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LETTERS TO THE EDITOR

AXISYMMETRIC VIBRATION OF INHOMOGENEOUS FREE CIRCULAR PLATES: AN UNUSUAL EXACT, CLOSED-FORM SOLUTION

I. ELISHAKOFF

Department of Mechanical Engineering, Florida Atlantic University, Boca Raton, Fl 33431-0991, *U.S.A*

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1. INTRODUCTION

In a companion study the free vibration of the circular plate, that was clamped at the edge, was investigated [1]. Harris [2] obtained closed-form solutions for the circular plate with free edge. His study appears to be very interesting. He considered a specific case of the variation of the thickness as a function of the radial co-ordinate r:

$$h(r) = h_0[1 - (r/R)^2], \tag{1}$$

leading to the bending stiffness

$$D(r) = D_0[1 - (r/R)^2]^3, \tag{2}$$

where R is the radius of the plate, since he observed that $D(r)$ satisfies the condition

$$D(R) = \mathrm{d}D(R)/\mathrm{d}r = 0. \tag{3}$$

The boundary conditions, demanding that the bending moment

$$M_r = -D(r)\left(\frac{\mathrm{d}^2 W}{\mathrm{d}r^2} + \frac{v}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\right) \tag{4}$$

and the shearing force

$$V_r = -D(r)\frac{\mathrm{d}}{\mathrm{d}r}\nabla^2 W - \frac{\mathrm{d}D(r)}{\mathrm{d}r}\left(\frac{\mathrm{d}^2 W}{\mathrm{d}r^2} + \frac{v}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\right) \tag{5}$$

are to vanish at the outer boundary, are identically satisfied. This led Harris [2] to the possibility to choose, *de facto* any expression of the mode shape. He chose the function

$$W(\rho) = \sum_{i=0}^{n} \rho^{2i}, \qquad \rho = r/R, \tag{6}$$

and obtained a closed-form expression for the natural frequency. The work by Harris [2] appears to the *only one* both for plates with constant thickness, as well as of variable thickness for which both the mode shape and the natural frequency are derived in the closed form. He not only obtained the expression for the mode shape, as did Lennox and Convey [3] but also for the eigenvalue. We pursue a somewhat different avenue with the attendant closed-form solution.

2. FORMULATION OF THE PROBLEM

Consider an inhomogeneous circular plate that is free at its ends. We pose the following inverse problem: find a distribution of the modulus of elasticity $E(\rho)$ and the material density $\rho(r)$, so that the free plate will possess the postulated mode shape

$$W(\rho) = 1 + \alpha\rho^2 + \beta\rho^4, \tag{7}$$

where α and β are parameters yet to be determined. If such a plate will be found, it is natural to visualize that its stiffness will be dependent upon the parameters α and β. Thus

$$D(\rho) = D(\rho, \alpha, \beta). \tag{8}$$

The free parameters α and β must be chosen in such a form that the conditions at $\rho = 1$,

$$D(\rho, \alpha, \beta) = 0, \qquad \frac{\mathrm{d}D(\rho, \alpha, \beta)}{\mathrm{d}\rho} = 0 \tag{9, 10}$$

are satisfied. This leads to two equations for α and β, and thus, to the attendant closed-form solution for the first non-zero natural frequency.

3. BASIC EQUATIONS

Governing differential equation, governing the free vibration of the inhomogeneous plate reads

$$D(r)r^3\Delta\Delta W + \frac{\mathrm{d}D}{\mathrm{d}r}\left(2r^3\frac{\mathrm{d}^3W}{\mathrm{d}r^3} + r^2(2+\nu)\frac{\mathrm{d}^2W}{\mathrm{d}r^2} - r\frac{\mathrm{d}W}{\mathrm{d}r}\right)$$
$$+ \frac{\mathrm{d}^2D}{\mathrm{d}r^2}\left(r^3\frac{\mathrm{d}^3W}{\mathrm{d}r^2} + \nu r^2\frac{\mathrm{d}W}{\mathrm{d}r}\right) - \rho h\omega^2 r^3 W = 0. \tag{11}$$

We represent the mass density and the stiffness by forms

$$\rho(r) = \sum_{i=0}^{m} c_i r^i, \qquad D(r) = \sum_{i=0}^{m+4} b_i r^i. \tag{12, 13}$$

Substitution of equation (12) and (13) into the governing differential equation leads to the result

$$\sum_{i=0}^{m+7} d_i r^i = 0, \tag{14}$$

where

$$d_0 = 0, \quad d_1 = 0, \quad d_2 = 2b_1\alpha(1+\nu), \quad d_3 = 64b_0\alpha + 8b_2\beta - a_0\omega^2,$$

$$d_4 = 132b_1\beta(1+\nu) + 18b_3\alpha(1+\nu), \quad d_5 = 32b_2\beta(7+\nu) + 32b_4\alpha(1+\nu) - a_0\alpha\omega^2, \quad (15)$$

$$d_6 = 20(17+3\nu)b_3\beta, \quad d_7 = 96(5+\nu)b_4\beta - a_0\beta\omega^2.$$

In order for equation (14) to be valid, all d_i's must vanish. We get six non-trivial equations for five coefficients b_0, b_1, b_2, b_3, b_4 and ω^2, i.e., a total of six unknowns. For non-triviality of the solution, the determinant should vanish. It turns out to be identically zero. Taking b_4 to be an undetermined coefficient, the solution is written as

$$b_0 = -[\alpha^2(1+\nu) - 6\beta(5+\nu)]b_4/4\beta^2, \quad b_1 = 0, \quad b_2 = 2\alpha b_4/\beta, \quad b_3 = 0, \quad (16)$$

with attendant natural frequency squared,

$$\omega^2 = 96(5+\nu)b_4/a_0. \quad (17)$$

Equation (16) yields the stiffness

$$D(\rho) = -\frac{[\alpha^2(1+\nu) - 6\beta(5+\nu)]b_4}{4\beta^2} + \frac{2\alpha b_4}{\beta}\rho^2 + b_4\rho^4. \quad (18)$$

We require that $D(1) = 0$, along with $D'(1) = 0$:

$$-\frac{\lfloor \alpha^2(1+\nu) - 6\beta(5+\nu) \rfloor b_4}{4\beta^2} + \frac{2\alpha b_4}{\beta} + b_4 = 0, \quad \frac{4\alpha b_4}{\beta} + 4b_4 = 0. \quad (19)$$

Solution of equations (19) and (20) yields

$$\alpha = -6, \quad \beta = 6. \quad (20)$$

Thus, the mode shape of the plate reads

$$W(\rho) = 1 - 6\rho^2 + 6\rho^4, \quad (21)$$

whereas the stiffness is

$$D(\rho) = b_4(1-\rho^2)^2. \quad (22)$$

4. CONCLUSION

This study presents a new closed-form solution for the natural frequency of an inhomogeneous circular plate that is free at its boundary. While Harris [2] dealt with the stiffness in the form $D(\rho) = D_0(1-\rho^2)^3$, here we uncover, by using inverse vibration analysis, a simpler possible expression for it, to yield a closed-form solution namely $D(\rho) = D_0(1-\rho^2)^2$.

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