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A FINITE ELEMENT FORMULATION FOR PLATES WITH WARPING

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1. INTRODUCTION

It has long been recognised that classical plate theory must be modified to include certain higher order effects like warping of the normal fibre. The first generation of the classical theories was given by Reissner [1], Bollé [2] and Mindlin [3]. Since then, there have been many further generalisations given by Naghdi [4], Essenberg [5], Nelson *et al.* [6], Reissner [7,8]. Perhaps the first general higher order theory resulting in 11 second order partial equations was given by Lo *et al.* [9,10]. In Lo's theory, the use of polynomial development leads to coupling in the matrices in the finite element formulation.

Hassis [11, 12] proposed, for plates and shells, a warping model based on developing the displacement using the normal modes of a geometrical beam associated with the normal fibre. Good estimation of the in-plane stresses, in-plane and out-of-plane displacements was observed. For the warping theory, the use of a development using the normal modes of the displacement leads to an uncoupled formulation; this is due to the orthogonality of the modes.

To complement the theoretical formulation of the warping theory [11, 12], a finite element formulation is presented here. The formulation is presented for homogeneous composite plates and for multilayered composite plates. It is shown that for homogeneous composite plates, the warping theory uses the same elementary sub-matrices which are used by the Mindlin theory.

2. FINITE ELEMENT FORMULATION

The warping theory developed in references [11,12] is summarized in Appendix A. The standard finite element technique is followed here. The total solution domain is discretised into NE sub-domains (elements) such that

$$U(d) = \sum_{n=1}^{NE} U^e(\mathbf{d}), \tag{1}$$

where U and U^e are the total internal strain energy of the system and the element respectively. The vector $\mathbf{d}$ of unknown displacement variable is defined by

$$[\mathbf{d}]^{\mathrm{T}} = [u_1\ u_2\ \beta_1\ \beta_2\ u_3\ W_1^n\ W_2^n]. \tag{2}$$

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The constitutive equations for the Lth layer can be written, in material co-ordinates (x, y), as

$$[\boldsymbol{\sigma}] = [\mathbf{c}][\boldsymbol{\varepsilon}] \quad \text{with } [\mathbf{c}] = \begin{bmatrix} c_{11} & c_{12} & c_{16} & 0 & 0 \\ c_{12} & c_{22} & c_{26} & 0 & 0 \\ c_{16} & c_{26} & c_{66} & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & c_{55} \end{bmatrix}, \tag{3}$$

The stress and strain vectors have five components and are ordered as follows:

$$\begin{aligned} [\boldsymbol{\sigma}]^{\mathrm{T}} &= [\sigma_{11} \;\; \sigma_{22} \;\; \sigma_{12} \;\; \sigma_{13} \;\; \sigma_{23}], \\ [\boldsymbol{\varepsilon}]^{\mathrm{T}} &= [\varepsilon_{11} \;\; \varepsilon_{22} \;\; \gamma_{12} \;\; \gamma_{13} \;\; \gamma_{23}]. \end{aligned} \tag{4}$$

The constitutive equations for the nth layer can be written, in used co-ordinates (x^1, x^2), as

$$[\bar{\boldsymbol{\sigma}}] = [\bar{\mathbf{c}}][\bar{\boldsymbol{\varepsilon}}] \tag{5}$$

with

$$[\bar{\mathbf{c}}] = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{12} & \bar{c}_{13} & 0 & 0 \\ \bar{c}_{12} & \bar{c}_{22} & \bar{c}_{23} & 0 & 0 \\ \bar{c}_{13} & \bar{c}_{23} & \bar{c}_{33} & 0 & 0 \\ 0 & 0 & 0 & \bar{c}_{44} & \bar{c}_{44} \\ 0 & 0 & 0 & \bar{c}_{44} & \bar{c}_{55} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{C}}_{\omega} & 0 \\ 0 & \bar{\mathbf{C}}_{\perp\omega} \end{bmatrix}.$$

$\bar{\mathbf{C}}_{\omega}$ are the in-plane components of the matrix $\bar{\mathbf{C}}$ and $\bar{\mathbf{C}}_{\perp\omega}$ are the interaction of the in- and the out-of-plane components of the matrix $\bar{\mathbf{C}}$. Using the stress resultants definition, the internal strain energy of an element due to extension, bending, warping and shear can be written as

$$\begin{aligned} U^e &= \frac{1}{2}\int_A [\boldsymbol{\varepsilon}^0 N + \boldsymbol{\kappa} M + \boldsymbol{\chi} P_n + \boldsymbol{\gamma}^0 T + \boldsymbol{\gamma}^n Q_n]\,\mathrm{d}A \\ &= \frac{1}{2}\int_A [\mathbf{d}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{Q}\mathbf{B}\mathbf{d}]\,\mathbf{d}A \end{aligned} \tag{6}$$

The matrices used in equation (6) are defined as follows:

$$[\mathbf{Q}] = \begin{bmatrix} A_e & A_{ef} & 0 & A^n_{ew} & 0 \\ A^{\mathrm{T}}_{ef} & A_f & 0 & A^n_{fw} & 0 \\ 0 & 0 & A_s & 0 & A^n_{sw} \\ A^{n\mathrm{T}}_{ew} & A^{n\mathrm{T}}_{fw} & 0 & A^n_w & 0 \\ 0 & 0 & A^{n\mathrm{T}}_{sw} & 0 & A^n_{sww} \end{bmatrix}, \qquad [\mathbf{B}] = \begin{bmatrix} L_E \\ L_B \\ L_s \\ L_W \\ L_{SW} \end{bmatrix}. \tag{7}$$

The generalised deformation parameters used in equation (6) are defined by

$$[\boldsymbol{\varepsilon}^0] = [\mathbf{L}_E]\mathbf{d}, \quad [\boldsymbol{\kappa}] = [\mathbf{L}_B]\mathbf{d}, \quad [\boldsymbol{\chi}] = [\mathbf{L}_W]\mathbf{d}, \quad [\boldsymbol{\gamma}^0] = [\mathbf{L}_S]\mathbf{d}, \quad [\boldsymbol{\gamma}^n] = [\mathbf{L}_{SW}]\mathbf{d}$$

with

$$[\mathbf{L}_E] = \begin{bmatrix} \frac{\partial}{\partial x^1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial x^2} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{L}_B] = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x^1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & 0 & 0 & 0 \end{bmatrix},$$

$$[\mathbf{L}_W] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x^1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x^2} \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} \end{bmatrix},$$

$$[\mathbf{L}_S] = \begin{bmatrix} 0 & 0 & 1 & 0 & \frac{\partial}{\partial x^1} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{\partial}{\partial x^2} & 0 & 0 \end{bmatrix}, \quad [\mathbf{L}_{SW}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[\mathbf{A}_e] = \int_{-h/2}^{h/2} [\bar{\mathbf{C}}_\omega]\,\mathrm{d}x^3, \qquad [\mathbf{A}_{ef}] = \int_{-h/2}^{h/2} x^3[\bar{\mathbf{C}}_\omega]\,\mathrm{d}x^3, \qquad [\mathbf{A}_f] = \int_{-h/2}^{h/2} (x^3)^2[\bar{\mathbf{C}}_\omega]\,\mathrm{d}x^3,$$

$$[\mathbf{A}^n_{ew}] = \int_{-h/2}^{h/2} \phi_n[\bar{\mathbf{C}}_\omega]\,\mathrm{d}x^3, \quad [\mathbf{A}^n_{fw}] = \int_{-h/2}^{h/2} \phi_n x^3[\bar{\mathbf{C}}_\omega]\,\mathrm{d}x^3, \quad [\mathbf{A}^n_w] = \int_{-h/2}^{h/2} (\phi_n)^2[\bar{\mathbf{C}}_\omega]\,\mathrm{d}x^3,$$

$$[\mathbf{A}_s] = \int_{-h/2}^{h/2} [\bar{\mathbf{C}}_{\perp\omega}]\,\mathrm{d}x^3, \quad [\mathbf{A}^n_{sw}] = \int_{-h/2}^{h/2} \phi_{n,3}[\bar{\mathbf{C}}_{\perp\omega}]\,\mathrm{d}x^3, \quad [\mathbf{A}^n_{ww}] = \int_{-h/2}^{h/2} (\phi_{n,3})^2[\bar{\mathbf{C}}_{\perp\omega}]\,\mathrm{d}x^3.$$

Using an interpolation function associated with nodes, equation (6) can be transformed into

$$U^e = \frac{1}{2}[\mathbf{a}^{\mathrm{T}}\mathbf{K}^e\mathbf{a}], \tag{8}$$

where $\mathbf{K}^e$ is the stiffness matrix for an element e ($\mathbf{K}^e$ includes extension, bending, warping and transverse shear effects) and $\mathbf{a}$ is the nodes displacement parameter.

For homogeneous plate, the matrix $\mathbf{Q}$ is reduced to

$$[\mathbf{Q}] = \begin{bmatrix} \mathbf{A}_e & 0 & 0 & 0 & 0 \\ 0 & \mathbf{A}_f & 0 & 0 & 0 \\ 0 & 0 & \mathbf{A}_s & 0 & \mathbf{A}^n_{sw} \\ 0 & 0 & 0 & \mathbf{A}^n_w & 0 \\ 0 & 0 & \mathbf{A}^{n\mathrm{T}}_{sw} & 0 & \mathbf{A}^n_{sww} \end{bmatrix}, \tag{9}$$

where $\mathbf{A}_e$ is relative to the extension effect which is associated with the displacement parameters (u_1, u_2), $\mathbf{A}_f$ is relative to the flexural effect which is associated with the displacement parameters (β_1, β_2), $\mathbf{A}_s$ is relative to the shear effect which is associated with the displacement parameter (β_1, β_2, u_3), A^n_{sw} is relative to the warping effect in shear which is associated with the displacement parameters (W^n_1, W^n_2), A^n_w is relative to the warping effect which is associated with the displacement parameters (W^n_1, W^n_2) and $\mathbf{A}^n_{sww}$ is relative to the shear effect in warping which is associated with the displacement parameters (W^n_1, W^n_2).

If the same interpolation is adopted for the rotations (β_1, β_2) and the warping parameters (W^n_1, W^n_2), the following relations are established:

$$[\mathbf{A}_w] = \frac{12\sum_n}{h^2}[\mathbf{A}_f], \qquad [\mathbf{A}_{sww}] = \Xi_n[\mathbf{A}_{s1}], \quad [\mathbf{A}_{sw}] = \theta_n[\mathbf{A}_{s1}]. \tag{10}$$

The coefficients used here are defined by [11, 12]

$$\sum_n = \frac{1}{h}\int_{-h/2}^{h/2} (\phi_n)^2\,\mathrm{d}x^3, \qquad \theta_n = \frac{1}{h}\int_{-h/2}^{h/2} (\phi_{n,3})\,\mathrm{d}x^3, \qquad \Xi_n = \frac{1}{h}\int_{-h/2}^{h/2} (\phi_{n,3})^2\,\mathrm{d}x^3. \tag{11}$$

Finally, the matrix $\mathbf{Q}$ becomes

$$[\mathbf{Q}] = \begin{bmatrix} \mathbf{A}_e & 0 & 0 & 0 & 0 \\ 0 & \mathbf{A}_f & 0 & 0 & 0 \\ 0 & 0 & \mathbf{A}_s & 0 & \theta_n[\mathbf{A}_s] \\ 0 & 0 & 0 & \frac{12\sum_n}{h^2}[\mathbf{A}_f] & 0 \\ 0 & 0 & \theta_n[\mathbf{A}_s]^{\mathrm{T}} & 0 & \Xi[\mathbf{A}_s] \end{bmatrix}, \tag{12}$$

The mass matrix for homogeneous composite plate takes the following form:

$$[M] = \begin{bmatrix} \mathbf{M}_c & & & \\ & \mathbf{M}_{fr} & & \\ & & \mathbf{M}_{fd} & \\ & & & \mathbf{M}_w \end{bmatrix} = \begin{bmatrix} \mathbf{M}_e & & & \\ & \mathbf{M}_{fr} & & \\ & & \mathbf{M}_{fd} & \\ & & & \frac{12\sum_n}{h^2}\mathbf{M}_{fr} \end{bmatrix}, \tag{13}$$

where $\mathbf{M}_e$ is the Mindlin's sub-mass matrix associated with the extension, $\mathbf{M}_{fr}$ is the Mindlin's sub-mass matrix associated with the rotational effect, M_{fd} is the Mindlin's sub-mass matrix associated with the deflection.

3. CONCLUSION

Finite element formulation is presented here for a Mindlin-warping model for plates. The stiffness and mass matrices are developed for homogeneous and multilayered composite plates. It is shown that for homogeneous plates, the stiffness matrix associated to the warping theory used the sub-matrices (multiplied by a coefficient) of the Mindlin theory. For numerical implementation, it is commercially attractive due to the case of use in software development and implementation in an existing general purpose program which is usually based on Mindlin theory.

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APPENDIX A

A.1. DISPLACEMENT FIELD

By neglecting the $\sigma 33$ effect, the warping theory uses the assumption of the displacement field in the following form:

$$\bar{U} = \begin{cases} U_1(x^1, x^2, x^3) = u_1(x^1, x^2) + x^3\beta_1(x^1, x^2) + \phi_n(x^3)W_1^n(x^1, x^2), \\ U_2(x^1, x^2, x^3) = u_2(x^1, x^2) + x^3\beta_2(x^1, x^2) + \phi_n(x^3)W_2^n(x^1, x^2), \\ U_3(x^1, x^2, x^3) = u_1(x^1, x^2), \end{cases} \tag{A1}$$

where (x^1, x^2) are the middle plane co-ordinates and x^3 is the normal co-ordinate to the middle plane (u_1, u_2, u_3) are the displacement components of the middle plane, (U_1, U_2, U_3) are the displacement components of a point of the plate. $\{\phi_n\}$ denotes the nth transverse mode inducing deformations of the normal fibre which is considered as a geometrical beam.

A.2. EQUATION OF MOTION

For a plate, the governing equilibrium equations are given by

$$\begin{aligned} N^{\alpha\lambda}_{,\lambda} + f^\alpha &= \rho h\ddot{u}^\alpha, \qquad & M^{\alpha\lambda}_{,\lambda} - T^\alpha + m^\alpha &= \rho\frac{h^3}{12}\ddot{\beta}^\alpha, \\ T^\alpha_{,\alpha} + f^3 &= \rho h\ddot{u}_3, \qquad & P^{\alpha\lambda}_{n,\lambda} - Q^\alpha_{,n} + f^\alpha_n &= \rho h\Sigma_n\ddot{W}^\alpha_n, \end{aligned} \tag{A2}$$

where ρ is the mass density, h is the thickness of the plate, f^α is the αth component of the inplane force vector, m^α is the αth component of the in-plane moment vector and f^α_n is the αth component of the projection of the in-plane force vector on the *n*th transverse normal mode.

The boundary conditions are given by

$$-N^{a\lambda}v_\lambda + f_s^\alpha = 0, \qquad -M^{\alpha\lambda}v_\lambda + m_s^\alpha = 0, -T^\alpha v_\alpha + f_s^3 = 0, \qquad -P_n^{\alpha\lambda}v_\alpha + (F_n^\alpha)_s = 0, \tag{A3}$$

The stress resultants using in equilibrium equations are defined as

$$\begin{bmatrix} N^{11} & N^{12} & N^{22} \\ M^{11} & M^{12} & M^{22} \\ P_n^{11} & P_n^{12} & P_n^{22} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} 1 \\ x^3 \\ \phi_n \end{bmatrix} [\sigma^{11} \ \ \sigma^{12} \ \ \sigma^{22}]\, dx^3,$$

$$\begin{bmatrix} T^1 & T^2 \\ Q_n^1 & Q_n^2 \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} 1 \\ \phi_{n,3} \end{bmatrix} [\sigma^{13} \ \ \sigma^{23}]\, dx^3. \tag{A.4}$$