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FUNDAMENTAL FREQUENCY OF A SQUARE MEMBRANE WITH A SQUARE CORE

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1. INTRODUCTION

Vibration of membranes is important in a variety of mechanical and physical devices. The resulting Helmholtz equation also governs the transmission of *TM* waves in electromagnetic waveguides. Gruner [1] used an eigenfunction matching method to study a square boundary with a square core. Gutierrez and Laura [2] used a Ritz method to study the related problem of a small, slightly rounded square core. The aim of the present Note is to compute the fundamental frequency for the whole range of core sizes and to derive asymptotic formulas in the cases of very large and very small cores.

We shall use a method similar to Gruner [3] but modified especially for square membranes. Such a method would also be more accurate for very small core sizes.

2. FORMULATION

Figure 1 shows the domain where all lengths have been normalized by the half-maximum width. The core size ratio is a. Due to symmetry we need to consider only the sub-regions I and II, each with their own co-ordinate axes as shown. The governing equation is

$$\nabla^2 w + k^2 w = 0, \tag{1}$$

where w is the displacement and k is the frequency normalized by (halfwidth) (density/tension per length)$^{1/2}$. The boundary conditions are that $w = 0$ on all the boundaries.

For the fundamental frequency, we note the symmetry $w_{\mathrm{I}}(x, y) = w_{\mathrm{I}}(y, x)$, together with the conditions $w_{\mathrm{I}}(x, 0) = 0$ and $w_{\mathrm{I}}(1 - a,\ 1 - a) = 0$. The general solution for region I is

$$w_{\mathrm{I}}(x, y) = \sum_{1}^{\infty} A_n [\sin(\alpha_n y) F_n(x) + \sin(\alpha_n x) F_n(y)], \tag{2}$$

where $\alpha_n = n\pi/(1 - a)$, A_n are coefficients to be determined, and if $s_n \equiv \sqrt{|k^2 - \alpha_n^2|}$,

$$F_n(x) \equiv \begin{cases} \sin(s_n x), & k \geqslant \alpha_n, \\ \sinh(s_n x), & k < \alpha_n. \end{cases} \tag{3}$$

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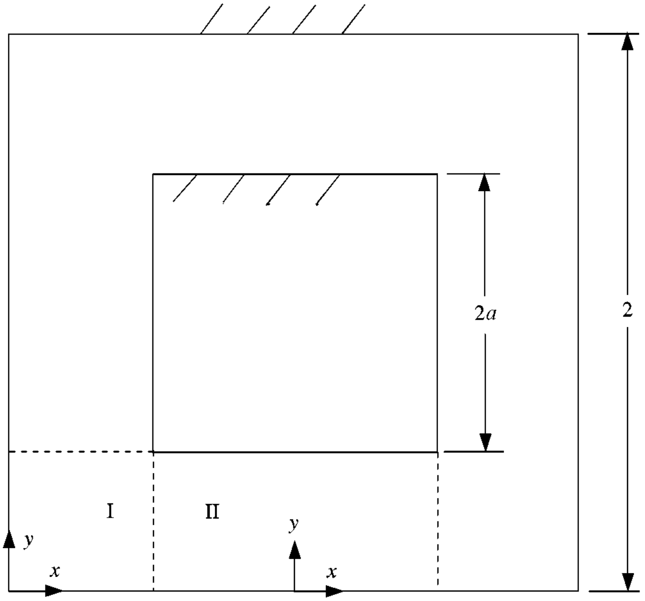

Figure 1. The square membrane with a square core.

For region II we note $w_{\mathrm{II}}(x, y) = w_{\mathrm{II}}(-x, y)$, together with $w_{\mathrm{II}}(x, 0) = w_{\mathrm{II}}(x, 1-a) = 0$. The general solution is

$$w_{\mathrm{II}}(x, y) = \sum_{1}^{\infty} B_n \sin(\alpha_n y) G_n(x), \tag{4}$$

where B_n are to be determined and

$$G_n(x) \equiv \begin{cases} \cos(s_n x), & k \geqslant \alpha_n, \\ \cosh(s_n x), & k < \alpha_n. \end{cases} \tag{5}$$

Now w_{I} and w_{II} are to be matched along their common boundary. Using $w_{\mathrm{I}}(1-a, y) = w_{\mathrm{II}}(-a, y)$ one obtains

$$B_n = \frac{F_n(1-a)}{G_n(-a)} A_n. \tag{6}$$

Matching of derivatives $w_{\mathrm{I}_x}(1-a, y) = w_{\mathrm{II}_x}(-a, y)$ gives

$$\sum A_n [\sin(\alpha_n y) F'_n(1-a) + \alpha_n (-1)^n F_n(y)] = \sum B_n \sin(\alpha_n y) G'_n(-a). \tag{7}$$

Multiplying by $\sin(\alpha_m y)$ and integrating from 0 to $1-a$ yield

$$A_m \left(\frac{1-a}{2}\right) F'_m(1-a) + \sum_{1}^{\infty} \alpha_n (-1)^n L_{mn} = B_m \left(\frac{1-a}{2}\right) G'_m(-a), \tag{8}$$

TABLE 1

Fundamental frequency

a	0	0·01	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1
k	2·2214	2·595	3·088	3·583	4·172	4·934	5·980	7·528	10·08	15·14	30·29	∞

where

$$L_{mn} = \begin{cases} \dfrac{1}{2}\left\{\dfrac{\sin[(1-a)(\alpha_m - s_n)]}{\alpha_m - s_n} - \dfrac{\sin[(1-a)(\alpha_m + s_n)]}{\alpha_m + s_n}\right\}, & k \geqslant \alpha_n, \\ \dfrac{\alpha_m(-1)^{m+1}}{\alpha_m^2 + s_m^2}\sinh[(1-a)s_n], & k < \alpha_n. \end{cases}$$

Truncating the series to N terms, equation (8) gives the algebraic system

$$A_m\left(\frac{1-a}{2}\right)\left[F'_m(1-a) - \frac{F_m(1-a)}{G_m(-a)}G'_m(-a)\right] + \sum_1^N A_n\alpha_n(-1)^n L_{mn} = 0,$$

$$m = 1 \text{ to } N. \tag{9}$$

For non-trivial solutions the determinant of the coefficients of A_n is set to zero. The lowest eigenvalue k is the fundamental frequency.

Table 1 shows the results. Usually $N = 20$ is adequate for a 3-figure accuracy. The value for $a = 0$ is $k = \pi/\sqrt{2}$ for a square membrane.

After k is found we set $A_1 = 1$ in equation (9) and solve for the rest of the A_n. Thus the eigenfunctions w_{I} and w_{II} can be obtained.

3. ASYMPTOTIC FORMULAS

In a previous paper on polygonal membranes with circular core [4], it was found that the frequency decays to the no-core frequency inverse logarithmically as the core tends to zero. A square core is expected to behave similarly. Figure 2 shows the computed k versus $|\ln a|^{-1}$ for $a = 0{\cdot}2, 0{\cdot}1, 0{\cdot}05, 0{\cdot}01$, with N increased to 150 when necessary. We see that, for small a, the tangent line is

$$k \sim \frac{\pi}{\sqrt{2}} + \frac{1{\cdot}65}{|\ln a|}, \quad a \approx 0. \tag{10}$$

Also shown in the figure are the results of Gruner [1] and Gutierrez and Laura [2], both inaccurate for small a. Figure 3 shows the level lines for $a = 0{\cdot}001$. The maximum displacement is at four symmetric points displaced from the center. Note that even for such small cores, the effects on frequency and displacement are considerable.

On the other hand, if a is close to unity, the geometry consists of four long strips plus four corners. The effective dimension is now the strip width $1 - a$ and the frequency is close to the strip frequency $\pi/(1 - a)$. Plotting $\pi - (1 - a)k$ for large a we find the asymptotic

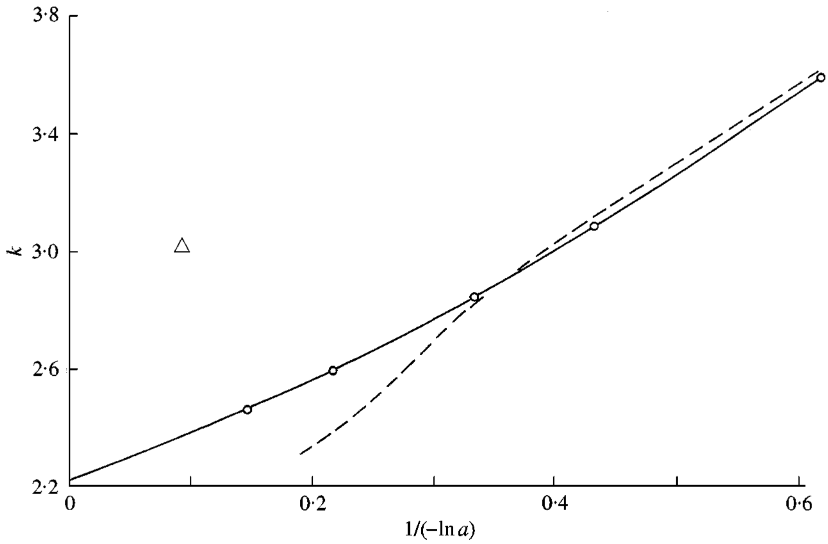

Figure 2. Frequency k versus $|\ln a|^{-1}$. ○, computed values which tend to $\pi/\sqrt{2}$ as $a \to 0$; △ from reference [2]; - - - -, from reference [1].

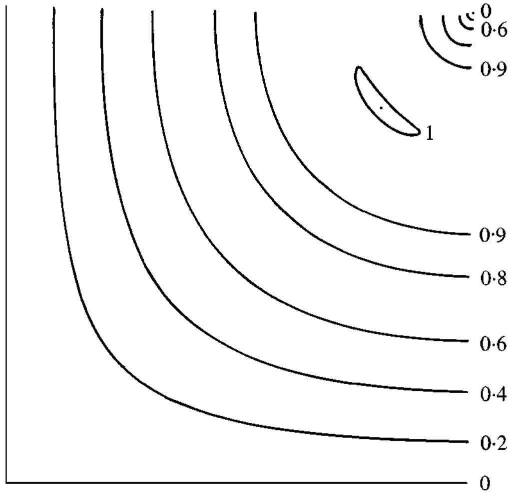

Figure 3. Level lines for $a = 0{\cdot}001$. Only 1/4 of membrane is shown.

formula

$$k \sim \frac{\pi}{1-a} - \frac{0{\cdot}1125}{1-a}, \quad a \approx 1, \tag{11}$$

where the last term is the effect of the corner. Figure 4 shows the comparison of the fundamental frequency with the asymptotic formulas. Note the infinite slope at $a = 0$.

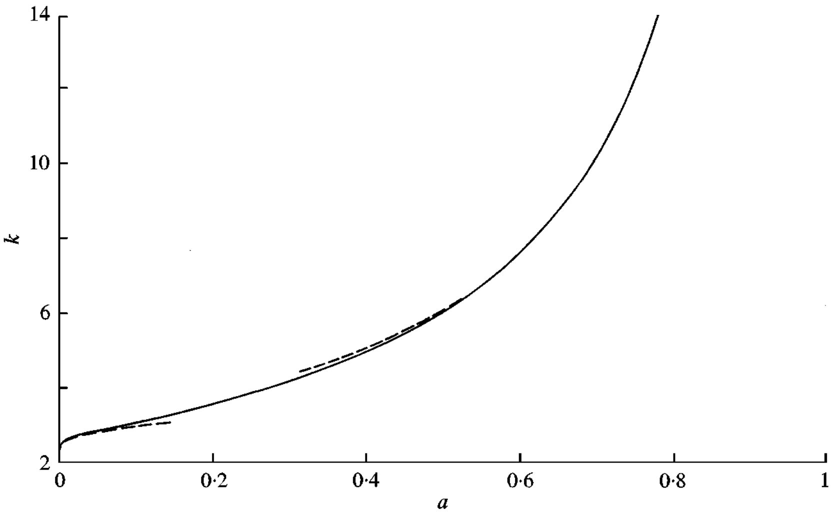

Figure 4. Fundamental frequency for square membrane with square core. Dashed lines (- - - -) are asymptotic formulas (10) or (11).

4. DISCUSSION

For very small cores most methods, including finite differences, would have serious scaling problems. The present paper uses a formulation which can be accurately solved for a as low as 10^{-3}. We indeed find that the frequency tends to the no-core frequency inverse logarithmically as $a \rightarrow 0$.

The effect of a corner is also of interest. Consider a long membrane strip with any number of right-angled turns anywhere. The fundamental frequency normalized by width without turns is π while with at least one turn the frequency decreases to $\pi - 0{\cdot}1125$.

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