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DETECTION OF DAMAGE IN BEAMS SUBJECTED TO DIFFUSED CRACKING

M. N. CERRI

Istituto di Scienza e Tecnica delle Costruzioni, Università di Ancona, Italy

AND

F. VESTRONI

Dipartimento di Ingegneria Strutturale e Geotecnica, Università di Roma "La Sapienza", Italy.
E-mail: *vestroni@scilla.ing.uniromal.it*

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This paper addresses the problem of identifying structural damage affecting one zone of a beam using measured frequencies. The beam model has a zone in which the stiffness is lower than the undamaged value. Damage is defined by three parameters: position, extension and degree, which need to be identified in the inverse problem. The solution of the direct problem is first obtained and the peculiarities of damage detection are examined. Two different procedures for damage identification are proposed, which use frequency measurements and take advantage of the peculiarities of the problem: the first procedure is based on the characteristic equation error and the second on the comparison between analytical and experimental frequency values. The identifiability and ill-conditioning properties are discussed by referring to cases with pseudo-experimental data.

1. INTRODUCTION

Techniques of structural identification have received considerable attention in recent decades, for different reasons. One of the most interesting applications involves the monitoring of structural integrity through the identification of damage [1, 2]. It is well known that damage modifies the dynamic response of a structure and, at the same time, that changes in its behavior may be associated with the decay of the system's mechanical properties [3–7]. Based on these considerations, various papers have examined the use of measured variations in dynamic behavior to detect structural damage. Particular attention has been focused on the use of frequencies only, on account of the simplicity of measuring them and, therefore, their experimental reliability [8–19].

As with most inverse problems, ill-conditioning complicates the search for a solution, since it strongly depends on the quantity and quality of experimental data [17, 19–21]. Another difficult aspect is the modelling of damage, even when the attention is limited to beam structures. Linear behavior is assumed here, both before and after damage; damage is therefore thought of as an open crack or decay of the mechanical properties of a small beam element and is represented by a more or less localized decrease in stiffness.

Damage detection has generally been viewed as a reconstruction problem, where the distribution of the stiffness parameter along the structure is completely unknown and the

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solution calls for a quantity of data that is seldom available. In previous papers [16, 17], starting from the classical problem of damage identification in a vibrating beam due to single cracks, it was pointed out that very few data are required for the evaluation of damage quantities. The same consideration holds good even when a discretized model is used [16], although this has not been taken into account. The peculiarity of damage detection is precisely the circumstance that only a few parameters need to be determined, since the damaged sections are very few, albeit unknown. The desired solution is such that the stiffness is known throughout to be equal to the undamaged value except in the few damaged zones. Most of these points have already appeared in different papers, albeit less than in some recent publications [16, 17, 22–24].

When the damage is concentrated, a rotational spring can accurately model the dynamic behavior of a damaged beam [5, 9, 12]. With this model damage is described by two parameters, location and degree, and it has been shown by means of experimental and pseudo-experimental data that in the inverse problem the two parameters can be suitably determined using not more than three frequencies [16, 17]. In several scenarios damage may be spread over a particular zone, albeit a small one; this occurs when several cracks are close to each other or when one part of an element is affected by a stiffness reduction due to cracking [18, 25], a problem that has not received particular attention until now.

The present paper addresses precisely this last problem of diffused damage in a vibrating beam. Three quantities are needed to represent diffused damage: location, extension and magnitude. First, the direct problem is solved and discussed, along with the characteristics of identifiability: it is shown that when the extension is limited three damage parameters cannot be identified, since the problem reverts to the previous case of concentrated damage described by only two parameters. Two different identification techniques are used, which take into account the peculiarity of the above problem: in the first the optimal parameter estimate is based on the characteristic equation, in the second on the error between the analytical and measured frequencies. The reliability of the results of the identification procedure depends more on the accuracy of the frequency data, than on their number, when it is sufficient, but also on the characteristics of the damage.

2. DIRECT PROBLEM

Before tackling the inverse problem, it is useful to analyze in depth the direct problem, i.e., the evaluation of the natural frequencies of a beam with a damaged zone. As a first step, a model able to represent such damage by means of a suitable number of parameters has to be introduced. The model of a beam with a segment damaged by diffused cracking, with the axis at distance X from the beam's middle span and with the length of the segment L^D can be represented by a beam with a zone of lower stiffness EI^D than the initial one EI^U (Figure 1). The parameters adopted to define the damage are the central axis position x and the length b of the damaged zone, and the damage degree coefficients; they are defined in adimensional form by

$$x = \frac{X}{L/2}, \qquad b = \frac{L^D}{L}, \qquad \beta = \frac{EI^U - EI^D}{EI^U}. \tag{1}$$

The governing equation of the free flexural vibrations of a uniform Euler–Bernoulli beam of mass per unit length (ρA) is given by

$$\rho A \frac{\partial^2 v(x,t)}{\partial t^2} + EI \frac{\partial^4 v(x,t)}{\partial x^4} = 0, \tag{2}$$

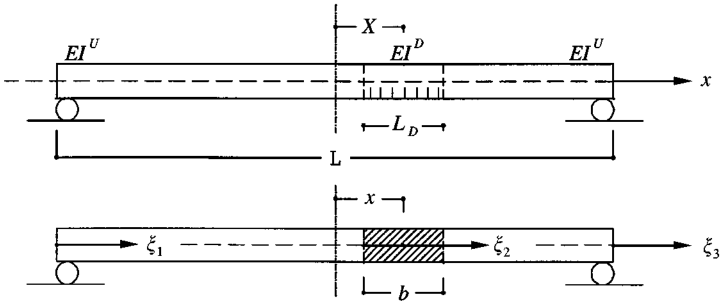

Figure 1. Model of beam with a zone damaged by diffuse cracking.

which admits as a solution $v(x, t) = V(x)\cos(\omega t - \varphi)$ which, substituted into equation (2) leads to

$$EI\,\frac{\mathrm{d}^4 V}{\mathrm{d}x^4} - \omega^2 \rho A V = 0, \tag{3}$$

where ω is the frequency of the motion.

The three beam segments can be treated separately; the equation for each segment is as follows:

$$\frac{\mathrm{d}^4 V_i}{\mathrm{d}\xi_i^4} - \lambda_i^4 V_i = 0, \quad i = 1, 2, 3, \tag{4}$$

where $\lambda_i^4 = \omega^2(\rho A/(EI)_i)L^4$ and $\xi_i = x_i/L$.

The solution of equation (4) for the three zones is written in the form

$$V_i(\xi_i) = A_{i1}\sinh\lambda_i\xi_i + A_{i2}\cosh\lambda_i\xi_i + A_{i3}\sin\lambda_i\xi_i + A_{i4}\cos\lambda_i\xi_i, \tag{5}$$

where A_{ij} are arbitrary constants.

The boundary conditions at the ends for a simply supported beam are

$$V_i(0) = 0 \qquad V_i''(0) = 0 \quad i = 1, 3, \tag{6}$$

while the compatibility conditions of the displacement, slope, moment and shear force at the boundary sections of two zones are

Section 1

$$V_1(\bar{\xi}_1) - V_2(0) = 0, \quad V_1''(\bar{\xi}_1) - (1-\beta)\cdot V_2''(0) = 0,$$

$$V_1'''(\bar{\xi}_1) - (1-\beta)V_2'''(0) = 0, \quad V_1'(\bar{\xi}_1) - V_2'(0) = 0,$$

Section 2

$$V_2'(\bar{\xi}_2) - V_3'(\bar{\xi}_3) = 0, \quad V_2'''(\bar{\xi}_2) - (1-\beta)\cdot V_3'''(\bar{\xi}_3) = 0,$$

$$V_2''(\bar{\xi}_2) - (1-\beta)\cdot V_3''(\bar{\xi}_3) = 0, \quad V_2(\bar{\xi}_2) - V_3(\bar{\xi}_3) = 0, \tag{7}$$

where the co-ordinates $\bar{\xi}_r$ can be suitably expressed in terms of the quantities x and b previously introduced: $\bar{\xi}_1 = L_1/L = 1/2(1 + x - b)$, $\bar{\xi}_2 = L_2/L = b$, $\bar{\xi}_3 = -L_3/L = -1/2(1 - x - b)$.

From conditions (4) and (5) and after some manipulations the characteristic equation for the problem is obtained

$$\det\begin{bmatrix} -\dfrac{1+\gamma^2}{2\gamma}\cos\lambda\bar{\xi}_1 & -\dfrac{1-\gamma^2}{2\gamma}\cosh\lambda\bar{\xi}_1 & \dfrac{1+\gamma^2}{2\gamma}\cos\lambda\gamma\bar{\xi}_2\cos\lambda\bar{\xi}_3 + \dfrac{1+\gamma^2}{2}\sin\lambda\gamma\bar{\xi}_2\sin\lambda\bar{\xi}_3 & \dfrac{1-\gamma^2}{2\gamma}\cos\lambda\gamma\bar{\xi}_2\cosh\lambda\bar{\xi}_3 + \dfrac{1-\gamma^2}{2}\sin\lambda\gamma\bar{\xi}_2\sinh\lambda\bar{\xi}_3 \\ -\dfrac{1+\gamma^2}{2}\sin\lambda\bar{\xi}_1 & -\dfrac{1-\gamma^2}{2}\sinh\lambda\bar{\xi}_1 & -\dfrac{1+\gamma^2}{2\gamma}\sin\lambda\gamma\bar{\xi}_2\cos\lambda\bar{\xi}_3 + \dfrac{1+\gamma^2}{2}\cos\lambda\gamma\bar{\xi}_2\sin\lambda\bar{\xi}_3 & -\dfrac{1-\gamma^2}{2\gamma}\sin\lambda\gamma\bar{\xi}_2\cosh\lambda\bar{\xi}_3 + \dfrac{1-\gamma^2}{2}\cos\lambda\gamma\bar{\xi}_2\sinh\lambda\bar{\xi}_3 \\ -\dfrac{1-\gamma^2}{2}\cos\lambda\bar{\xi}_1 & -\dfrac{1-\gamma^2}{2\gamma}\cosh\lambda\bar{\xi}_1 & \dfrac{1-\gamma^2}{2\gamma}\cosh\lambda\gamma\bar{\xi}_2\cos\lambda\bar{\xi}_3 - \dfrac{1-\gamma^2}{2}\sinh\lambda\gamma\bar{\xi}_2\sin\lambda\bar{\xi}_3 & \dfrac{1+\gamma^2}{2\gamma}\cosh\lambda\gamma\bar{\xi}_2\cosh\lambda\bar{\xi}_3 - \dfrac{1+\gamma^2}{2}\sinh\lambda\gamma\bar{\xi}_2\sinh\lambda\bar{\xi}_3 \\ -\dfrac{1-\gamma^2}{2}\sin\lambda\bar{\xi}_1 & -\dfrac{1+\gamma^2}{2}\sinh\lambda\bar{\xi}_1 & -\dfrac{1-\gamma^2}{2\gamma}\sinh\lambda\gamma\bar{\xi}_2\cos\lambda\bar{\xi}_3 + \dfrac{1-\gamma^2}{2}\cosh\lambda\gamma\bar{\xi}_2\sin\lambda\bar{\xi}_3 & -\dfrac{1+\gamma^2}{2\gamma}\sinh\lambda\gamma\bar{\xi}_2\cosh\lambda\bar{\xi}_3 + \dfrac{1+\gamma^2}{2}\cosh\lambda\gamma\bar{\xi}_2\sinh\lambda\bar{\xi}_3 \end{bmatrix} = 0, \tag{8}$$

where $\lambda^4 = \omega^2(\rho A/(EI)^U)L^4$ and $\gamma = 1/(1-\beta)^{1/4}$.

Equation (8) represents a relationship among the damage parameters x, b, β and the eigenvalue λ, and so the frequency of the harmonic motion, can be written in a compact form as

$$g(\lambda, x, b, \beta) = 0. \tag{9}$$

The characteristic equation (9) makes it possible to determine the vibration frequencies λ_r^D for a beam with a damaged zone defined by the parameters x, b, β. For a beam with a damaged zone extending over a length b equal to 0·2, equation (9) is used to evaluate the first three frequencies for different degrees and locations of damage. Parameters x and b are geometrically restrained to be $x \leqslant 1 - b$, as the damaged zone remains inside the beam. In the case considered ($b = 0{\cdot}2$) the x position of the axis therefore ranges within the interval [0, 0·8].

Figure 2 shows, in particular, the development of the ratio $\Delta\omega_r/\omega_r^U$ for the first three modes of vibration ($r = 1, 2, 3$), as defined by

$$\frac{\Delta\omega_r}{\omega_r^U} = \left[1 - \left(\frac{\lambda_r^D}{\lambda_r^U}\right)^2\right]. \tag{10}$$

As would be expected, frequencies decrease regularly with the degree of damage and are lower when the damaged segment is close to an inflection point of the eigenfunction, where the flexural curvature is zero, and on the contrary is greater where the curvature is greatest. The trend of the curves is very similar to those already found for localized damage [17]; however, even when the damage here is close to points with zero curvature (the boundary, the midspan and one-third of the length for the first, second and third mode, respectively), a small decrease in frequency is found, due to the extension of the damaged zone.

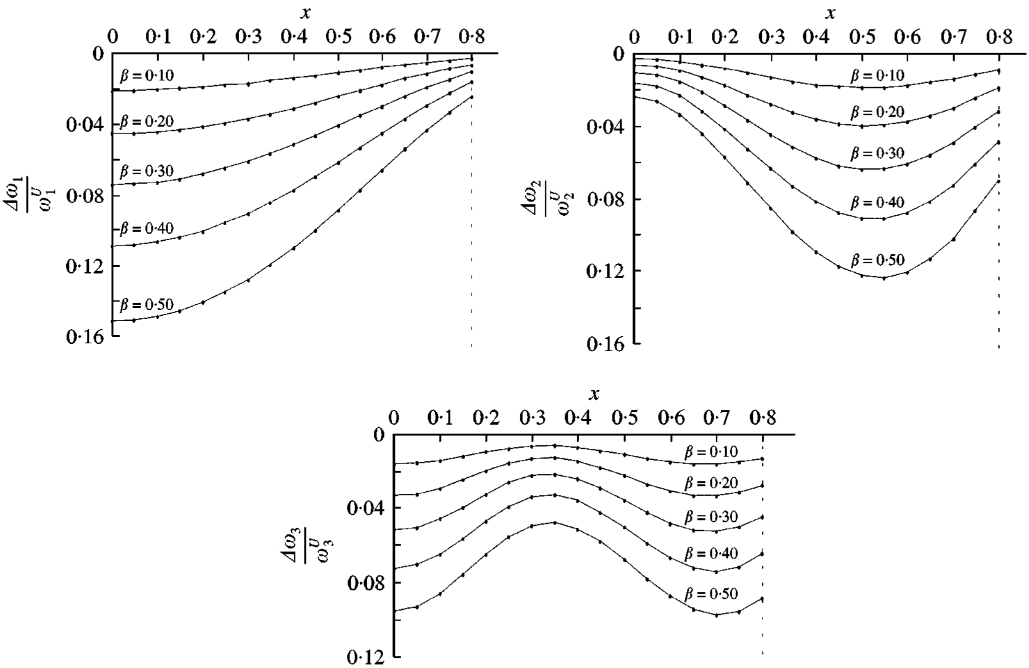

Figure 2. Relative variations of the first three frequencies versus the positions of damage with assigned length (b = 0·20) for different β intensities.

When the extension is limited in comparison with the length of the beam, it is possible to simplify the model by introducing concentrated damage with an equivalent deformability. In this case a torsional spring with stiffness K represents both the extension and the entity of the damage. In order to determine the spring parameter K the following procedure can be used: the relative rotation φ^D between the sections delimiting the damaged zone is expressed as

$$\varphi^D = \varphi^U + \Delta\varphi, \tag{11}$$

where φ^U represents the relative rotation in undamaged conditions and $\Delta\varphi$ represents the increase in rotation due to damage. Rotations φ^D and φ^U are expressed by the ratios

$$\varphi^D = \frac{M}{EI^D} L_D, \quad \varphi^U = \frac{M}{EI^U} L_D. \tag{12}$$

By means of equation (11) the increase in the rotation $\Delta\varphi$ can be expressed in terms of the damage parameters

$$\Delta\varphi = M \frac{L}{EI^U} b \frac{\beta}{(1-\beta)}. \tag{13}$$

In the case of localized damage $\Delta\varphi = M/K$, the non-dimensional stiffness $k = K/(EI^U/L)$ of an equivalent torsional spring can therefore be written as a function of b and β

$$k = \frac{(1-\beta)}{b\beta}. \tag{14}$$

Figure 3 shows the curves of the function $k = k(\beta, b)$ for some assigned values of parameter b. In this case, the damage is defined by two parameters only, the position x and the intensity k; however, different combinations of limited extension and intensity correspond to a given k value and in practice furnish the same variation of frequencies.

The characteristic equation for a simply supported beam with torsional spring k is expressed as

$$4k \sin\lambda \sinh\lambda + \lambda[\sin\lambda(\cosh\lambda - \cosh\lambda x) + \sinh\lambda(\cos\lambda - \cos\lambda x)] = 0, \qquad (15)$$

which can be written in a compact form as

$$k g_1(\lambda) - g_2(\lambda, x) = 0. \qquad (16)$$

In this particular problem, equation (16) can be solved for k in an explicit form as a function of position x

$$k = \frac{g_2(\lambda, x)}{g_1(\lambda)}. \qquad (17)$$

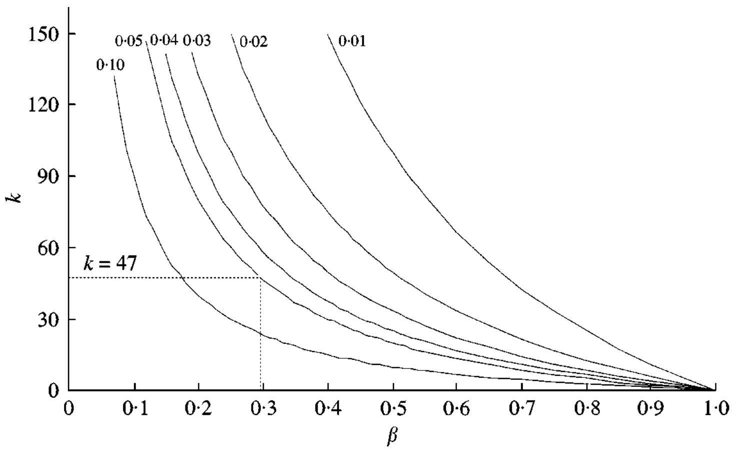

Figure 3. Curve $k = k(\beta, b)$ for values of parameter b ranging in 0·01–0·10.

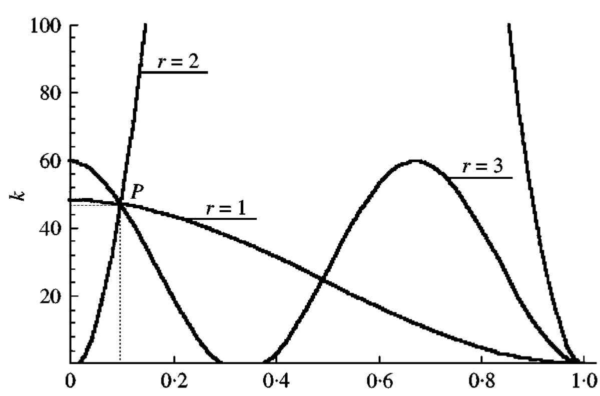

Figure 4. Curves k_r as a function of parameter x for assigned λ_r.

This equation makes it possible to analyze the uniqueness of the problem of evaluating the damage parameters k and x. For an assigned value λ_r the ratio $g_2(x, \lambda_r)/g_1(\lambda_r)$ exclusively depends on the parameter x and determines a curve in the plane of parameters x and k, the locus of points which satisfy the characteristic equation for the rth vibration mode. Two values of λ_r are sufficient to determine x and k when there is only one point, P, at which the curves $k_r(x)$ associated with the two different λ_r meet as already observed [8, 13].

Figure 4 shows the functions $k_r(x)$, $r = 1, 2, 3$, related to assigned values of λ_r due to damage defined by the parameter values $k = 47$, $x = 0{\cdot}10$. The knowledge of the first and second frequencies furnishes two curves which cross only in the right solution. But a pair of frequencies does not always define one solution, as occurs for the curves $k_1(x)$ and $k_3(x)$. In any case, even when two frequencies determine two solutions, the addition of a further frequency defines the solution exactly.

3. INVERSE PROBLEM

The identification of damage parameters on the basis of known experimental frequencies belongs to the class of inverse problems. The damage identification procedure generally comprises two parts:

- determination of the beam model in the undamaged condition
- determination of the damage parameters when the beam is damaged.

The attention is here focused on the second phase, on the assumption that the representative model for the undamaged beam has already been determined.

From the study of the direct problem, it is known that for a given set of measured frequencies, different conditions are satisfied by the true solution. Two of these conditions are considered in the following:

$$g(\lambda_r, x, b, \beta) = 0 \quad \forall r \tag{18}$$

according to equation (9), and the classical comparison between experimental ω_e and analytical frequencies expressed as functions of damage parameters x, b and β:

$$\omega_{e,r} - \omega_r(x, b, \beta) = 0 \quad \forall r. \tag{19}$$

Condition (18) can be specialized in the case of concentrated damage due to the explicit expression of $k_r(x)$:

$$k_r(x) = g_2(\lambda_r, x)/g_1(\lambda_r). \tag{20}$$

The first procedure for damage identification is based on the eigenvalue equation and will be known as the *modal equation procedure*; the second is based on the frequency comparison and known as *the response quantities procedure*.

3.1. PROCEDURE BASED ON THE MODAL EQUATION

In this procedure the inverse problem for the determination of damage parameters is based on the *characteristic equation* that governs free flexural oscillations of the damaged beam and is generally an implicit function in the eigenvalue λ and in the damage parameters. In the case of a simply supported beam with damage extended over a zone with finite length, the *characteristic equation* (9) is really an implicit function of the three parameters and is different from the case of concentrated damage governed by the two-parameter equation (16). Evaluation of the three damage parameters requires at least a system of three equations

$$g_r(x, b, \beta) = 0 \quad \text{with } r = p, s, t \tag{21}$$

associated with three experimental eigenvalues λ_r^D. Since the system is non-linear, it can exhibit more than one solution and another frequency should be used in addition to ensure a unique solution.

In the three-dimensional parameter space any one of the g_r functions represents a surface whose points satisfy the corresponding λ_r^D eigenvalue equation. The intersection curve C_{ps} of the two surfaces g_p and g_s related to two different eigenvalues p and s, represents the locus of the parameters that at the same time satisfy the experimental eigenvalues λ_p^D and λ_s^D. Repeating this procedure for another couple of eigenvalues, for example pth and tth, another curve C_{pt} is obtained and, finally, the solution point of the problem is given as the intersection of the two curves in the space of the damage parameters.

In order to analyze the characteristics of the problem and the procedure, the inverse problem of damage detection is solved for different cases, using pseudo-experimental data, i.e., the frequencies obtained from the direct problem in undamaged conditions and in some damaged conditions defined by three damage parameters. The beam for the case study has the following geometrical and mechanical characteristics: length $L = 1000$ mm, area $A = 15\ \text{mm}^2$, moment of inertia $I = 31{\cdot}25\ \text{mm}^4$, modulus of elasticity $E = 2{\cdot}1 \times 10^5\ \text{N/mm}^2$ and density $\rho = 0{\cdot}784 \times 10^{-8}\ \text{N s}^2/\text{mm}^4$.

Three different cases of damage among several studied are presented: they are illustrative of the main features of the inverse problem. The first case is characterized by a limited extension of the damaged zone, and the other two by larger damaged zones in two different positions. The frequency values for the undamaged beam and their variations associated with the three cases of damage are shown in Table 1 and used in the identification procedure. From the frequency values in the undamaged conditions ω_r^U and in the damaged conditions $\omega_r^D = \omega_r^U - \Delta\omega_r$, it is possible to obtain the pseudo-experimental damaged eigenvalues $\lambda_r^D = \lambda_r^U(\omega_r^D/\omega_r^U)^{1/2}$.

For the first case of damage, specified by parameters $x = 0{\cdot}375$, $b = 0{\cdot}05$, and $\beta = 0{\cdot}30$, Figures 5(a) and 5(b), respectively, represent the curves locus of the points that satisfy the characteristic equations written for the first and second frequencies (C_{12}), and for the first and third frequencies (C_{13}), respectively, projected on the plane x–β. By superposing these curves, as in Figure 6, the intersection in the space of damage parameters gives the solution to the inverse problem. Figures 5(a) and 5(b) show that each curve is defined in a small range of the parameter x within the interval [0, 1]. In particular, the curve C_{12} that satisfies the first and second frequencies is defined within an interval between the value $x_{lim} = 0{\cdot}36$—associated with the position of the limit case of concentrated damage that satisfy the assigned first and second frequencies—and a value of the damage position associated with the maximum damage extension compatible with the condition $x \leqslant 1 - b$. The curve C_{13} related to the first and third frequencies is defined within the interval between the two values $x_{lim} = 0{\cdot}28$ and $0{\cdot}374$, both associated with the two positions of the concentrated damage that satisfy the characteristic equation for the first and third frequencies. The presence of two vertical asymptotes, corresponding to small b values, should not surprise since for localized damage there are two different positions of a torsional spring k that satisfy the assigned first and third frequencies, which are not able to localize damage uniquely.

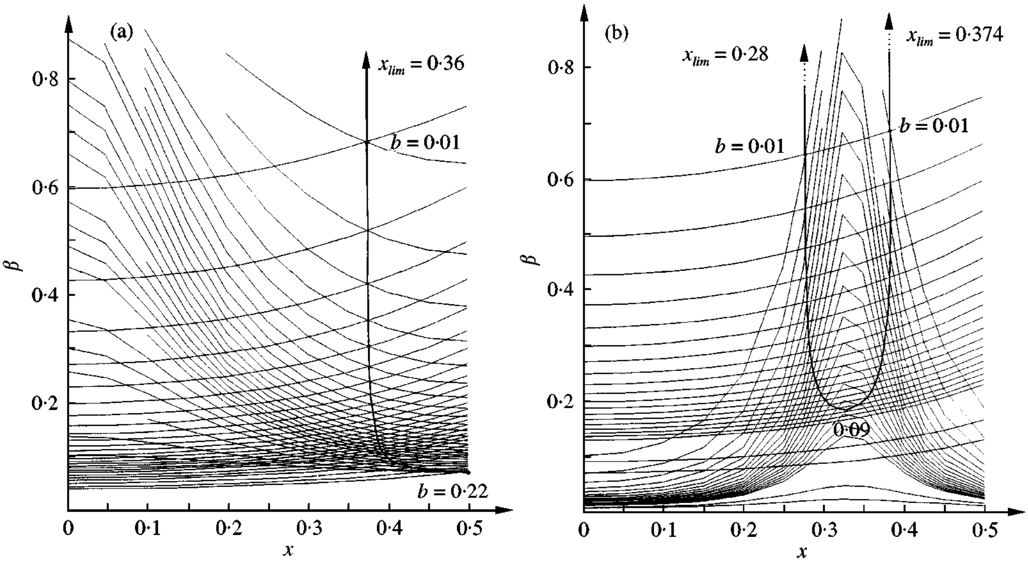

Figure 5. Loci of the three parameter values for which (a) $g_1 = 0$ and $g_2 = 0$; (b) $g_1 = 0$ and $g_3 = 0$.

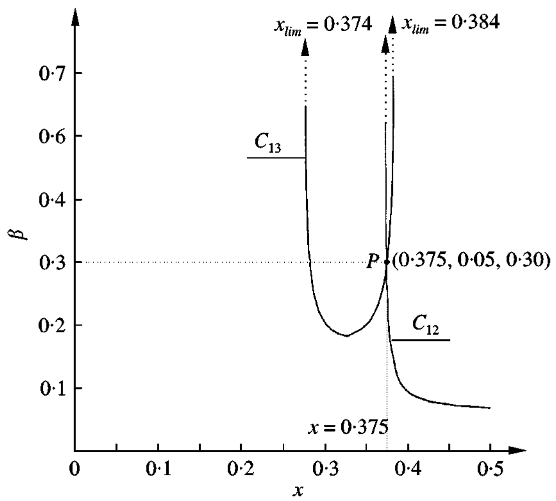

Figure 6. Projection in the x–β plane of the point P solution of the inverse problem.

Both curves show patterns in which high values of β are associated with small values of b and the limit location coincides with the case of concentrated damage. On the contrary, small values of β and large b values are related to locations around the zone of vanishing curvature in the modal shape where the damage effect is small.

It is possible to observe that, in the case of limited damage extension and in the absence of errors, the position of damage x is determined exactly, as are the other two parameters b and β. However, the inverse problem of evaluating parameters b and β is badly conditioned, because the cross is almost singular. Therefore, when damage extends over a small segment, it is advisable to abandon the two distinct damage parameters b and β and instead to tackle the problem of determining the equivalent parameter k—damage entity—which results in a well-conditioned problem, with reference to the model of a beam with concentrated damage. In this case expected value of k corresponding to the couple b and β is 47, as indicated in Figure 3.

The three curves $k_r(x)$, $r = 1, 2, 3$, in Figure 7 do not intersect exactly at one point; although the problem under consideration is pseudo-experimental, there is no exact solution, since the interpretative model or interpretation model with concentrated damage is different from that generating pseudo-experimental data. It is nevertheless possible to obtain an optimum estimate of the two parameters that minimize the difference among the three curves; the estimated values practically coincide with the expected values ($x = 0{\cdot}375$ and $k = 47$). In the inverse problem, of course, it is not possible to obtain b and β from the value of k, but all the couples determined by the horizontal line indicated in Figure 3 in the range of small b satisfy the characteristic equation for the assigned three eigenvalues with good approximation. As clearly outlined by the direct problem, a greater amount of data could not furnish better results concerning b and β, which for small extension damage are scarcely observable.

In the case of damage extending over a length $b = 0{\cdot}20$, it is possible to use the procedure already described, thus obtaining in the x–β plane the curve locus of the points that simultaneously satisfy the characteristic equations of the first and second modes and of the first and third modes (Figures 8(a) and 8(b)). The intersection of the curves in the damage parameter space makes it possible to determine the unique solution, defined by the expected parameter values: $x = 0{\cdot}375$, $b = 0{\cdot}20$, and $\beta = 0{\cdot}30$ (Figure 9). The curves are similar to those in the previous damage case, but in this case, where the damage is more extended, the intersection between the two curves is better defined and the three parameters can be determined exactly, again in the absence of model or experimental errors.

The third example considered is the case of a beam with extended damage, $b = 0{\cdot}20$, with its axis in a different position close to the middle of the beam $x = 0{\cdot}10$. Figures 10(a) and 10(b) show the curve locus of the points satisfying the characteristic equations of the first and second frequencies, and of the first and third frequencies, projected on the x–β plane and Figures 10(c) and 10(d) show the same curves on the x–b plane. An almost perfect superposition of part of the two curves can be appreciated from Figures 11(a) and 11(b). It is useful to analyze the intersection of the curves in the two planes, x–β and x–b; indeed, the superposition on both sections indicates that in the presence of an even small error, it becomes practically impossible to obtain one single solution to the inverse problem with the amount of data considered, but there is a surface in the damage parameter space where each point $P(x, b, \beta)$ almost simultaneously satisfies all three given frequencies. For the damage

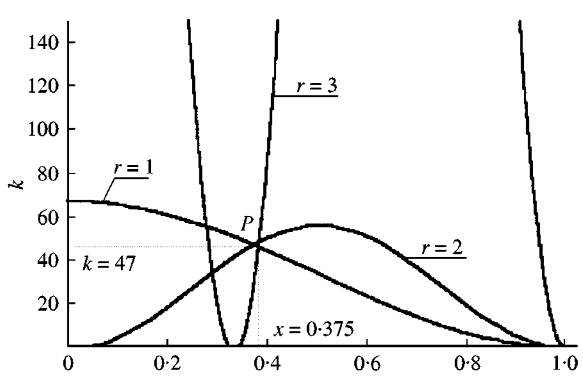

Figure 7. Curves k_r, $r = 1, 2, 3$, for small extension damage.

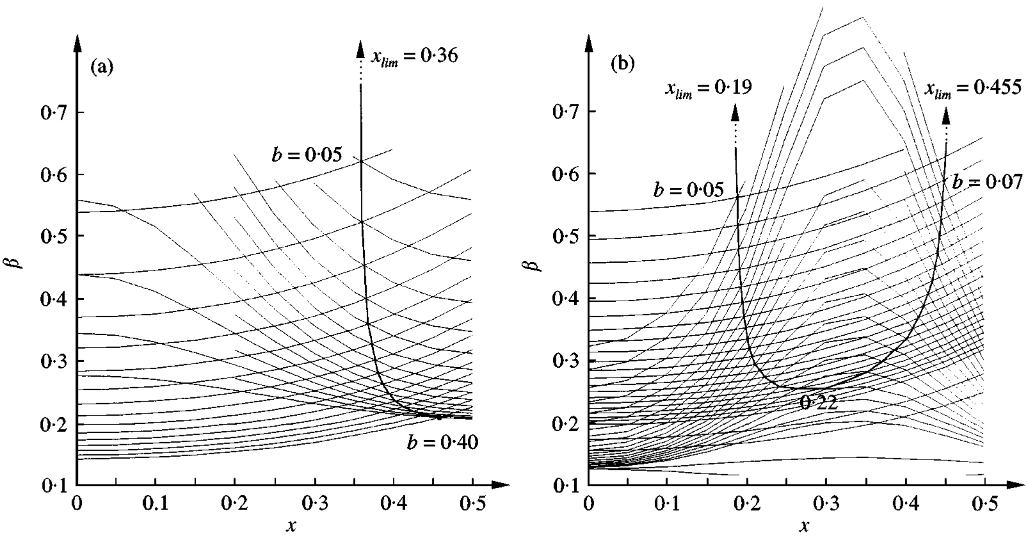

Figure 8. Loci of the three parameter values for which (a) $g_1 = 0$ and $g_2 = 0$; (b) $g_1 = 0$ and $g_3 = 0$.

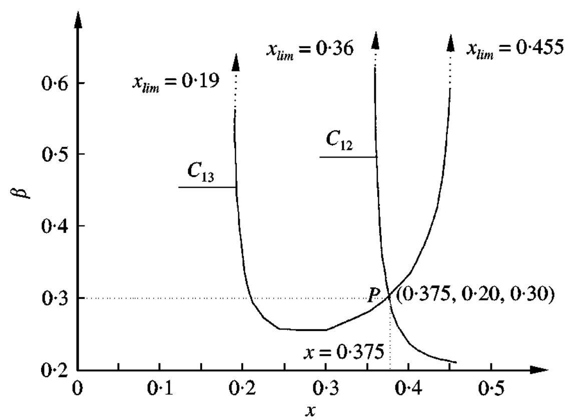

Figure 9. Projection in the x–β plane of the point P solution of the inverse problem.

position considered, the problem is very ill-conditioned and practically indeterminate, although in a limited parameter range. This is nevertheless a different situation from that of the first case, where indetermination only concerned b and β, owing to the redundant parameter of the interpretation model, which cannot be observable. In the present case, a greater amount of data could lead to a better defined unique solution.

The effect of errors in the experimental data is analyzed in Figures 12(a) and 12(b), where the curves C_{12} and C_{13} are reported for the case of damage already discussed in Figure 9. The noise corrupted frequencies are reported in Table 1, cases 2a and 2b refer to a smaller and a greater error respectively. The curves are affected by error, but not very much. When three frequencies only are used, a solution is found which is different from the exact one, but close to it. The identified damage parameters in the case of a small error are $(x, b, \beta) = (0{\cdot}39, 0{\cdot}17, 0{\cdot}33)$, while in the case of greater error they are $(0{\cdot}40, 0{\cdot}14, 0{\cdot}38)$, when the exact values are $(0{\cdot}375, 0{\cdot}20, 0{\cdot}30)$. It can be observed that the identified damage parameters b and β are different from the exact values, but identifying a damage globally equivalent to the real condition, because b is smaller than the true value while β is larger.

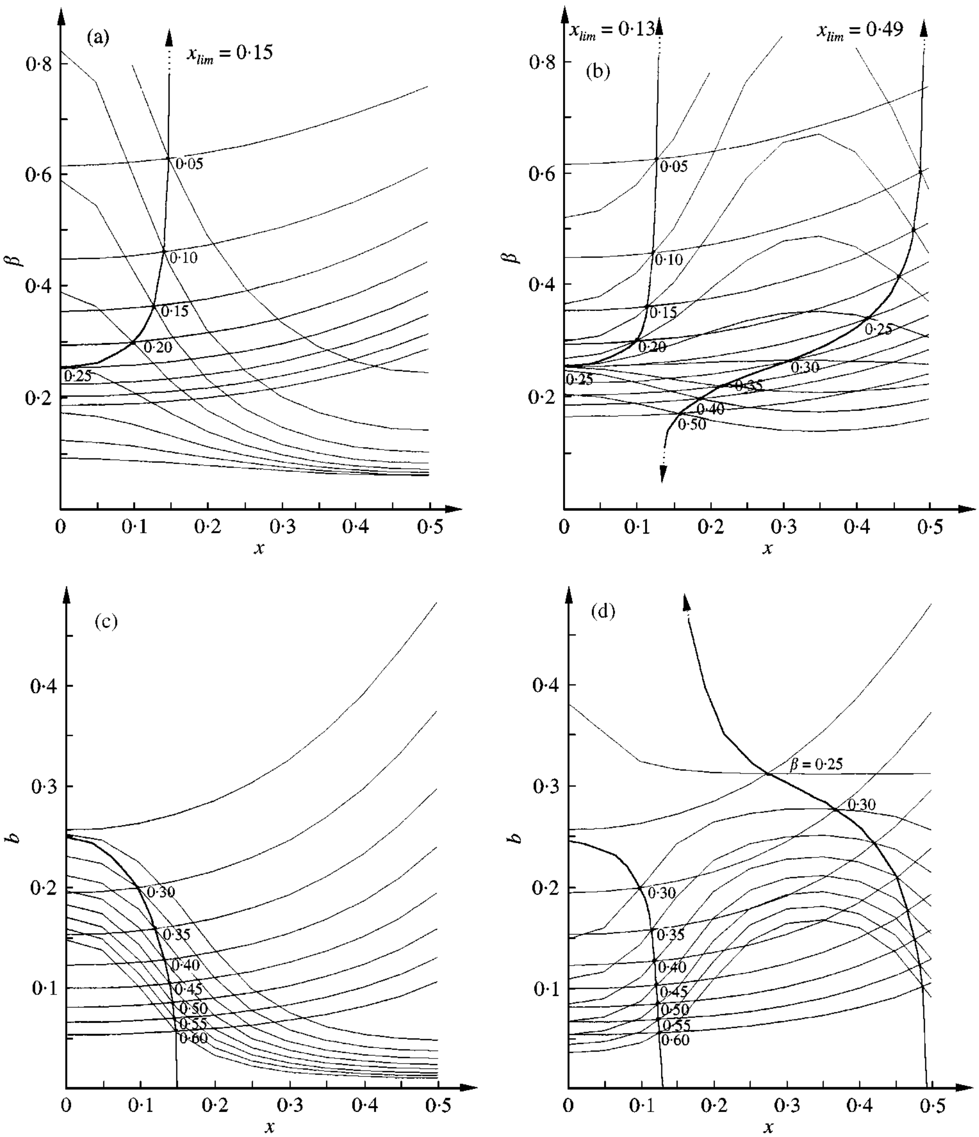

Figure 10. Loci of the three parameter values for which (a, c) $g_1 = 0$ and $g_2 = 0$; (b, d) $g_1 = 0$ and $g_3 = 0$.

When four frequencies are used, that is a number greater than that strictly necessary, any pair C_{rs} and C_{pt} do not cross at the same point as it happens in absence of error. The optimal values of damage parameters can then be obtained as the minimum of the objective function:

$$\tilde{l}(x) = \sum_{r,s,p,t} ((\beta_{rs} - \beta_{pt})^2 + (b_{rs} - b_{pt})^2), \tag{22}$$

where rs and pt must be any, but different, couple for r, s, p, $t = 1, 4$.

For the case of greater error previously discussed (2b in Table 1) the use of four frequencies defines three different points, furnishing a best estimate of parameters sufficiently close to the exact values (Figure 12(c)).

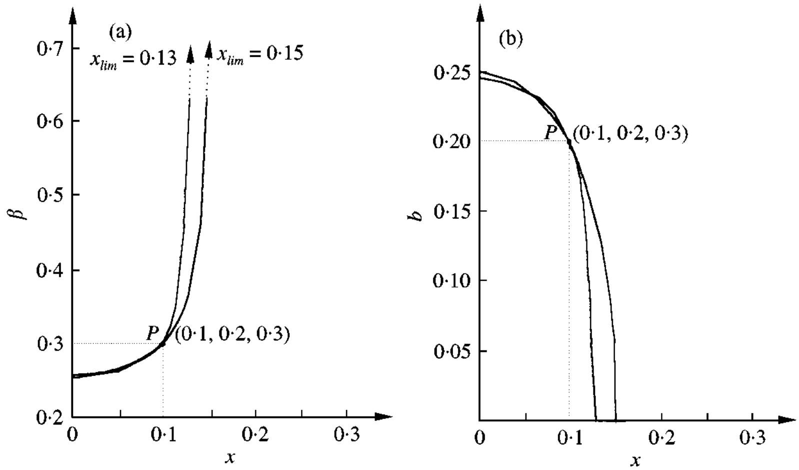

Figure 11. Projection of the point P, solution of the inverse problem (a) in the plane x–β; (b) in the plane x–b.

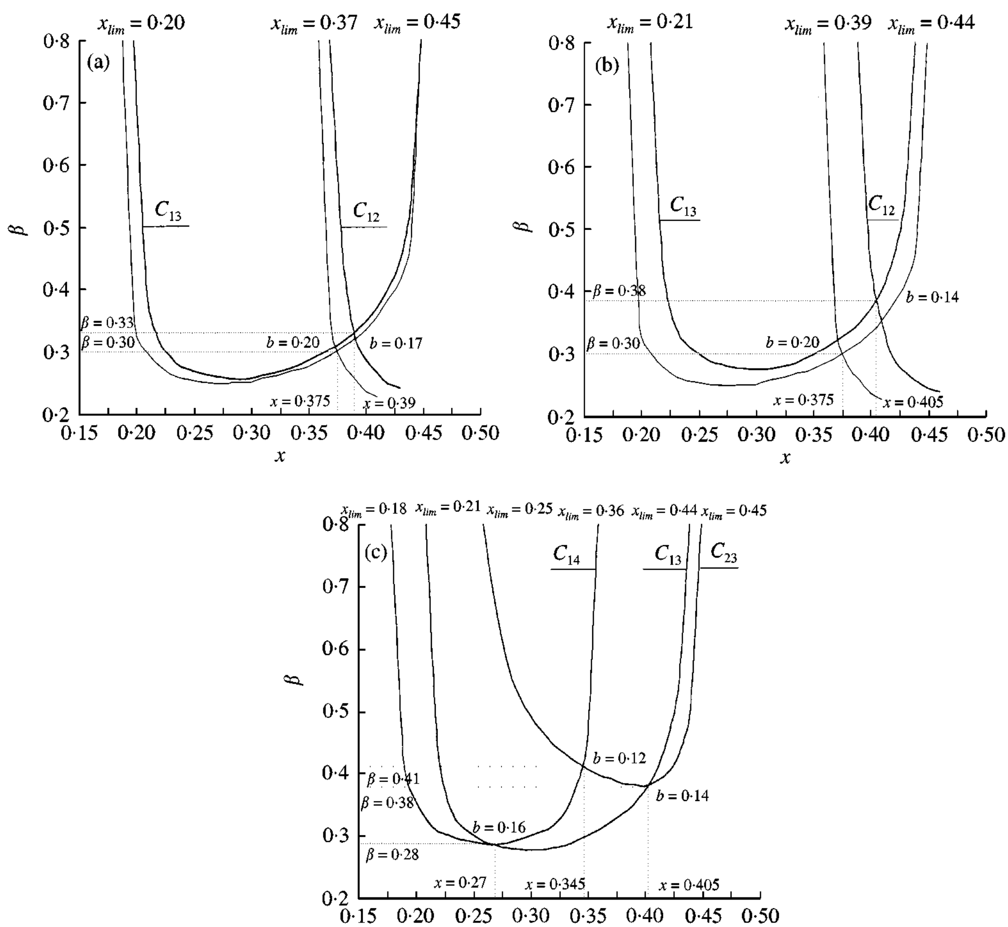

Figure 12. Curves C_{rs} for the cases 2a and 2b (Table 1) using three frequencies (a and b, thin lines are without error, case 2) and curves C_{rs} for the case 2b using four frequencies (c).

TABLE 1

Frequencies for the undamaged beam and variations associated to three cases of damage free of error (1–3) *and two cases with error* (2a–2b)

				Natural frequencies (rad/s)				
				ω_1	ω_2	ω_3	ω_4	ω_5
Undamaged				73·728	294·91	663·55	1179·6	1843·2
Case number	Damage position x	Damage extension b	Damage coefficient β	Variations of natural frequencies (rad/s) $\Delta\omega_1$	$\Delta\omega_2$	$\Delta\omega_3$	$\Delta\omega_4$	$\Delta\omega_5$
1	0·375	0·05	0·30	1·070	5·075	0·750	12·01	32·50
2	0·375	0·20	0·30	3·977	16·09	14·44	43·09	61·42
3	0·100	0·20	0·30	5·341	4·543	30·17	40·80	66·96
2a	0·375	0·20	0·30	3·900	17·15	13·00	—	—
2b	0·375	0·20	0·30	3·700	17·60	10·10	45·00	—

In the inverse problem solution, the error in the data is usually amplified strongly, in this case it can be noticed that this is not very pronounced, a mean error of 0·23 and 0·54% in frequency measurements in the cases 2a and 2b, respectively, becomes 2·5 and 5·2% in the damage parameters.

3.2. PROCEDURE BASED ON THE RESPONSE COMPARISON

To solve the damage identification by means of the procedure based on the response comparison, it is useful to introduce an objective function defined by the difference between experimental and analytical frequency values:

$$l(x, b, \beta) = \sum_r \left| \frac{\omega_r^D - \omega_r^D(x, b, \beta)}{\omega_r^U} \right|^2, \tag{23}$$

where ω_r^D represents the experimental value of the damaged rth frequency, $\omega_r^D(x, b, \beta)$ represents the analytical value as a function of damage parameters, and ω_r^U represents the frequency of the undamaged beam.

The identification of the damage parameters coincides with the search for the minimum of function (23). In the case of the beam with a damaged zone, as in the case of concentrated damage, it is possible to use a two-phase procedure [17, 23]. The objective function

$$\tilde{l}(x) = \min_{b,\beta} l(x, b, \beta) \tag{24}$$

as a function of parameter x only is initially determined from the minimization of equation (23) with respect to parameters b and β. For an assigned damage location x, the function $\tilde{l}(x)$ determines the best values of damage parameters such that to minimize the error between experimental and analytical frequencies. The solution to the inverse problem can then be given by the minimum of $\tilde{l}(x)$. If the problem exhibits only one minimum, it is possible to determine one single value $\bar{x}$ satisfying the condition $\tilde{l}(\bar{x}) = 0$ for a number of frequencies equal to or greater than three.

Let us consider again the second case of damage described in section 3.1 above, which gave a well-defined solution to the data in Table 1. Figure 13(a) shows the function $\tilde{l}(x)$ associated with this case of damage; it reaches its minimum at the expected parameter values: $x = 0{\cdot}375$, $b = 0{\cdot}20$ and $\beta = 0{\cdot}30$. The minimum of the function $\tilde{l}(x)$ is very well defined, and the exact solution is clearly furnished. For the first case in Table 1 characterized by equal parameters x and b but with limited extension $b = 0{\cdot}05$, the same identification procedure gives the curve $\tilde{l}(x)$ shown in Figure 13(b). The curve has a shape similar to that shown in Figure 13(a), but the minimum has a smaller resolution.

In Figures 14(a) and 14(b), the objective function $\tilde{l}(x)$ for the third case of Table 1 with damage located close to the midspan is drawn. The curve shown in Figure 14(a) is obtained considering the first three frequencies, while the curve shown in Figure 14(b) is obtained using the fourth frequency instead of the second. In Figure 14(a), two minimum points are present. The absolute minimum gives the exact solution of the inverse problem. In Figure 14(b), a lot of minimum points are present, but also in this case the exact solution is given by the absolute minimum at $x = 0{\cdot}10$. The use of higher frequencies introduces irregularities in the objective function, and sometimes multiple solutions, as can be argued from Figure 14 and it is clear in Figure 7, due to the higher number of zero point in the $k_r(x)$ for increasing r.

In the presence of model and experimental errors, the minimum of $\tilde{l}(x)$ still gives the most appropriate solution. In order to limit the interference of such errors in identifying the

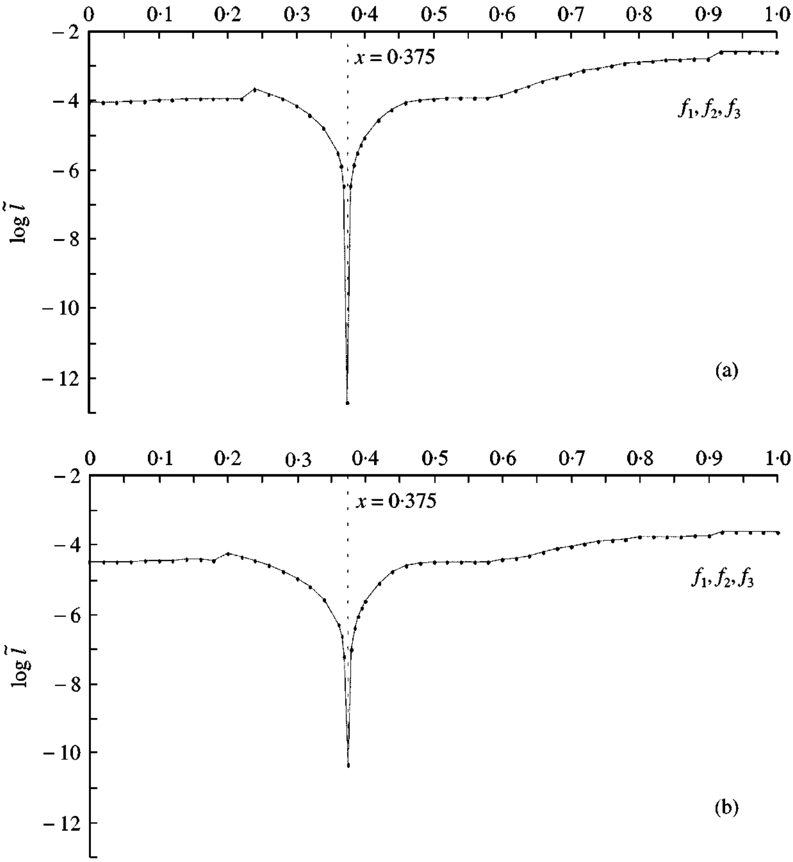

Figure 13. Objective function log $\tilde{l}(x)$ for two damage cases: (a) $x = 0{\cdot}375$, $b = 0{\cdot}20$ and $\beta = 0{\cdot}30$; (b) $x = 0{\cdot}375$, $b = 0{\cdot}05$ and $\beta = 0{\cdot}30$.

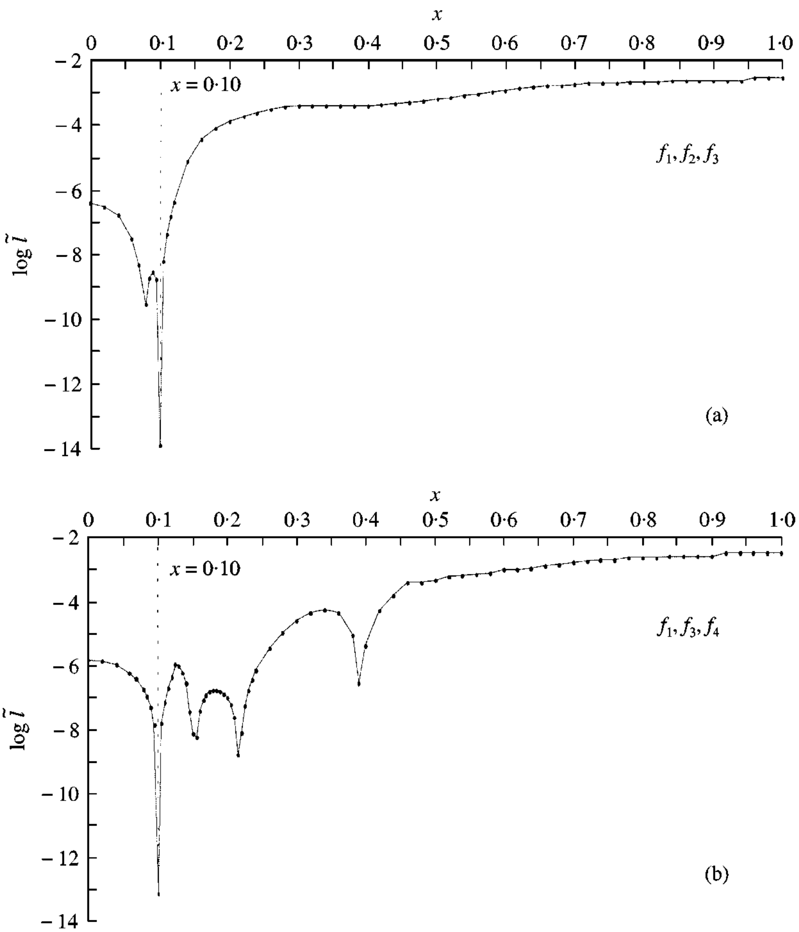

Figure 14. Objective function log $\tilde{l}(xc)$ for a damage case ($x = 0{\cdot}10$, $b = 0{\cdot}20$ and $\beta = 0{\cdot}30$) with different sets of frequencies.

solution, a larger number of experimental frequencies than is strictly necessary may be used. If the same problem is dealt with by using four frequencies, only one minimum—the absolute minimum—remains just in the right position. This can be very useful in the real word because the distribution of errors cannot realistically alter the $\tilde{l}(x)$ function in such way that the true minimum becomes unrecognizable. On the contrary, this could happen when several minima are present; a relative minimum corrupted by noise can appear as the absolute minimum. However, in accordance with the consideration described above—that few frequencies, about three, are sufficient to identify a unique solution—, it is advisable to have as precise as possible an estimate of the experimental frequencies in order to reach an accurate solution to the inverse problem.

This second procedure of damage parameter identification based on the minimization of an objective function, is without doubt more easily generalized than the procedure based on the modal equation, especially when errors are present and complex structures, which need to be modelled by large discrete models, are tackled. The procedure illustrated in section 3.1 has nevertheless enabled us to assess more clearly which data can be used to find a solution to the inverse problem and has shown that situations such as the limited extension of damage and its position mean that the solution is not well determined. In this respect, when a real problem with experimental data has to be solved, it is very useful to study the similar inverse problem with pseudo-experimental data, which can give information about the uniqueness and ill-conditioning of the specific problem.

4. CONCLUSIONS

A model of a damaged beam is defined that is useful to evaluate damage caused by diffused cracking in one zone of a beam. First, the relations among the damage parameters—position, extension and degree—and the frequency variations in the first vibration modes are considered. In the case of concentrated damage, a rotational spring k is frequently used to represent the increment in flexibility introduced by damage. The characteristic equation can be solved with respect to k and this circumstance has suggested a very simple damage detection technique, already presented in references [17, 23]. For the present case of diffused damage, the characteristic equation is an implicit relation among the damage parameters and an extension of the earlier approach is proposed.

Two identification procedures are used: in the first the optimal value of parameters is such as to satisfy at best the characteristic equation written for no less than three measured frequencies; in the second procedure the optimal value is such as to minimize an objective function based on the error between analytical and experimental values of three or more frequencies.

Illustrative examples with pseudo-experimental data are used to discuss the identifiability and ill-conditioning of the problem. Similarly to the case of concentrated damage, few data are sufficient to solve the inverse problem, specifically, few more than three frequencies. The presence of experimental errors produces a modification in the damage parameter estimate, but the error in the data is not strongly amplified in the solution. Thus, it is not the quantity but the quality of the experimental data that is important in achieving reliable results. The problem is generally ill-conditioned, but when the extension is small, one parameter can no longer be observable and the problem reverts to the case of concentrated damage with only two parameters to identify. It is shown that the identifiability properties depend strongly on the characteristics of the damage, especially its location.

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