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LETTERS TO THE EDITOR

RESURRECTION OF THE METHOD OF SUCCESSIVE APPROXIMATIONS TO YIELD CLOSED-FORM SOLUTIONS FOR VIBRATING INHOMOGENEUOS BEAMS

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1. INTRODUCTION

The method of successive approximations apparently was pioneered by Picard [1] and later employed by Hadamard [2], Hohenemser and Prager [3], Kolousek [4], Ananiev [5], Nowacki [6], Birger and Panovko [7], Biderman [8] and others. Presently, it almost never appears in the texts on vibration, due to development of the versatile finite element method. The present study attempts to inject new life to it, yet not for its original purpose of approximate solutions. Rather, a closed-form solution is obtained by the resurrected method. Moreover, the solutions obtained as intermediate approximations of the mode shape are shown to become closed-form solutions for the *inhomogeneous* beams.

The differential equation that governs the free vibrations of the beams of variable cross-section reads

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI(x)\frac{\mathrm{d}^2W}{\mathrm{d}x^2}\right)-\omega^2\rho A(x)W(x)=0. \tag{1}$$

Birger and Mavliutov [9] suggest to replace this equation by the equivalent integral equation. As they mention (p. 401) "equation (1) can be brought to the form of an integral equation, which gives series of advantages for the approximate solution." The amplitude values of the shear force $V_y(x)$ and the bending moment $M_z(x)$ read

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(EI(x)\frac{\mathrm{d}^2W}{\mathrm{d}x^2}\right)=-V_y(x), \qquad EI(x)\,\mathrm{d}^2W/\mathrm{d}x^2=-M_z(x). \tag{2, 3}$$

Birger and Mavliutov [9] integrate equation (1) between x and L. For specificity, we consider the cantilever beam. We note that the shearing force at $x=L$ is absent. We get

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(EI(x)\frac{\mathrm{d}^2W}{\mathrm{d}x^2}\right)=\omega^2\int_x^L\rho A(x_1)W(x_1)\,\mathrm{d}x_1. \tag{4}$$

We repeat the operation of integration utilizing the condition $M_z(L)=0$, to get

$$EI(x)\frac{\mathrm{d}^2W}{\mathrm{d}x^2}=\omega^2\int_x^L\int_{x_1}^L\rho A(x_2)W(x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1. \tag{5}$$

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We solve the direct vibration problem. Since it is assumed that the stiffness $EI(x)$ is known, we divide both sides of equation (5) by it, and integrate the result twice between zero to x. In view of the conditions

$$W(0)=0, \quad \mathrm{d}W(0)/\mathrm{d}x=0, \tag{6}$$

we obtain, for the cantilever beam,

$$W(x)=\omega^2\int_0^x\int_0^{x_1}\frac{1}{EI\,(x)}\int_{x_2}^L\int_{x_3}^L \rho A(x_4)W(x_4)\,\mathrm{d}x_4\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1. \tag{7}$$

This is a homogeneous integral equation that is equivalent to differential equation (1) and the appropriate boundary conditions. It is an integral equation, since the unknown function appears under the integral sign. In a short form, equation (7) can be written as

$$W=\omega^2 KW, \tag{8}$$

where KW is the integral operator, represented by the right-hand side of equation (7). It is easy to see that $W(x)$ is the solution of equation (8), as is the function $CW(x)$ where C is an arbitrary constant. Equation (8) possesses the trivial solution $W(x)\equiv 0$, yet for some values of $\omega^2=\omega_1^2, \omega_2^2$, etc. it has non-trivial solutions, with $\omega_1, \omega_2, \ldots$ being natural frequencies. If one tries as $W(x_1)$, the function

$$W_1(x)=(x/L)^2, \tag{9}$$

one gets the second approximation

$$W_{(2)}=\omega_1^2 KW_{(1)}. \tag{10}$$

Were $W_{(1)}$ an exact solution, then the functions $W_{(2)}$ and $W_{(1)}$ would coincide for all values of x. Yet $W_{(1)}$ is not an exact solution. Hence,

$$W_2(x)\neq W_1(x). \tag{11}$$

Birger and Mavliutov [9] impose the condition

$$W_{(2)}(L)=W_{(1)}(L)=1 \tag{12}$$

resulting in

$$\omega_{(1)}^2=[KW_{(1)}(L)]^{-1}=\left[\int_0^L\int_0^{z_1}\frac{1}{EI(x)}\int_{x_2}^L\int_{x_3}^L \rho A\,\frac{x_4^2}{L^2}\,\mathrm{d}x_4\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1\right]^{-1}. \tag{13}$$

According to reference [9], "usually already first approximation yields an error not exceeding 2–5%. It can be shown that the process of successive approximations always converges to the first natural frequency. Obtaining by this means of consequent frequencies and modes requires performing the orthogonalization process." Note that Collatz [10], and Ponomarev *et al.* [11] utilize another, integral criterion for determining the successive approximations of the natural frequency.

2. EVALUATION OF THE EXAMPLE BY BIRGER AND MAVLIUTOV

We evaluate a particular example with two objectives in mind: (1) illustration of the method of successive approximations and (2) use of its *intermediate* results in the subsequent section of inhomogeneous beams.

Birger and Mavliutov [9] consider the example of a blade in form of a cantilever beam. They write: "for construction of the approximate model the cross-section of the blade can be assumed to be constant." For determination of fundamental vibration frequency they used the integral method,

$$\omega_{(1)}^2 = [KW_{(1)}(L)]^{-1} = \left[\frac{\rho A}{EI}\int_0^L \int_0^{x_1} \int_{x_2}^L \int_{x_3}^L \left(\frac{x_4}{L}\right)^2 \mathrm{d}x_4\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1\right]^{-1}. \tag{14}$$

This corresponds to the first approximation in equation (9). We find

$$\begin{aligned} KW_{(1)}(x) &= \frac{\rho A}{EI}\int_0^L \int_0^{x_1} \int_{x_2}^L \int_{x_3}^L \left(\frac{x_4}{L}\right)^2 \mathrm{d}x_4\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &= \frac{\rho A}{EI}\frac{1}{L^2}\int_0^x \int_0^{x_1} \left(\frac{1}{4}L^4 - \frac{1}{3}L^3 x_2 + \frac{1}{12}x_2^4\right)\mathrm{d}x_2\,\mathrm{d}x_1 \end{aligned} \tag{15}$$

or

$$KW_{(1)}(x) = \rho AL^4[\tfrac{1}{8}(x/L)^2 - \tfrac{1}{18}(x/L)^3 + \tfrac{1}{360}(x/L)^6]/EI. \tag{17}$$

The value of $KW_{(1)}(x)$ at $x = L$ equals

$$KW_{(1)}(L) = \tfrac{13}{180}\rho AL^4/EI. \tag{18}$$

The first approximation of the natural frequency becomes

$$\omega_{(1)}^2 = \tfrac{180}{13}EI/\rho AL^4. \tag{19}$$

For further refinement we calculate

$$W_{(2)}(x) = \omega_{(1)}^2 KW_{(1)}(x) = KW_{(1)}(x)/KW_{(1)}(L) = \tfrac{180}{13}[\tfrac{1}{8}(x/L)^2 - \tfrac{1}{18}(x/L)^3 + \tfrac{1}{360}(x/L)^6]. \tag{20}$$

Now,

$$\begin{aligned} KW_{(2)}(x) &= \frac{\rho A}{EI}\int_0^L \int_0^{x_1} \int_{x_2}^L \int_{x_3}^L \left\{\frac{180}{13}\left[\frac{1}{8}\left(\frac{x_4}{L}\right)^2 - \frac{1}{18}\left(\frac{x_4}{L}\right)^3 + \frac{1}{360}\left(\frac{x_4}{L}\right)^6\right]\right\}\mathrm{d}x_4\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &= \frac{\rho AL^4}{EI}\left[\frac{1}{131\,040}\left(\frac{x}{L}\right)^{10} - \frac{1}{1092}\left(\frac{x}{L}\right)^7 + \frac{1}{208}\left(\frac{x}{L}\right)^6 - \frac{71}{1092}\left(\frac{x}{L}\right)^3 + \frac{59}{416}\left(\frac{x}{L}\right)^2\right] \end{aligned} \tag{21}$$

leading to the second approximation

$$\omega_{(2)}^2 = [KW_{(2)}(L)]^{-1} = \frac{8190}{661}\frac{EI}{\rho AL^4} \approx 12{\cdot}39\,\frac{EI}{\rho AL^4} \tag{22}$$

Note that Birger and Mavliutov [9] do not give an expression for $KW_{(2)}(x)$; they quote a factor of 12.23 in equation (22). For further refinement we calculate

$$W_{(3)}(x) = \omega^2_{(2)} KW_{(2)}(x) = KW_{(2)}(x)/KW_{(2)}(L)$$
$$= \frac{8190}{661}\left[\frac{1}{131\,040}\left(\frac{x}{L}\right)^{10} - \frac{1}{1092}\left(\frac{x}{L}\right)^{7} + \frac{1}{208}\left(\frac{x}{L}\right)^{6} - \frac{71}{1092}\left(\frac{x}{L}\right)^{3} + \frac{59}{416}\left(\frac{x}{L}\right)^{2}\right]. \quad (23)$$

We get

$$KW_{(2)}(x) = \frac{\rho A}{EI}\int_0^L\int_0^{x_1}\int_{x_2}^L\int_{x_3}^L W_{(3)}(x_4)\,\mathrm{d}x_4\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1$$
$$= \frac{\rho A L^4}{EI}\left[\frac{1}{254\,077\,824}\left(\frac{x}{L}\right)^{14} - \frac{1}{698\,016}\left(\frac{x}{L}\right)^{11} + \frac{1}{84\,608}\left(\frac{x}{L}\right)^{10} - \frac{71}{74\,032}\left(\frac{x}{L}\right)^{7}\right.$$
$$\left. + \frac{413}{84\,608}\left(\frac{x}{L}\right)^{6} - \frac{45\,541}{698\,016}\left(\frac{x}{L}\right)^{3} + \frac{12\,031}{84\,608}\left(\frac{x}{L}\right)^{2}\right], \quad (24)$$

leading to the third approximation

$$\omega^2_{(3)} = \frac{15\,879\,864}{1\,244\,461}\frac{EI}{\rho A L^4} \approx 12{\cdot}36\,\frac{EI}{\rho A L^4}, \quad (25)$$

which nearly coincides with the exact solution [12], associated with the factor $\approx 1{\cdot}875^4 \approx 12{\cdot}3596$.

3. REINTERPRETATION OF THE INTEGRAL METHOD FOR INHOMOGENEOUS BEAMS

As we saw above, the integral method can be effectively utilized for the approximate solution of eigenvalue problems. Its new twist will be presented here for obtaining *closed-form* solutions by the integral method.

We pose the following problem: find closed-form solutions of beams of variable inertial coefficient, and variable stiffness

$$\delta(x) = \rho(x)A(x), \quad D(x) = E(x)I(x) \quad (26)$$

so that the beam possesses the pre-selected mode shape $\psi(x)$. To this end we rewrite equation (5) by identifying the mode shape $W(x)$ with the selected function $\psi(x)$, in conjunction with equation (26)

$$D(x) = \left[\omega^2\int_x^L\int_{x_1}^L \delta(x_2)\psi(x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1\right]\Big/\psi''(x). \quad (27)$$

By substituting various functions $\psi_j(x)$ into equation (27) we obtain appropriate expressions of the stiffness $D(x)$. We adopt functions

$$\psi_j(x) = (x/L)^{j+4} - \tfrac{1}{6}(j+2)(j+3)(j+4)(x/L)^3 + \tfrac{1}{2}(j+1)(j+3)(j+4)(x/L)^2,$$
$$\text{for } j \geqslant 0 \quad (28)$$

that are proportional to the displacement of the uniform beam under the load $(x/L)^j$.

Consider a particular example. Let the width $b(x)$ of the cross-section of the beam be constant and equal b, whereas the height varies linearly,

$$h(x) = (h_1 - h_0)x/L + h_0, \tag{29}$$

where h_0 is the height at $x = 0$, while h_1 is the height at $x = L$. Then the cross-sectional area is

$$A(x) = bh(x) = bh_0\left(1 + \frac{h_1 - h_0}{h_0}\frac{x}{L}\right) \tag{30}$$

or, with α defining the ratio of heights

$$\alpha = h_1/h_0, \tag{31}$$

we have

$$A(x) = bh(x) = A_0[1 + (1 - \alpha)x/L], \tag{32}$$

where A_0 is the cross-sectional area at $x = 0$. The moment of inertia reads

$$I(x) = bh(x)^3/12 = I_0[1 + (1 - \alpha)x/L]^3, \tag{33}$$

where

$$I_0 = bh_0^3/12 \tag{34}$$

is the moment of inertia at $x = 0$. The modulus of elasticity is written as

$$E(x) = E_0 e(x), \tag{35}$$

where E_0 is the modulus of elasticity at $x = 0$. The stiffness becomes

$$D(x) = D_0 e(x)[1 - (1 - \alpha)x/L]^3, \tag{36}$$

where D_0 is the stiffness at the origin

$$D_0 = E_0 I_0. \tag{37}$$

Let the density be given as a polynomial of mth order,

$$\rho(x) = \rho_0 \varphi(x), \tag{38}$$

where

$$\varphi(x) = 1 + \beta_1(x/L) + \beta_2(x/L)^2 + \beta_3(x/L)^3 + \cdots + \beta_m(x/L)^m. \tag{39}$$

Equation (27) is rewritten as

$$e(x)[1-(1-\alpha)x/L]^3 = \left[\omega^2\rho_0 A_0 \int_x^L \int_{x_1}^L \varphi(x)[1-(1-\alpha)x/L]\psi(x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1\right]\Big/ D_0\psi''. \tag{40}$$

We define the natural frequency as

$$\omega^2 = \gamma D_0/\rho_0 A_0 L^4, \tag{41}$$

where γ the arbitrary parameter. Since the factor multiplying γ is a known quantity, one can maintain that an arbitrary value of the natural frequency can be obtained by designing a material with a given modulus of elasticity. Indeed, in order for the beam to have an arbitrary natural frequency as given in equation (38), the modulus of elasticity must take the form

$$e(x) = \gamma \tilde{e}(x), \tag{42}$$

where $\tilde{e}(x)$ reads

$$\tilde{e}(x) = \frac{E(x)}{E(0)}, \quad E(x) = \frac{1}{L^4} \frac{\left[\int_x^L \int_{x_1}^L \varphi(x_2)[1-(1-\alpha)x_2/L]\psi(x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 \right]}{\psi''[1-(1-\alpha)x/L]^3}. \tag{43}$$

The quantity $\tilde{e}(x)$ can be designated as a *parent modulus of elasticity*. The function $e(x) = \gamma\tilde{e}(x)$ then produces a one-parameter *family* of elastic moduli. Consider now the particular cases.

4. UNIFORM MATERIAL DENSITY

Let the density of the beam's material be constant:

$$\rho(x) = \rho_0, \quad \beta_j = 0, \quad j = 1, 2, \ldots, m. \tag{44}$$

The parent elastic modulus is

$$\tilde{e}_0(\xi) = \frac{26 + 16\xi + 6\xi^2 - 4\xi^3 + \xi^4}{26[1-(1-\alpha)\xi]^2}. \tag{45}$$

As is seen we obtain a rational expression for any α, except $\alpha = 1$, corresponding to the beam of the uniform cross-section, in which case

$$\tilde{e}_0(\xi) = (26 + 16\xi + 6\xi^2 - 4\xi^3 + \xi^4)/26 \tag{46}$$

and the parent modulus of elasticity becomes a polynomial expression. Expressions (45) and (46) correspond to function (28) with $j = 0$, thus, subscript 0 in equations (45) and (46) respectively. Figure 1 depicts the parent modulus of elasticity for values $\alpha_1 = \frac{1}{3}$, $\alpha_2 = 0{\cdot}5$ and $\alpha_3 = 1$. Fixing j at unity in equation (28) yields

$$\tilde{e}_1(\xi) = \frac{132 + 82\xi + 32\xi^2 - 18\xi^3 + 2\xi^4 + \xi^5}{66[1-(1-\alpha)\xi]^2(2+\xi)}. \tag{47}$$

Figure 2 portrays variation of $\tilde{e}_1(\xi)$, for $\alpha_1 = \frac{1}{3}$, $\alpha_2 = 0{\cdot}5$ and $\alpha_3 = 1$.

For $j = 2$ in equation (28) we obtain

$$\tilde{e}_2(\xi) = \frac{3(413 + 258\xi + 103\xi^2 - 52\xi^3 + 3\xi^4 + 2\xi^5 + \xi^6)}{413[1-(1-\alpha)\xi]^2(2 + 2\xi + \xi^2)}. \tag{48}$$

Figure 3 shows the function $\tilde{e}_2(\xi)$ for various values of α and $j = 2$. For $j = 3$, one gets

$$\tilde{e}(\xi) = \frac{1016 + 673\xi + 258\xi^2 - 121\xi^3 + 4\xi^4 + 3\xi^5 + 2\xi^6 + \xi^7}{254[1-(1-\alpha)\xi]^2(4 + 3\xi + 2\xi^2 + \xi^3)}. \tag{49}$$

Figure 4 illustrates the dependence of $\tilde{e}_3(\xi)$ on ξ for different values of α, for $j = 3$.

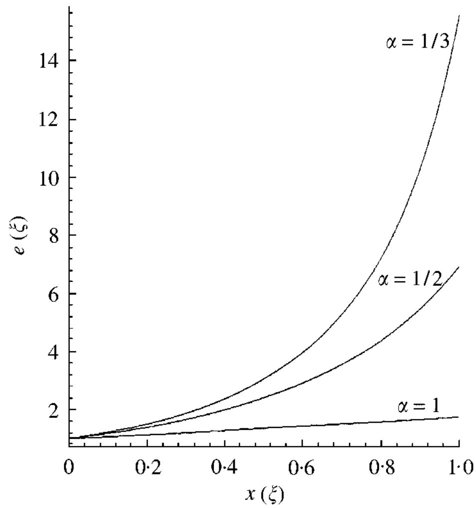

Figure 1. Variation of parent modulus of elasticity when $j = 0$.

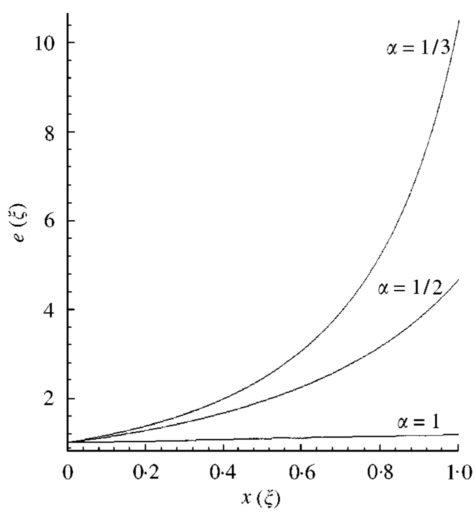

Figure 2. Variation of parent modulus of elasticity when $j = 1$.

5. LINEARLY VARYING DENSITY

Let the material density vary as

$$\rho(x) = \rho_0(1 + \beta x/L), \tag{50}$$

whereas the cross-sectional area and the moment of inertia are varying as in equations (32) and (34) respectively. The following results are obtained for the parent modulus of elasticity, for $j = 0$:

$$\tilde{e}_0(\xi) = [182 + 142\beta + (112 + 102\beta)\xi + (42 + 62\beta)\xi^2 - (28 - 22\beta)\xi^3 + (7 - 18\beta)\xi^4 + 5\beta\xi^5]/\{(182 + 142\beta)[1 - (1 - \alpha)\xi]^2\}. \tag{51}$$

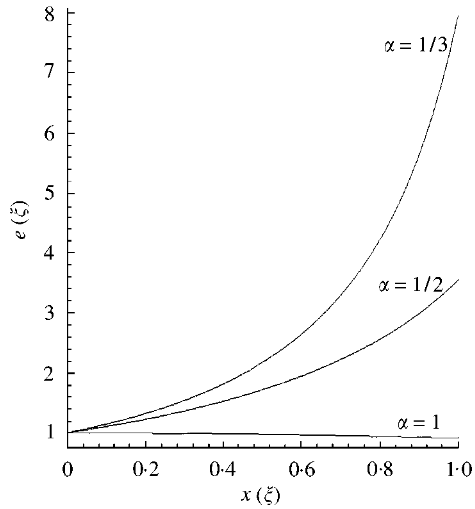

Figure 3. Variation of parent modulus of elasticity when $j = 2$.

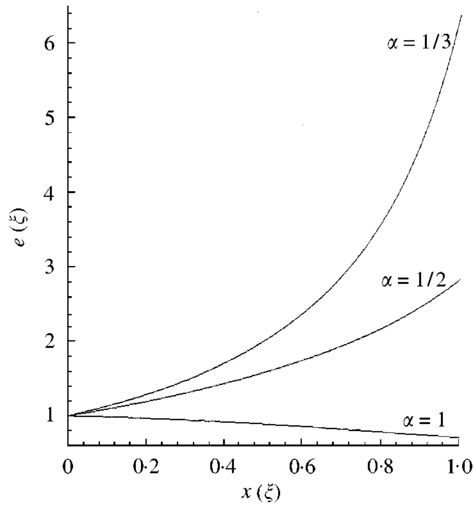

Figure 4. Variation of parent modulus of elasticity when $j = 3$.

Again, for a beam with uniform cross-section the variation is polynomial, while for values of $\alpha \neq 1$, the rational expression is obtained. For $j = 1$, 2 and 3 we arrive at

$$\tilde{e}_1(\xi) = 2[528 + 413\beta + (328 + 298\beta)\xi + (128 + 183\beta)\xi^2 - (72 - 68\beta)\xi^3 + (8 - 47\beta)\xi^4 + (4 + 6\beta)\xi^5 + 3\beta\xi^6]/\{(528 + 413\beta)[1 - (1 - \alpha)\xi]^2(2 + \xi)\}, \quad (52)$$

$$\tilde{e}_2(\xi) = 3[3717 + 2912\beta + (2322 + 2107\beta)\xi + (927 + 1302\beta)\xi^2 - (468 - 497\beta)\xi^3 + (27 - 308\beta)\xi^4 + (18 + 21\beta)\xi^5 + (9 + 14\beta)\xi^6 + 7\beta\xi^7]/\{(3717 + 2912\beta)[1 - (1 - \alpha)\xi]^2(3 + 2\xi + \xi^2)\}, \quad (53)$$

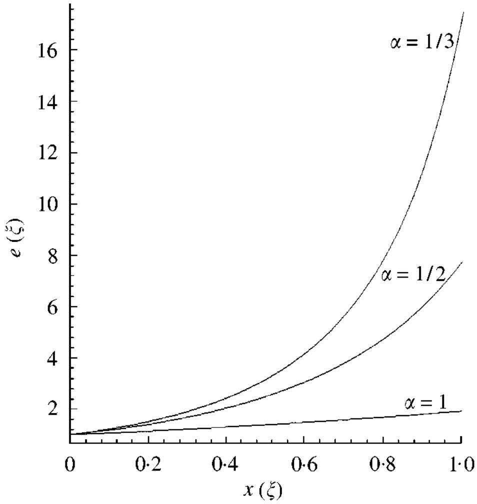

Figure 5. Variation of parent modulus of elasticity for the beam with linearly varying material density ($\beta = 1, j = 0$).

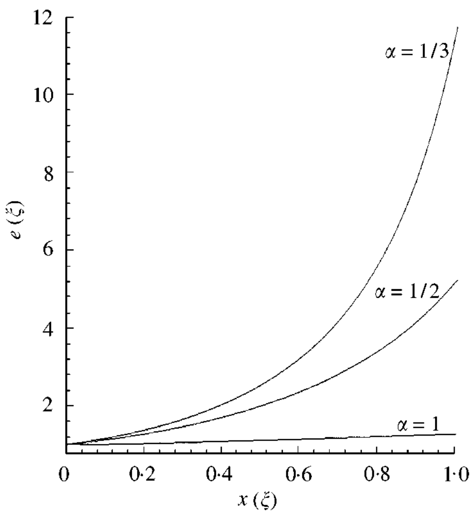

Figure 6. Variation of parent modulus of elasticity for the beam with linearly varying material density ($\beta = 1, j = 1$).

$$\tilde{e}_3(\xi) = [5080 + 3984\beta + (3185 + 2888\beta)\xi + (1290 + 1792\beta)\xi^2 + (605 - 696\beta)\xi^3 + (20 - 400\beta)\xi^4 + (15 + 16\beta)\xi^5 + (10 + 12\beta)\xi^6 + (5 + 8)\xi^7 + 4\beta\xi^8]/\{(1270 + 996\beta)[1 - (1 - \alpha)\xi]^2(4 + 3\xi + 2\xi^2 + \xi^3)\}. \quad (54)$$

Figures 5–8 depict some of the dependencies of the parent modulus of elasticity for various values of α, $\beta = 1$.

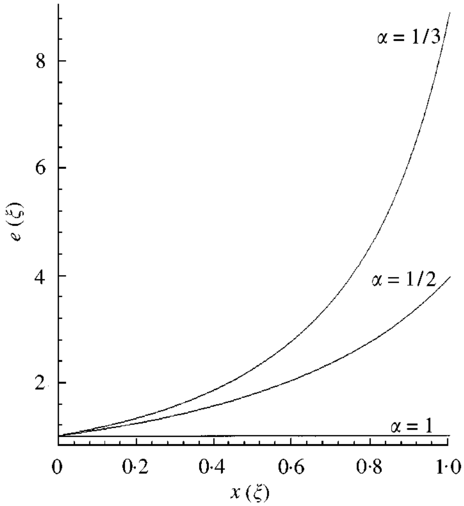

Figure 7. Variation of parent modulus of elasticity for the beam with linearly varying material density ($\beta = 1, j = 2$).

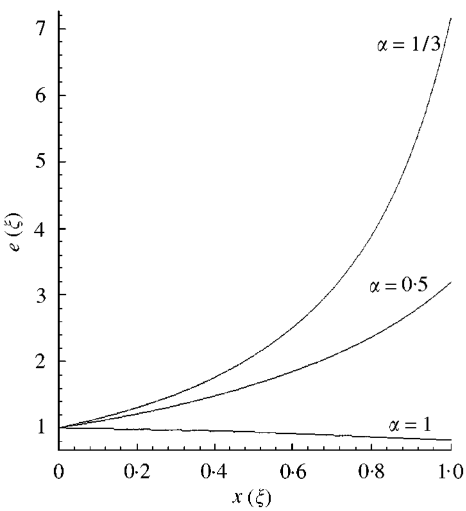

Figure 8. Variation of parent modulus of elasticity for the beam with linearly varying material density ($\beta = 1, j = 3$).

6. PARABOLICALLY VARYING DENSITY

Let the density vary as

$$\rho(x) = \rho_0 [1 + \beta_1 x/L + \beta_2 (x/L)^2]. \tag{55}$$

The expressions for the parent modulus of elasticity reads, for $j = 0$,

$$\tilde{e}_0(\xi) = [728 + 568\beta_1 + 465\beta_2 + (448 + 408\beta_1 + 362\beta_2)\xi$$

$$+ (168 + 248\beta_1 + 259\beta_2)\xi^2 - (112 - 88\beta_1 - 156\beta_2)\xi^3$$

$$+ (28 - 72\beta_1 + 53\beta_2)\xi^4 + (20\beta_1 - 50\beta_2)\xi^5$$

$$+ 15\beta_2\xi^6]/\{(728 + 568\beta_1 + 465\beta_2)[1 - (1 - \alpha)\xi]^2\}. \tag{56}$$

Again, this is a rational expression for $\alpha \neq 1$ and polynomial expression for $\alpha = 1$. For $j = 1, 2$ and 3 the expressions of $\tilde{e}(\xi)$ are

$$\begin{aligned}\tilde{e}_1(\xi) = 2[&1584 + 1239\beta_1 + 1016\beta_2 + (984 + 894\beta_1 + 793\beta_2)\xi \\ &+ (384 + 549\beta_1 + 570\beta_2)\xi^2 - (216 - 204\beta_1 - 347\beta_2)\xi^3 \\ &+ (24 - 141\beta_1 + 124\beta_2)\xi^4 + (12 + 18\beta_1 - 99\beta_2)\xi^5 \\ &+ (9\beta_1 + 7\beta_2)\xi^6 + 7\beta_2\xi^7]/\{(1584 + 1239\beta_1 + 1016\beta_2)[1 - (1 - \alpha)\xi]^2(2 + \xi)\},\end{aligned} \tag{57}$$

$$\begin{aligned}\tilde{e}_2(\xi) = 3[&18\,585 + 14\,560\beta_1 + 11\,952\beta_2 + (11\,610 + 10\,535\beta_1 + 9344\beta_2)\xi \\ &+ (4635 + 6510\beta_1 + 6736\beta_2)\xi^2 - (2340 - 2485\beta_1 - 4128\beta_2)\xi^3 \\ &+ (135 - 1540\beta_1 + 1520\beta_2)\xi^4 + (90 + 105\beta_1 - 1088\beta_2)\xi^5 \\ &+ (45 + 70\beta_1 + 84\beta_2)\xi^6 + (35\beta_1 + 56\beta_2)\xi^7 \\ &+ 28\beta_2\xi^8]/\{(18\,585 + 14\,560\beta_1 + 11\,952\beta_2)[1 - (1 - \alpha)\xi]^2(3 + 2\xi + \xi^2)\},\end{aligned} \tag{58}$$

$$\begin{aligned}\tilde{e}_3(\xi) = [&55\,880 + 43\,824\beta_1 + 36\,000\beta_2 + (35\,035 + 31\,768\beta_1 + 28\,176\beta_2)\xi \\ &+ (14\,190 + 19\,712\beta_1 + 20\,352\beta_2)\xi^2 - (6655 - 7656\beta_1 - 12\,528\beta_2)\xi^3 \\ &+ (220 - 4400\beta_1 + 4704\beta_2)\xi^4 + (165 + 176\beta_1 - 3120\beta_2)\xi^5 \\ &+ (110 + 132\beta_1 + 144\beta_2)\xi^6 + (55 + 88\beta_1 + 108\beta_2)\xi^7 + (44\beta_1 + 72\beta_2)\xi^8 \\ &+ 36\beta_2\xi^9]/\{(13\,970 + 10\,956\beta_1 + 9000\beta_2)[1 - (1 - \alpha)\xi]^2(4 + 3\xi + 3\xi^2 + \xi^3)\}.\end{aligned} \tag{59}$$

7. CAN SUCCESSIVE APPROXIMATIONS SERVE AS MODE SHAPES?

Now, we pose a somewhat provocative question. The first approximation of the mode shape in equation (9) does not satisfy all boundary conditions. It satisfies geometric conditions, but not the essential ones. Yet, two subsequent approximations, utilized in this study are given in equations (20) and (23); they satisfy all boundary conditions. Can they serve as *exact* mode shapes of some inhomogeneous beams? The reply is affirmative. Substituting equation (20) into equation (27) we get, for the beam with uniform density,

$$\tilde{e}(\xi) = \frac{3(416 + 301\xi + 186\xi^2 + 71\xi^3 - 44\xi^4 + 3\xi^5 + 2\xi^6 + \xi^7)}{416(3 + 2\xi + \xi^2)}, \tag{60}$$

Likewise, substitution of equation (23) into equation (27) yields

$$\begin{aligned}\tilde{e}(\xi) = (&367\,392 + 265\,575\xi + 163\,758\xi^2 + 61\,941\xi^3 - 39\,876\xi^4 + 3270\xi^5 \\ &+ 2112\xi^6 + 954\xi^7 - 204\xi^8 + 3\xi^9 + 2\xi^{10} + \xi^{11})/[122\,464(3 + 2\xi + \xi^2)].\end{aligned} \tag{61}$$

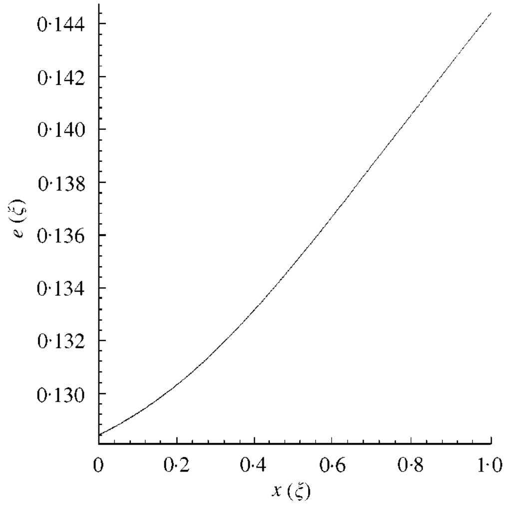

Figure 9. Variation of parent modulus of elasticity produced by the second approximation serving as an exact mode shape of a beam with uniform material density.

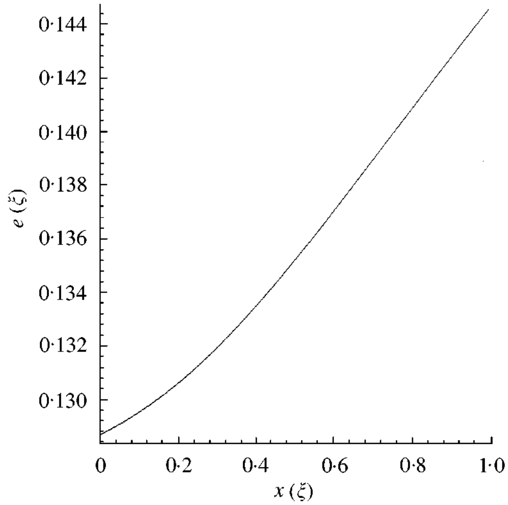

Figure 10. Variation of parent modulus of elasticity produced by the third approximation serving as an exact mode shape of a beam with uniform material density.

Likewise, one can obtain the closed-form solutions for the beams with linearly or parabolically varying material density. We do not reproduce these formulae to save space. Figures 9–12 depict the variations of the modulus of elasticity. Figures 9 and 10 are associated with constant density, whereas Figures 11 and 12 depict the variations of modulus of elasticity for the linearly varying density, $\beta = 1$.

8. CONCLUSION

As we have demonstrated a method that was designed and used for decades for *approximate* evaluation of the natural frequencies can be "twisted" to yield *closed-form* solutions.

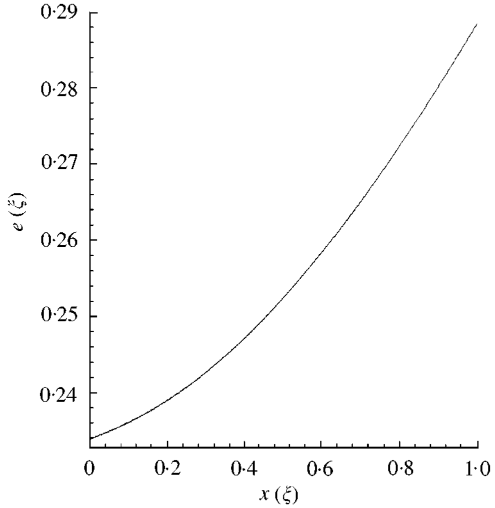

Figure 11. Variation of parent modulus of elasticity produced by the second approximation serving as an exact mode shape of a beam with linearly varying material density ($\beta = 1$).

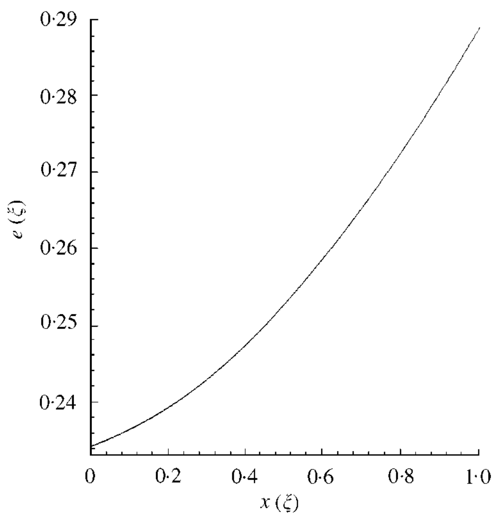

Figure 12. Variation of parent modulus of elasticity produced by the third approximation serving as an exact mode shape of a beam with linearly varying material density ($\beta = 1$).

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