formulas to predict the longitudinal mode shapes of one-step rod with continuously varying cross-section are thus presented. The fundamental solutions that satisfy the normalization conditions are derived and used to establish the frequency equations for one-step rods with classical and non-classical boundary conditions. Using the fundamental solutions of each step rod and a recurrence formula developed in this paper, a new exact approach for determining the natural frequencies and mode shapes of multi-step non-uniform rods is proposed. The governing differential equation for the longitudinal vibration of non-uniform rods is similar to that of wave propagation through ducts with varying cross-section. Therefore, the exact solutions presented in this paper can be used to investigate such problems. Numerical examples demonstrate that the calculated longitudinal natural frequencies and mode shapes are in good agreement with the experimental data and the results determined by FEM, and the proposed procedure is an efficient and exact method.

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