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NON-LINEAR VIBRATION CHARACTERISTICS OF CLAMPED LAMINATED SHALLOW SHELLS

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This paper examines non-linear free vibration characteristics of first and second vibration modes of laminated shallow shells with rigidly clamped edges. Non-linear equations of motion for the shells based on the first order shear deformation and classical shell theories are derived by means of Hamilton's principle. We apply Galerkin's procedure to the equations of motion in which eigenvectors for first and second modes of linear vibration obtained by the Ritz method are employed as trial functions. Then simultaneous non-linear ordinary differential equations are derived in terms of amplitudes of the first and second vibration modes. Backbone curves for the first and second vibration modes are solved numerically by the Gauss–Legendre integration method and the shooting method respectively. The effects of lamination sequences and transverse shear deformation on the behavior are discussed. It is also shown that the motion of the first vibration mode affects the response for the second vibration mode.

1. INTRODUCTION

Fiber-reinforced plastics (FRP) are superior in some mechanical properties (e.g., high stiffness-to-weight ratio, high strength-to-weight ratio) to isotropic materials, and have been extensively used in many industrial fields. Therefore, there have been a number of papers concerned with non-linear vibrations of laminated plates and shells.

Chia [1, 2] and Sathyamoorthy [3] collected and reviewed the comprehensive literature dealing with non-linear vibrations of laminated plates. Hui [4] investigated non-linear free vibrations of antisymmetrically laminated plates with geometric imperfections by using Lindstedt's perturbation method. A single-mode analysis of the non-linear dynamic free response of a curved, simply supported orthotropic panel, which is based on the Donnell–Mushtari–Vlasov shell theory, was studied by Raouf and Palazotto [5]. The effects of transverse shear deformation, rotatory inertia and geometrically initial imperfection on non-linear vibration and postbuckling of antisymmetric angle-ply cylindrical thick panels and generally laminated circular cylindrical thick shells with non-uniform boundary conditions were discussed by Fu and Chia [6, 7]. Xu *et al.* [8] derived non-linear equations of transverse motion for a generally laminated, truncated conical shell and solved the non-linear vibration problem by the method of harmonic balance. Cheung and Fu [9] presented non-linear static and dynamic analysis of the symmetric cross-ply shallow spherical shell based on the Timoshenko–Mindlin kinematic hypothesis. On isotropic materials, Kobayashi and Leissa [10] examined large-amplitude vibrations of doubly curved thick shallow shells by the Gauss–Legendre integration method, and the non-linear vibration and dynamic instability of thin shallow spherical and conical shells subjected to periodic transverse and in-plane loads were reported by Ye [11]. However, no papers have been presented for non-linear free vibration characteristics of the first and second (asymmetric first) vibration modes of generally laminated shallow shells reported here.

This paper examines non-linear vibration characteristics of the first and second vibration modes of generally laminated shallow shells with rigidly clamped edge condition. For the purpose of this study, non-linear equations of motion for the shells based on first order shear deformation theory as well as classical shell theory are derived by means of Hamilton's principle. Next, we discretize the non-linear governing equation by using Galerkin's procedure. However, when the laminations sequence is symmetric angle-ply, the mode shapes are complex (see, for example, references [12, 13]) and it is difficult to properly choose trial functions for Galerkin's procedure. Thus, we use the linear strain–displacement relations and calculate eigenvectors of the shells by the Ritz method. By employing the eigenvectors for linear first and second vibration modes as trial functions, we apply Galerkin's procedure to the non-linear governing equation. Then simultaneous non-linear ordinary differential equations in terms of the first and second modes are derived. It is explained clearly that responses for the second vibration mode are affected by a quadratic non-linear term that is not considered in a single-mode analysis. Non-linear dynamic behaviors of the first and second modes of the shells are solved numerically by the Gauss–Legendre integration method and the shooting method, respectively. The effects of shear deformation and lamination sequence on the behavior are discussed.

2. EQUATIONS OF MOTION

Figure 1 shows a laminated shallow shell of rectangular planform, which consists of N layers of an orthotropic sheet, with lengths a and b, thickness h and radii of curvature R_x and R_y. The co-ordinate system (x, y, z) is taken in the midsurface of the shell, as shown in

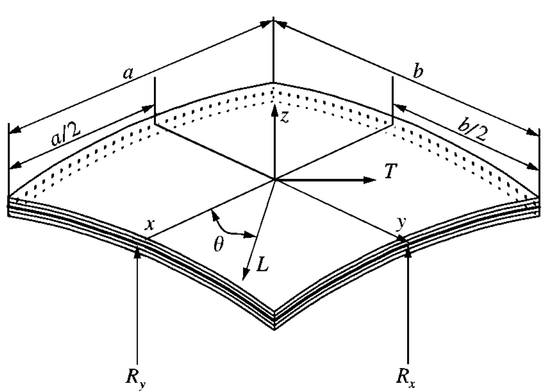

Figure 1. Geometry of a laminated shallow shell and co-ordinate systems.

the figure. The principal directions of elasticity are denoted by L and T, and θ_k is the angle between L- and x-axis in the kth layer. The displacement components are u, v and w at an arbitrary point of the shell in the x, y and z directions respectively.

According to the first order shear deformation theory (FSDT) and the classical shell theory (CST), it is assumed that the in-plane displacements u and v are linear functions of co-ordinate z, and that the transverse displacement w is constant through the thickness of the shell. While the CST adopts Kirchhoff's hypothesis, the FSDT does not adopt it (i.e., it cannot be assumed that normals to the midsurface remain normal to it after deformation). The displacement field based on the FSDT or CST can be given in the following form:

$$u = u_0 + \Delta z\psi_x - (1-\Delta)zw_{0,x}, \quad v = v_0 + \Delta z\psi_y - (1-\Delta)zw_{0,y}, \quad w = w_0, \tag{1}$$

where u_0, v_0 and w_0 are the displacements at the midsurface, ψ_x and ψ_y are the rotations of the midsurface about the y- and x-axis respectively. Indicator Δ is the tracing constant which takes 1 and 0 for the FSDT and CST respectively. The non-linear strain–displacement relations of the shallow shell can be written as

$$\begin{gathered}\varepsilon_x = \varepsilon_x^0 + z\kappa_x, \quad \varepsilon_y = \varepsilon_y^0 + z\kappa_y, \quad \varepsilon_z = 0, \quad \varepsilon_{xy} = \varepsilon_{xy}^0 + z\kappa_{xy}, \\ \varepsilon_{xz} = \Delta(w_{0,x} - u_0/R_x + \psi_x), \quad \varepsilon_{yz} = \Delta(w_{0,y} - v_0/R_y + \psi_y)\end{gathered} \tag{2}$$

in which

$$\begin{gathered}\varepsilon_x^0 = u_{0,x} + w_0/R_x + w_{0,x}^2/2, \quad \varepsilon_y^0 = v_{0,y} + w_0/R_y + w_{0,y}^2/2, \\ \varepsilon_{xy}^0 = u_{0,y} + v_{0,x} + w_{0,x}w_{0,y},\end{gathered} \tag{3}$$

$$\begin{gathered}\kappa_x = \Delta\psi_{x,x} - (1-\Delta)w_{0,xx}, \quad \kappa_y = \Delta\psi_{y,y} - (1-\Delta)w_{0,yy}, \\ \kappa_{xy} = \Delta(\psi_{x,y} + \psi_{y,x}) - 2(1-\Delta)w_{0,xy}.\end{gathered} \tag{4}$$

In these equations, the subscripts following a comma stand for partial differentiation.

The constitutive relations of the shell can be expressed as follows:

$$\begin{Bmatrix}\mathbf{N}\\ \mathbf{M}\end{Bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B} & \mathbf{D}\end{bmatrix}\begin{Bmatrix}\boldsymbol{\varepsilon}^0\\ \boldsymbol{\kappa}\end{Bmatrix}, \tag{5}$$

$$\begin{Bmatrix}Q_y\\ Q_x\end{Bmatrix} = \begin{bmatrix}S_{44} & S_{45}\\ S_{45} & S_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_{yz}\\ \varepsilon_{xz}\end{Bmatrix}, \tag{6}$$

where

$$\mathbf{N} = \{N_x, N_y, N_{xy}\}^{\mathrm{T}}, \quad \mathbf{M} = \{M_x, M_y, M_{xy}\}^{\mathrm{T}}$$

$$\mathbf{A} = \begin{bmatrix}A_{11} & A_{12} & A_{16}\\ A_{12} & A_{22} & A_{26}\\ A_{16} & A_{26} & A_{66}\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}B_{11} & B_{12} & B_{16}\\ B_{12} & B_{22} & B_{26}\\ B_{16} & B_{26} & B_{66}\end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix}D_{11} & D_{12} & D_{16}\\ D_{12} & D_{22} & D_{26}\\ D_{16} & D_{26} & D_{66}\end{bmatrix}, \tag{7}$$

$$\boldsymbol{\varepsilon}^0 = \{\varepsilon_x^0, \varepsilon_y^0, \varepsilon_{xy}^0\}^{\mathrm{T}}, \quad \boldsymbol{\kappa} = \{\kappa_x, \kappa_y, \kappa_{xy}\}^{\mathrm{T}}$$

and N, M and Q are the stress, moment and shear stress resultants respectively. Constants A_{ij}, B_{ij}, D_{ij} and S_{ij} are the stiffness coefficients of the shell, which are derived from

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & 0 & 0 & C_{26} \\ 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & 0 & 0 & C_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{Bmatrix}, \tag{8}$$

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} C_{ij}^{(k)}(1, z, z^2)\,\mathrm{d}z, \quad i, j = 1, 2, 6, \tag{9}$$

$$S_{ij} = K^2 \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} C_{ij}^{(k)}\,\mathrm{d}z, \quad i, j = 4, 5, \tag{10}$$

in which $C_{ij}^{(k)}$ are the stiffness matrix elements expressing the stress–strain relation in the kth layer, K^2 is the shear correction factor and h_k is the distance from the midsurface to the upper surface of the kth layer.

The kinetic energy of the shell can be written as

$$T = \frac{\rho h}{2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \left\{ {u_{0,t}}^2 + {v_{0,t}}^2 + {w_{0,t}}^2 + \varDelta \frac{h^2}{12} ({\psi_{x,t}}^2 + {\psi_{y,t}}^2) \right\} \mathrm{d}x\,\mathrm{d}y, \tag{11}$$

where ρ is mass density of the shell. The strain energy of the shell is given by

$$U = \frac{1}{2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (N_x \varepsilon_x^0 + N_y \varepsilon_y^0 + N_{xy} \varepsilon_{xy}^0 + Q_x \varepsilon_{xz} + Q_y \varepsilon_{yz} + M_x \kappa_x + M_y \kappa_y + M_{xy} \kappa_{xy})\,\mathrm{d}x\,\mathrm{d}y. \tag{12}$$

By substituting equations (11) and (12) into Hamilton's principle

$$\int_{t_0}^{t_1} \delta(T - U)\,\mathrm{d}t = 0, \tag{13}$$

and taking the variation in consideration of equations (2)–(4), the governing equations are derived as follows:

$$\rho h u_{0,tt} = N_{x,x} + N_{xy,y} + \varDelta \frac{Q_x}{R_x}, \quad \rho h v_{0,tt} = N_{xy,x} + N_{y,y} + \varDelta \frac{Q_y}{R_y}, \tag{14}$$

$$\begin{aligned} \rho h w_{0,tt} = {} & -\frac{N_x}{R_x} - \frac{N_y}{R_y} + \varDelta (Q_{x,x} + Q_{y,y}) + (1 - \varDelta)(M_{x,xx} + 2M_{xy,xy} + M_{y,yy}) \\ & + (N_x w_{0,x} + N_{xy} w_{0,y})_{,x} + (N_{xy} w_{0,x} + N_y w_{0,y})_{,y}, \end{aligned} \tag{15}$$

$$\Delta \frac{\rho h^3}{12} \psi_{x,tt} = \Delta(M_{x,x} + M_{xy,y} - Q_x), \quad \Delta \frac{\rho h^3}{12} \psi_{y,tt} = \Delta(M_{xy,x} + M_{y,y} - Q_y). \tag{16}$$

For convenience of the analysis, the following dimensionless quantities are introduced:

$$\xi = \frac{2x}{a}, \quad \eta = \frac{2y}{b}, \quad U = \frac{u_0}{h}, \quad V = \frac{v_0}{h}, \quad W = \frac{w_0}{h},$$

$$\alpha = \frac{a}{b}, \quad H = \frac{h}{a}, \quad r_x = \frac{R_x}{a}, \quad r_y = \frac{R_y}{a}, \tag{17}$$

$$\mathbf{A}^* = \frac{4a^2}{D_0}\mathbf{A}, \quad \mathbf{B}^* = \frac{4a^2}{D_0 h}\mathbf{B}, \quad \mathbf{D}^* = \frac{4a^2}{D_0 h^2}\mathbf{D}, \quad S^*_{ij} = \frac{4a^2}{D_0} S_{ij}, \quad \tau = \frac{1}{a^2}\sqrt{\frac{D_0}{\rho h}}\, t$$

in which

$$D_0 = \frac{E_T h^3}{12(1 - \nu_{LT}\nu_{TL})}, \tag{18}$$

and τ is non-dimensional time. By using equation (17), equations (14)–(16) can be rewritten in non-dimensional forms as

$$F_U = U_{,\tau\tau} - \bar{N}_{\xi,\xi} - \alpha \bar{N}_{\xi\eta,\eta} - \Delta \frac{\bar{Q}_\xi}{2r_x} = 0, \tag{19}$$

$$F_V = V_{,\tau\tau} - \bar{N}_{\xi\eta,\xi} - \alpha \bar{N}_{\eta,\eta} - \Delta \frac{\bar{Q}_\eta}{2r_y} = 0, \tag{20}$$

$$F_W = W_{,\tau\tau} + \frac{\bar{N}_\xi}{2r_x} + \frac{\bar{N}_\eta}{2r_y} - \Delta(\bar{Q}_{\xi,\xi} + \alpha \bar{Q}_{\eta,\eta}) - 2(1-\Delta)H(\bar{M}_{\xi,\xi\xi} + 2\alpha \bar{M}_{\xi\eta,\xi\eta} + \alpha^2 \bar{M}_{\eta,\eta\eta})$$

$$- 2H(\bar{N}_\xi W_{,\xi} + \alpha \bar{N}_{\xi\eta} W_{,\eta})_{,\xi} - 2H\alpha(\bar{N}_{\xi\eta} W_{,\xi} + \alpha \bar{N}_\eta W_{,\eta})_{,\eta} = 0, \tag{21}$$

$$F_X = \Delta\left(\frac{\psi_{x,\tau\tau}}{12} - \bar{M}_{\xi,\xi} - \alpha \bar{M}_{\xi\eta,\eta} - \frac{\bar{Q}_\xi}{2H}\right) = 0, \tag{22}$$

$$F_Y = \Delta\left(\frac{\psi_{y,\tau\tau}}{12} - \bar{M}_{\xi\eta,\xi} - \alpha \bar{M}_{\eta,\eta} - \frac{\bar{Q}_\eta}{2H}\right) = 0, \tag{23}$$

where

$$\begin{Bmatrix} \bar{\mathbf{N}} \\ \bar{\mathbf{M}} \end{Bmatrix} = \begin{bmatrix} \mathbf{A}^* & \mathbf{B}^* \\ \mathbf{B}^* & \mathbf{D}^* \end{bmatrix} \begin{Bmatrix} \bar{\boldsymbol{\varepsilon}}^0 \\ \bar{\boldsymbol{\kappa}} \end{Bmatrix}, \quad \begin{Bmatrix} \bar{Q}_y \\ \bar{Q}_x \end{Bmatrix} = \begin{bmatrix} S^*_{44} & S^*_{45} \\ S^*_{45} & S^*_{55} \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_{yz} \\ \bar{\varepsilon}_{xz} \end{Bmatrix},$$

$$\bar{\mathbf{N}} = \{\bar{N}_\xi, \bar{N}_\eta, \bar{N}_{\xi\eta}\}^{\mathrm{T}}, \quad \bar{\mathbf{M}} = \{\bar{M}_\xi, \bar{M}_\eta, \bar{M}_{\xi\eta}\}^{\mathrm{T}},$$

$$\bar{\boldsymbol{\varepsilon}}^0 = \{\bar{\varepsilon}^0_\xi, \bar{\varepsilon}^0_\eta, \bar{\varepsilon}^0_{\xi\eta}\}^{\mathrm{T}}, \quad \bar{\boldsymbol{\kappa}} = \{\bar{\kappa}_\xi, \bar{\kappa}_\eta, \bar{\kappa}_{\xi\eta}\}^{\mathrm{T}},$$

$$\bar{\varepsilon}^0_\xi = U_{,\xi} + \frac{W}{2r_x} + HW^2_{,\xi}, \quad \bar{\varepsilon}^0_\eta = \alpha V_{,\eta} + \frac{W}{2r_y} + H\alpha^2 W^2_{,\eta},$$

$$\bar{\varepsilon}^0_{\xi\eta} = \alpha U_{,\eta} + V_{,\xi} + 2H\alpha W_{,\xi} W_{,\eta}, \quad \bar{\varepsilon}_{\xi z} = \Delta\left(W_{,\xi} - \frac{U}{2r_x} + \frac{\psi_x}{2H}\right), \tag{24}$$

$$\bar{\varepsilon}_{\eta z} = \Delta\left(\alpha W_{,\eta} - \frac{V}{2r_y} + \frac{\psi_y}{2H}\right), \quad \bar{\kappa}_\xi = \Delta\psi_{x,\xi} - 2(1-\Delta)HW_{,\xi\xi},$$

$$\bar{\kappa}_\eta = \Delta\alpha\psi_{y,\eta} - 2(1-\Delta)H\alpha^2 W_{,\eta\eta}, \quad \bar{\kappa}_{\xi\eta} = \Delta(\alpha\psi_{x,\eta} + \psi_{y,\xi}) - 4(1-\Delta)H\alpha W_{,\xi\eta}.$$

3. LINEAR ANALYSIS

In this section, linear strain–displacement relations are considered. Linear natural frequencies and eigenfunctions of laminated shallow shells based on the FSDT and CST are obtained using the Ritz method.

The maximum kinetic energy of the shell is given by

$$T_{max} = \frac{D_0 H^2 \lambda^2}{8\alpha} \int_{-1}^{1}\int_{-1}^{1} \left\{U^2 + V^2 + W^2 + \frac{\Delta}{12}(\psi_x^2 + \psi_y^2)\right\} \mathrm{d}\xi\,\mathrm{d}\eta, \tag{25}$$

in which λ is the non-dimensional linear natural frequency, which is related to the linear natural frequency ω by $\lambda = \omega a^2(\rho h/D_0)^{1/2}$. The maximum strain energy of the shell is expressed in the matrix form

$$U_{max} = \frac{D_0 H^2}{8\alpha}\int_{-1}^{1}\int_{-1}^{1}\left(\begin{Bmatrix}\bar{\boldsymbol{\varepsilon}}_l \\ \bar{\boldsymbol{\kappa}}\end{Bmatrix}^{\mathrm{T}} \begin{bmatrix}\mathbf{A}^* & \mathbf{B}^* \\ \mathbf{B}^* & \mathbf{D}^*\end{bmatrix} \begin{Bmatrix}\bar{\boldsymbol{\varepsilon}}_l \\ \bar{\boldsymbol{\kappa}}\end{Bmatrix} + \Delta \begin{Bmatrix}\bar{\varepsilon}_{\eta z} \\ \bar{\varepsilon}_{\xi z}\end{Bmatrix}^{\mathrm{T}} \begin{bmatrix}S^*_{44} & S^*_{45} \\ S^*_{45} & S^*_{55}\end{bmatrix} \begin{Bmatrix}\bar{\varepsilon}_{\eta z} \\ \bar{\varepsilon}_{\xi z}\end{Bmatrix}\right)\mathrm{d}\xi\,\mathrm{d}\eta, \tag{26}$$

where

$$\boldsymbol{\varepsilon}_l = \{\varepsilon^0_{\xi l}, \varepsilon^0_{\eta l}, \varepsilon^0_{\xi\eta l}\}^{\mathrm{T}} = \{U_{,\xi} + W/(2r_x), \alpha V_{,\eta} + W/(2r_y), \alpha U_{,\eta} + V_{,\xi}\}^{\mathrm{T}}. \tag{27}$$

In the present study, we consider shallow shells with rigidly clamped edges. The boundary conditions are

$$W = U = V = \psi_x = \psi_y = 0 \text{ at } \xi = \pm 1 \text{ and } \eta = \pm 1 \quad \text{(for FSDT)},$$
$$W = U = V = W_{,\xi} = W_{,\eta} = 0 \text{ at } \xi = \pm 1 \text{ and } \eta = \pm 1 \quad \text{(for CST)}. \tag{28}$$

The displacements of the shell are approximated using power functions [14–16] which satisfy the boundary conditions (28) as

$$U = \sum_{i=1}^{I} \sum_{j=1}^{J} a_{ij} U_{\xi i}(\xi) U_{\eta j}(\eta) = \sum_{i=1}^{I} \sum_{j=1}^{J} a_{ij} \xi^{i-1}(1-\xi^2)\eta^{j-1}(1-\eta^2),$$

$$V = \sum_{i=1}^{I} \sum_{j=1}^{J} b_{ij} V_{\xi i}(\xi) V_{\eta j}(\eta) = \sum_{i=1}^{I} \sum_{j=1}^{J} b_{ij} \xi^{i-1}(1-\xi^2)\eta^{j-1}(1-\eta^2),$$

$$W = \sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} W_{\xi i}(\xi) W_{\eta j}(\eta) = \sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} \xi^{i-1}(1+\xi)^{\beta}(1-\xi)^{\beta}\eta^{j-1}(1+\eta)^{\beta}(1-\eta)^{\beta}, \quad (29)$$

$$\psi_x = \sum_{i=1}^{I} \sum_{j=1}^{J} d_{ij} \psi_{x\xi i}(\xi) \psi_{x\eta j}(\eta) = \sum_{i=1}^{I} \sum_{j=1}^{J} d_{ij} \xi^{i-1}(1-\xi^2)\eta^{j-1}(1-\eta^2),$$

$$\psi_y = \sum_{i=1}^{I} \sum_{j=1}^{J} e_{ij} \psi_{y\xi i}(\xi) \psi_{y\eta j}(\eta) = \sum_{i=1}^{I} \sum_{j=1}^{J} e_{ij} \xi^{i-1}(1-\xi^2)\eta^{j-1}(1-\eta^2),$$

in which β takes 1 and 2 for the FSDT and CST respectively, and a_{ij}, b_{ij}, c_{ij}, d_{ij} and e_{ij} are unknown coefficients. Substituting equation (29) into equations (25) and (26) and using the conditions for a stationary value of Lagrangian $L = T_{max} - U_{max}$,

$$\frac{\partial L}{\partial a_{ij}} = \frac{\partial L}{\partial b_{ij}} = \frac{\partial L}{\partial c_{ij}} = \frac{\partial L}{\partial d_{ij}} = \frac{\partial L}{\partial e_{ij}} = 0, \quad (30)$$

the frequency equation is derived as

$$\begin{bmatrix} U^{U}_{aijkl} & U^{U}_{bijkl} & U^{U}_{cijkl} & U^{U}_{dijkl} & U^{U}_{eijkl} \\ & U^{V}_{bijkl} & U^{V}_{cijkl} & U^{V}_{dijkl} & U^{V}_{eijkl} \\ & & U^{W}_{cijkl} & U^{W}_{dijkl} & U^{W}_{eijkl} \\ & & & U^{\psi_x}_{dijkl} & U^{\psi_x}_{eijkl} \\ & \mathrm{Sym.} & & & U^{\psi_y}_{eijkl} \end{bmatrix} \begin{Bmatrix} a_{kl} \\ b_{kl} \\ c_{kl} \\ d_{kl} \\ e_{kl} \end{Bmatrix}$$

$$= \lambda^2 \begin{bmatrix} T^{U}_{aijkl} & 0 & 0 & 0 & 0 \\ & T^{V}_{bijkl} & 0 & 0 & 0 \\ & & T^{W}_{cijkl} & 0 & 0 \\ & & & T^{\psi_x}_{dijkl} & 0 \\ & \mathrm{Sym.} & & & T^{\psi_y}_{eijkl} \end{bmatrix} \begin{Bmatrix} a_{kl} \\ b_{kl} \\ c_{kl} \\ d_{kl} \\ e_{kl} \end{Bmatrix} \quad (k = 1, 2, \ldots, I;\ l = 1, 2, \ldots, J). \quad (31)$$

The elements of the matrix are given in Appendix A. The non-dimensional frequency λ is obtained as the eigenvalue of the equation, and the eigenfunctions of linear vibration are determined by calculating eigenvectors a_{kl}, b_{kl}, c_{kl}, d_{kl} and e_{kl}. In the case of the CST, it is noted that d_{kl} and e_{kl} vanish.

4. NON-LINEAR ANALYSIS

In this paper, we examine non-linear vibration characteristics of first and second vibration modes of the shells. Thus, we assume that the displacements can be expressed by using the eigenfunctions of two vibration modes which are obtained in the previous section:

$$U = \sum_{i=1}^{I}\sum_{j=1}^{J}\{U_1(\tau)a_{ij}^{(1)} + U_2(\tau)a_{ij}^{(2)}\}U_{\xi i}(\xi)U_{\eta j}(\eta),$$

$$V = \sum_{i=1}^{I}\sum_{j=1}^{J}\{V_1(\tau)b_{ij}^{(1)} + V_2(\tau)b_{ij}^{(2)}\}V_{\xi i}(\xi)V_{\eta j}(\eta),$$

$$W = \sum_{i=1}^{I}\sum_{j=1}^{J}\{W_1(\tau)c_{ij}^{(1)} + W_2(\tau)c_{ij}^{(2)}\}W_{\xi i}(\xi)W_{\eta j}(\eta), \tag{32}$$

$$\psi_x = \sum_{i=1}^{I}\sum_{j=1}^{J}\{X_1(\tau)d_{ij}^{(1)} + X_2(\tau)d_{ij}^{(2)}\}\psi_{x\xi i}(\xi)\psi_{x\eta j}(\eta),$$

$$\psi_y = \sum_{i=1}^{I}\sum_{j=1}^{J}\{Y_1(\tau)e_{ij}^{(1)} + Y_2(\tau)e_{ij}^{(2)}\}\psi_{y\xi i}(\xi)\psi_{y\eta j}(\eta),$$

where time functions $U(\tau)$, $V(\tau)$, $W(\tau)$, $X(\tau)$ and $Y(\tau)$ with subscripts 1 and 2 are amplitudes of the first and second modes respectively. In a similar way, coefficients a_{ij}, b_{ij}, c_{ij}, d_{ij} and e_{ij} with superscripts (1) and (2) denote eigenvectors of the first and second modes respectively. By substituting equation (32) into the equations of motion (19)–(23) and applying Galerkin's procedure,

$$\int_{-1}^{1}\int_{-1}^{1} F_U \sum_{i=1}^{I}\sum_{j=1}^{J} a_{ij}^{(l)}U_{\xi i}U_{\eta j}\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \quad \int_{-1}^{1}\int_{-1}^{1} F_V \sum_{i=1}^{I}\sum_{j=1}^{J} b_{ij}^{(l)}V_{\xi i}V_{\eta j}\,\mathrm{d}\xi\,\mathrm{d}\eta = 0,$$

$$\int_{-1}^{1}\int_{-1}^{1} F_W \sum_{i=1}^{I}\sum_{j=1}^{J} c_{ij}^{(l)}W_{\xi i}W_{\eta j}\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \quad \int_{-1}^{1}\int_{-1}^{1} F_X \sum_{i=1}^{I}\sum_{j=1}^{J} d_{ij}^{(l)}\psi_{x\xi i}\psi_{x\eta j}\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \tag{33}$$

$$\int_{-1}^{1}\int_{-1}^{1} F_Y \sum_{i=1}^{I}\sum_{j=1}^{J} e_{ij}^{(l)}\psi_{y\xi i}\psi_{y\eta j}\,\mathrm{d}\xi\,\mathrm{d}\eta = 0 \quad (l = 1,\, 2),$$

the following equations are obtained:

$$C_{U1}^{(1)}U_{1,\tau\tau} + C_{U2}^{(1)}U_1 + C_{U3}^{(1)}V_1 + C_{U4}^{(1)}W_1 + C_{U5}^{(1)}X_1 + C_{U6}^{(1)}Y_1 + C_{U7}^{(1)}W_1^2 + C_{U8}^{(1)}W_2^2 = 0,$$

$$C_{V1}^{(1)}V_{1,\tau\tau} + C_{V2}^{(1)}U_1 + C_{V3}^{(1)}V_1 + C_{V4}^{(1)}W_1 + C_{V5}^{(1)}X_1 + C_{V6}^{(1)}Y_1 + C_{V7}^{(1)}W_1^2 + C_{V8}^{(1)}W_2^2 = 0,$$

$$\begin{aligned} &C_{W1}^{(1)}W_{1,\tau\tau} + C_{W2}^{(1)}U_1 + C_{W3}^{(1)}V_1 + C_{W4}^{(1)}W_1 + C_{W5}^{(1)}X_1 + C_{W6}^{(1)}Y_1 + C_{W7}^{(1)}U_1W_1 \\ &\quad + C_{W8}^{(1)}V_1W_1 + C_{W9}^{(1)}U_2W_2 + C_{W10}^{(1)}V_2W_2 + C_{W11}^{(1)}W_1^2 + C_{W12}^{(1)}W_2^2 + C_{W13}^{(1)}W_1X_1 \\ &\quad + C_{W14}^{(1)}W_1Y_1 + C_{W15}^{(1)}W_2X_2 + C_{W16}^{(1)}W_2Y_2 + C_{W17}^{(1)}W_1^3 + C_{W18}^{(1)}W_1W_2^2 = 0, \end{aligned} \tag{34}$$

$$C_{X1}^{(1)}X_{1,\tau\tau} + C_{X2}^{(1)}U_1 + C_{X3}^{(1)}V_1 + C_{X4}^{(1)}W_1 + C_{X5}^{(1)}X_1 + C_{X6}^{(1)}Y_1 + C_{X7}^{(1)}W_1^2 + C_{X8}^{(1)}W_2^2 = 0,$$

$$C_{Y1}^{(1)}Y_{1,\tau\tau} + C_{Y2}^{(1)}U_1 + C_{Y3}^{(1)}V_1 + C_{Y4}^{(1)}W_1 + C_{Y5}^{(1)}X_1 + C_{Y6}^{(1)}Y_1 + C_{Y7}^{(1)}W_1^2 + C_{Y8}^{(1)}W_2^2 = 0;$$

$$C_{U1}^{(2)}U_{2,\tau\tau} + C_{U2}^{(2)}U_2 + C_{U3}^{(2)}V_2 + C_{U4}^{(2)}W_2 + C_{U5}^{(2)}X_2 + C_{U6}^{(2)}Y_2 + C_{U7}^{(2)}W_1W_2 = 0,$$

$$C_{V1}^{(2)}V_{2,\tau\tau} + C_{V2}^{(2)}U_2 + C_{V3}^{(2)}V_2 + C_{V4}^{(2)}W_2 + C_{V5}^{(2)}X_2 + C_{V6}^{(2)}Y_2 + C_{V7}^{(2)}W_1W_2 = 0,$$

$$C_{W1}^{(2)}W_{2,\tau\tau} + C_{W2}^{(2)}U_2 + C_{W3}^{(2)}V_2 + C_{W4}^{(2)}W_2 + C_{W5}^{(2)}X_2 + C_{W6}^{(2)}Y_2 + C_{W7}^{(2)}U_1W_2$$

$$+ C_{W8}^{(2)}V_1W_2 + C_{W9}^{(2)}U_2W_1 + C_{W10}^{(2)}V_2W_1 + C_{W11}^{(2)}W_1W_2 + C_{W12}^{(2)}W_1X_2 \quad (35)$$

$$+ C_{W13}^{(2)}W_1Y_2 + C_{W14}^{(2)}W_2X_1 + C_{W15}^{(2)}W_2Y_1 + C_{W16}^{(2)}W_1^2W_2 + C_{W17}^{(2)}W_2^3 = 0,$$

$$C_{X1}^{(2)}X_{2,\tau\tau} + C_{X2}^{(2)}U_2 + C_{X3}^{(2)}V_2 + C_{X4}^{(2)}W_2 + C_{X5}^{(2)}X_2 + C_{X6}^{(2)}Y_2 + C_{X7}^{(2)}W_1W_2 = 0,$$

$$C_{Y1}^{(2)}Y_{2,\tau\tau} + C_{Y2}^{(2)}U_2 + C_{Y3}^{(2)}V_2 + C_{Y4}^{(2)}W_2 + C_{Y5}^{(2)}X_2 + C_{Y6}^{(2)}Y_2 + C_{Y7}^{(2)}W_1W_2 = 0.$$

Coefficients ($C_{U1}^{(1)}$, $C_{V1}^{(1)}$, etc.) in the above equations are obtained by integrating equation (33) numerically by the use of the software package *Mathematica* [17]. In equations (34) and (35), X_i and Y_i vanish for the case of the CST.

If in-plane and rotatory inertias in equations (34) and (35) are neglected, two sets of ordinary differential equations in terms of the transverse displacements W_1 and W_2 are derived by eliminating U_1, U_2, V_1, V_2, X_1, X_2, Y_1 and Y_2 from equations (34) and (35):

$$W_{1,\tau\tau} + \omega_1^2W_1 + G_1W_1^2 + G_2W_2^2 + G_3W_1^3 + G_4W_1W_2^2 = 0,$$
$$W_{2,\tau\tau} + \omega_2^2W_2 + G_5W_1W_2 + G_6W_1^2W_2 + G_7W_2^3 = 0, \quad (36)$$

where ω_i and G_i are the non-dimensional linear natural frequencies and the non-dimensional coefficients of the non-linear terms respectively. As can be seen from equation (36), $W_2 = 0$ as long as the shell does not possess internal resonances between the first and second modes when W_1 oscillates. However, when W_2 oscillates, W_1 is activated by the non-linear term $G_2W_2^2$. Thus, the motion of the first mode affects the response for the second mode through the non-linear terms $G_5W_1W_2$ and $G_6W_1^2W_2$. The effect of the quadratic non-linear term $G_5W_1W_2$ on the non-linear vibration of the second mode is not considered in a single-mode analysis with considers only the second mode. In the case of flat plates, there are no quadratic non-linear terms (e.g., $G_2W_2^2$ and so on), and the modal interaction between the first and second modes does not occur. Consequently, this modal interaction is specific to shells.

Since this paper does not treat shells with internal resonances between the first and second modes, W_2 can be neglected to examine non-linear free vibrations for the first mode:

$$W_{1,\tau\tau} + \omega_1^2W_1 + G_1W_1^2 + G_3W_1^3 = 0. \quad (37)$$

Performing some manipulations, the equation expressing the relationship between non-linear frequency ω_{nl} and the amplitude W_1 is derived from equation (37) as [10]

$$\omega_{nl} = \frac{\pi}{\int_{W_{1\,min}}^{W_{1\,max}} \frac{\mathrm{d}W_1}{\sqrt{C - \omega_1^2 W_1^2 - \frac{2}{3}G_1 W_1^3 - \frac{1}{2}G_3 W_1^4}}}, \tag{38}$$

in which

$$C = \omega_1^2 W_{1\,max}^2 + \tfrac{2}{3}G_1 W_{1\,max}^3 + \tfrac{1}{2}G_3 W_{1\,max}^4, \tag{39}$$

and $W_{1,max}$ and $W_{1\,min}$ are the maximum and minimum amplitudes respectively. Using the conditions,

$$\begin{aligned} W_1 &= W_{1\,max}, \quad W_{1,\tau} = 0, \\ W_1 &= W_{1\,min}, \quad W_{1,\tau} = 0, \end{aligned} \tag{40}$$

non-linear frequencies, which are dependent upon the amplitude, of the first mode are calculated by applying the Gauss–Legendre integration method to equation (38). For further details, the reader should refer to reference [10].

On the other hand, in order to examine non-linear vibration characteristics for the second mode, we use the shooting method [18, 19]. According to the shooting method, we convert equation (36) into first-order differential equations

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}\hat{t}} = \mathbf{Z}, \tag{41}$$

where

$$\begin{aligned} \mathbf{z} &= [z_1, z_2, z_3, z_4]^{\mathrm{T}} = [W_1, \dot{W}_1, W_2, \dot{W}_2]^{\mathrm{T}}, \\ \mathbf{Z} &= [Z_1, Z_2, Z_3, Z_4]^{\mathrm{T}}, \\ Z_1 &= z_2, \\ Z_2 &= -\{\omega_1^2 z_1 + (G_1 + G_3 z_1) z_1^2 + (G_2 + G_4 z_1) z_3^2\}/\omega_2^2, \\ Z_3 &= z_4, \\ Z_4 &= -z_3 - \{(G_5 + G_6 z_1) z_1 z_3 + G_7 z_3^3 - F\cos(\omega_{nl}/\omega_2)\hat{t}\}/\omega_2^2. \end{aligned} \tag{42}$$

Here, $\hat{t}$ is a new time variable which is defined as $\hat{t} = \omega_2 \tau$, a dot denotes the derivative with respect to $\hat{t}$, and F and ω_{nl} are the amplitude and frequency of an additional harmonic excitation. We calculate frequency–response curves corresponding to backbone curves on the assumption that F is very small.

5. NUMERICAL RESULTS AND DISCUSSION

Some numerical examples on non-linear free vibration behavior of laminated shallow shells with rigidly clamped edges are presented in this section. It is assumed that the shells consist of graphite-epoxy layers and each layer has the same thickness and the following material properties:

$$E_L = 138\ \text{GPa}, \quad E_T = 8{\cdot}96\ \text{GPa}, \quad G_{LT} = 7{\cdot}1\ \text{GPa}, \quad G_{LZ} = E_T/2, \quad \nu_{LT} = 0{\cdot}3.$$

Unless specified, the FSDT is employed in the following numerical examples and the shear correction factor K^2 is taken as $K^2 = 5/6$.

5.1. NON-LINEAR VIBRATION OF THE FIRST VIBRATION MODE

Not only linear natural frequencies but also non-linear frequencies of the shells are dependent upon the number of series (29). Table 1 shows convergence characteristics of linear and non-linear frequencies as the number of terms I and J of equation (29) increases. The calculation is carried out for a symmetric laminated angle-ply shallow shell ($\theta = 45°/-45°/45$) with square planform. Two types of the thickness ratios ($H = 0{\cdot}01, 0{\cdot}1$) are considered here. With an increase in number of the series, the linear and non-linear frequencies converge within three or four significant digits. They can be regarded as well-converged if the series are equal to or larger than $I \times J = 8 \times 8$. Therefore, $I \times J = 8 \times 8$ is used in the following numerical examples. On the other hand, λ_1 is obtained by considering the effects of in-plane and rotatory inertias (the Ritz method), whereas ω_1 is obtained by neglecting the effects (Galerkin's procedure). While ω_1 agrees fairly with λ_1 for the thin shell ($H = 0{\cdot}01$), ω_1 is slightly higher than λ_1 for the thick shell. However, the errors are less than 1%, and it can be said that the analysis neglecting the in-plane and rotatory inertias is valid to study the non-linear vibration.

Non-linear vibration characteristics for the first vibration mode of the shells are shown in Figures 2–5. The linear natural frequencies ω_1 of the shells used in Figures 2–4 and 5 are listed in Tables 2 and 3 respectively.

TABLE 1

Convergence of frequency parameters on the first vibration mode of symmetric angle-ply laminated shallow shells ($\theta = 45°/-45°/45°$, $r_x = r_y = 10$, $\alpha = 1$)

$H(=h/a)$	$I \times J$	$\omega_1(\lambda_1)$	ω_{nl}: $W_{1\,max} = 1$	ω_{nl}: $W_{1\,max} = 2$
0·01	4×4	142·9 (142·9)	140·1	196·1
	6×6	140·2 (140·1)	147·0	213·4
	8×8	140·1 (140·1)	147·1	213·7
	10×10	140·1 (140·1)	147·1	213·7
0·1	4×4	63·27 (62·90)	89·23	139·6
	6×6	63·17 (62·80)	88·78	138·7
	8×8	63·16 (62·79)	88·70	138·5
	10×10	63·15 (62·78)	88·71	138·5

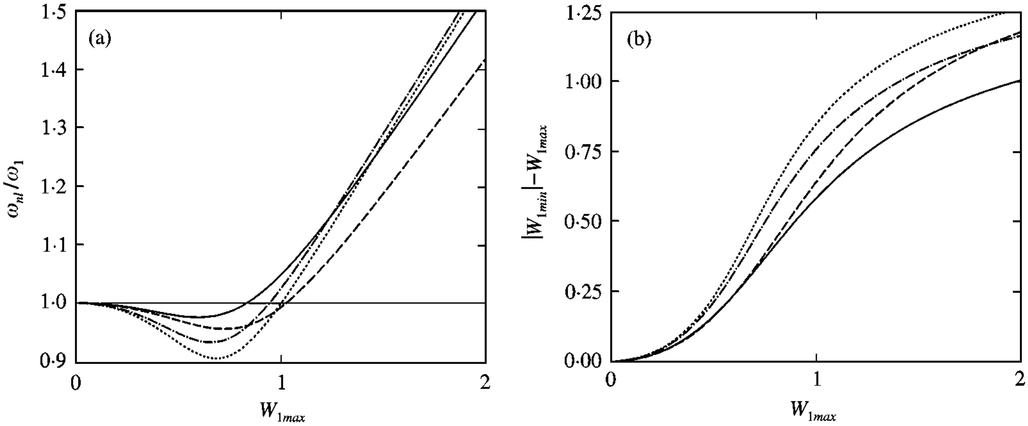

Figure 2. Effect of lamination sequence on non-linear vibration of the first mode ($\alpha = 1$, $r_x = 10$, $r_x/r_y = 1$, $H = 0{\cdot}01$): (a) ω_{nl}/ω_1 versus $W_{1\,max}$, (b) $|W_{1\,min}| - W_{1\,max}$ versus $W_{1\,max}$: ——, $\theta = 45°/-45°/45°$; – – – – –, $\theta = 0°/90°/0°$; –·–·–·–, $\theta = 45°/-45°$; ·······, $\theta = 0°/90°$.

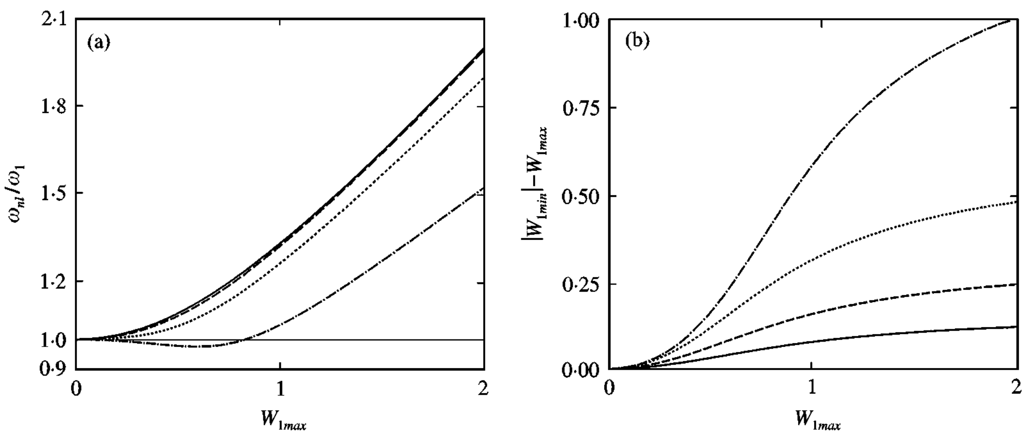

Figure 3. Effect of radius of curvature on non-linear vibration of the first mode ($\theta = 45°/-45°/45°$, $\alpha = 1$, $r_x/r_y = 1$, $H = 0{\cdot}01$): (a) ω_{nl}/ω_1 versus $W_{1\,max}$, (b) $|W_{1\,min}| - W_{1\,max}$ versus $W_{1\,max}$: ——, $r_x = 100$; – – – – –, $r_x = 50$; ·······, $r_x = 25$; –·–·–·–, $r_x = 10$.

Figure 2 presents the effect of lamination sequences of spherical shells on the non-linear free vibrations. It was reported in reference [10] that absolute value of maximum inward displacement $W_{1\,min}$ is larger than that of maximum outward displacement $W_{1\,max}$ in a vibratory cycles. Figures 2(a) and 2(b) show the ratio of the non-linear frequency to the linear frequency ω_{nl}/ω_1 and the difference between the maximum inward and outward displacements $|W_{1\,min}| - W_{1\,max}$ respectively. It is found from Figure 2(a) that soft-spring behavior appears in all lamination sequences. We observe the tendency that the increase or decrease in the frequency is more pronounced for the lamination sequences which include the effect of bending–stretching coupling ($\theta = 45°/-45°$, $0°/90°$) than for the lamination sequences which do not include the effect ($\theta = 0°/90°/0°$, $45°/-45°/45°$). It can be seen in Figure 2(b) that the difference $|W_{1\,min}| - W_{1\,max}$ increases radically with an increase of $W_{1\,max}$.

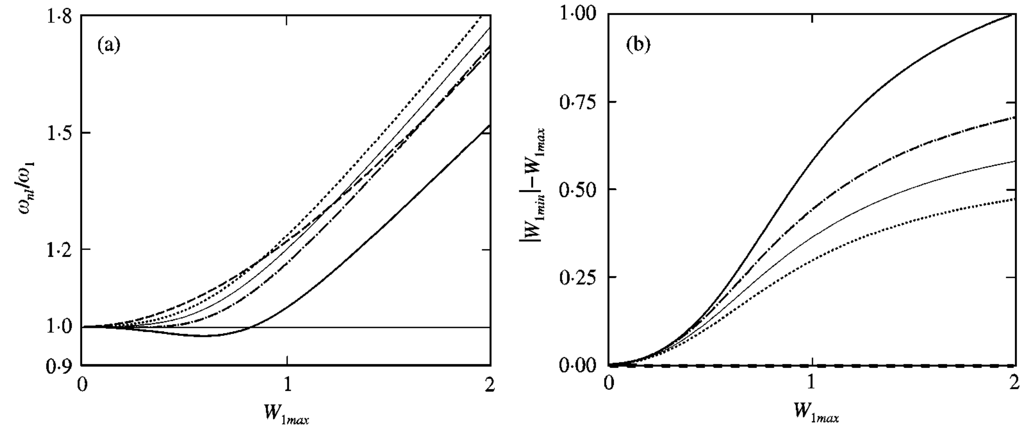

Figure 4. Effects of curvature and aspect ratios on non-linear vibration of the first mode ($\theta = 45°/-45°/45°$, $r_x = 10$, $H = 0{\cdot}01$): (a) ω_{nl}/ω_1 versus $W_{1\,max}$, (b) $|W_{1\,min}| - W_{1\,max}$ versus $W_{1\,max}$: ——, $r_x/r_y = 1$, $\alpha = 1$; —— $r_x/r_y = 0$, $\alpha = 1$; - - - - -, $r_x/r_y = -1$, $\alpha = 1$; –·–·–·–, $r_x/r_y = 1$, $\alpha = 1{\cdot}5$; ·······, $r_x/r_y = 1$, $\alpha = 2$.

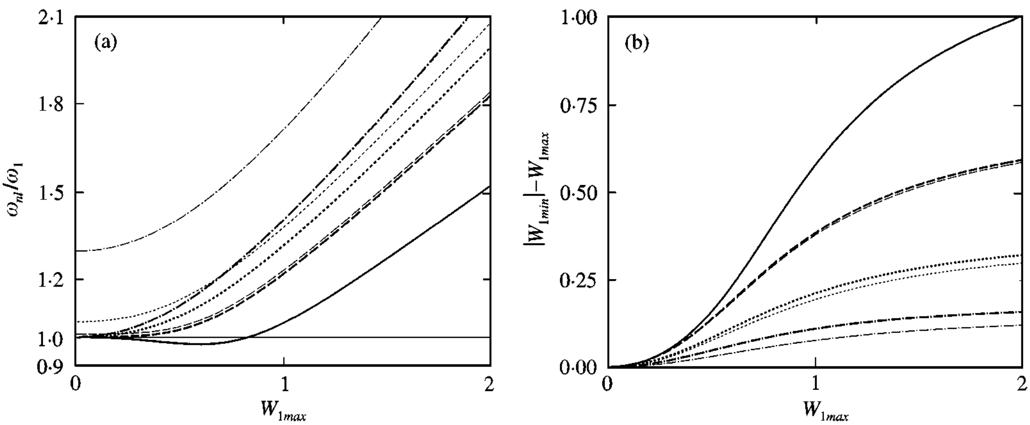

Figure 5. Effect of thickness ratio on non-linear vibration of the first mode ($\theta = 45°/-45°/45°$, $\alpha = 1$, $r_x = 10$, $r_x/r_y = 1$): (a) ω_{nl}/ω_1 versus $W_{1\,max}$, (b) $|W_{1\,min}| - W_{1\,max}$ versus $W_{1\,max}$: ——, $H = 0{\cdot}01$; - - - - -, $H = 0{\cdot}02$; ·······, $H_x = 0{\cdot}04$; –·–·–·–, $H_x = 0{\cdot}1$.

Figure 3 shows the effect of radius of curvature on the non-linear dynamic properties of a spherical symmetric laminated shallow shell ($\theta = 45°/-45°/45°$). As the radius of curvature increases, the hard-spring response becomes strong, and the soft-spring behavior does not appear in the shells with $r_x \geqslant 25$. However, it should be noted that there is the difference $|W_{1\,min}| - W_{1\,max}$ in the shells with $r_x \geqslant 25$. Though the backbone curves for $r_x = 100$ almost agree with those for $r_x = 50$, $|W_{1\,min}| - W_{1\,max}$ for $r_x = 50$ is nearly the double of that for $r_x = 100$ during $1 \leqslant W_{1\,max} \leqslant 2$.

In Figure 4 the effects of curvature ratio (r_x/r_y) and aspect ratio of a symmetric laminated shallow shell with $r_x = 10$ on the non-linear vibration is presented. It is significant to note that $r_x/r_y = 1$, 0 and -1 correspond to spherical, cylindrical and hyperbolic paraboloid shells respectively. The backbone curves do not exhibit soft-spring behavior except the shell with $r_x/r_y = 1$ and $\alpha = 1$. With decreases in the curvature ratio or increase in the aspect ratio, the effect of quadratic non-linear term G_2 becomes smaller and $|W_{1\,min}| - W_{1\,max}$

TABLE 2

Frequency parameters ω_1 of shallow shells used in Figures 2–4

θ	α	r_x	r_x/r_y	ω_1	Figures
45°/−45°/45°	1	10	1	140·1	2, 3, 4
0°/90°/0°	1	10	1	150·9	2
45°/−45°	1	10	1	129·3	2
0°/90°	1	10	1	133·9	2
45°/−45°/45°	1	25	1	93·01	3
45°/−45°/45°	1	50	1	83·97	3
45°/−45°/45°	1	100	1	81·54	3
45°/−45°/45°	1	10	0	103·2	4
45°/−45°/45°	1	10	−1	98·81	4
45°/−45°/45°	1·5	10	1	178·6	4
45°/−45°/45°	2	10	1	248·1	4

TABLE 3

Comparison of frequency parameters ω_1 of symmetric angle-ply laminated shallow shells ($\theta = 45°/-45°/45°$, $r_x = r_y = 10$, $\alpha = 1$)

	ω_1	
H	CST	FSDT
0·01	140·3	140·1
0·02	99·48	98·50
0·04	86·04	81·77
0·1	81·86	63·16

decreases. Especially, there does not exist $|W_{1\,min}| - W_{1\,max}$ for the hyperbolic paraboloid shell (because of $G_2 = 0$).

Figure 5 shows the non-linear vibration characteristics for symmetric angle-ply laminated shells with different thickness ratios. In order to examine the effect of the transverse shear deformation, the results obtained by the FSDT (thick line) are compared with those by the CST (thin line). The comparison of linear natural frequencies is tabulated in Table 3. In Figure 5(a), ω_1 denotes the linear natural frequency obtained by the FSDT. It is seen in Figures 5(a) and 5(b) that the hard-spring behavior becomes stronger and $|W_{1\,min}| - W_{1\,max}$ decreases with an increase of the thickness ratio. Further, with the increase, the results obtained by the FSDT deviate from those by the CST, and it can be said that the FSDT should be adopted for non-linear vibration analyses of moderately thick shells ($H \geqslant 0{\cdot}04$).

5.2. NON-LINEAR VIBRATION OF THE SECOND VIBRATION MODE

First of all, the influence of the first vibration mode on non-linear vibrations of the second mode is examined. The two-mode analysis considering both W_1 and W_2 is compared in

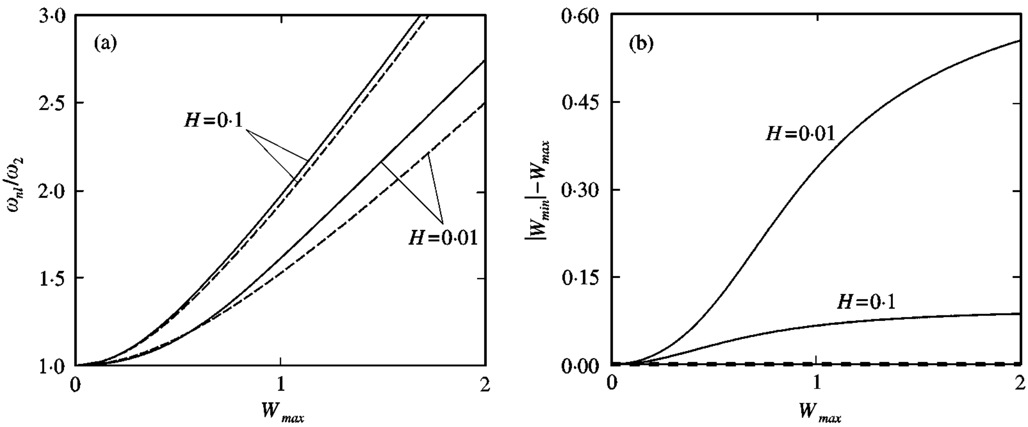

Figure 6. Comparison of single- and two-mode analyses on non-linear vibration of the second mode ($\theta = 45°/-45°/45°$, $\alpha = 1$, $r_x = 10$, $r_x/r_y = 1$): (a) ω_{nl}/ω_2 versus W_{max}, (b) $|W_{min}| - W_{max}$ versus W_{max}: ——, two-mode analysis; - - - - -, single-mode analysis.

Figure 6 with the single-mode analysis neglecting W_1 (considering only W_2). In this calculation, we treat symmetric angle-ply laminated shells ($\theta = 45°/-45°/45°$) with $H = 0{\cdot}01$ and $0{\cdot}1$ and use $I \times J = 8 \times 8$. In Figure 6, W_{max} is defined as $W_{max} = (W_1 + W_2)_{max}$, and solid and broken lines denote the results obtained by the two- and single-mode analyses respectively. It is seen in Figure 6(a) that the frequency ratios obtained by the single-mode analysis are smaller than those obtained by the two-mode analysis. The difference between maximum and minimum displacements is shown in Figure 6(b). The difference between them is considered in the two-mode analysis, but the difference is neglected in the single-mode analysis (i.e., $|W_{min}| = W_{max}$). Therefore, we can see the necessity of the first vibration mode for studying non-linear dynamic properties of the second vibration mode of the shell. Furthermore, because the errors on the backbone curve and the difference, $|W_{min}| - W_{max}$, are more pronounced for the thin shell ($H = 0{\cdot}01$) than for the thick shell ($H = 0{\cdot}1$), we conclude that the influence of the first vibration mode becomes stronger for the thin shell than for thick shell.

The convergence of the present series solutions for linear and non-linear frequencies of the second mode in the case of symmetric angle-ply laminated shells ($\theta = 45°/-45°/45°$) with $H = 0{\cdot}01$ and $0{\cdot}1$ is shown in Table 4. The maximum error between the last two columns is less than 1%. In the calculation for the second mode, $I \times J = 8 \times 8$ is used from a practical point of view.

Figures 7, 8, 9 and 10 present, respectively, the effects of lamination sequence, radius of curvature, and aspect ratios, and thickness ratio on the non-linear vibration of the second mode. Tables 5 and 6 show the linear natural frequencies ω_2 of the shells used in Figures 7–9 and 10 respectively. The linear natural frequencies of the second and third modes of antisymmetric angle-ply ($\theta = 45°/-45°$) and asymmetric cross-ply ($\theta = 0°/90°$) laminated shells are close to each other when $\alpha = 1$. In order to examine the non-linear vibration of the second mode of such shells, we must consider the third mode in the analysis. Since the present paper does not treat the shells with one-to-one internal resonance, we use $\alpha = 1{\cdot}25$ in Figure 7.

It is found from Figures 7(a)–10(a) that all responses of the second mode are hard-spring type because the frequency ratio ω_{nl}/ω_2 increases with an increase of the amplitude. It is also observed that a tendency on the frequency ratio of the second mode is different from that of

TABLE 4

Convergence of frequency parameters on the second vibration mode of symmetric angle-ply laminated shallow shells ($\theta = 45°/-45°/45°$, $r_x = r_y = 10$, $\alpha = 1$)

			ω_{nl}	
$H(=h/a)$	$I \times J$	$\omega_2(\lambda_2)$	$W_{max} = 1$	$W_{max} = 2$
0·01	4 × 4	184·5 (184·5)	248·4	396·2
	6 × 6	171·1 (171·0)	274·4	467·4
	8 × 8	171·0 (170·9)	276·1	471·1
	10 × 10	171·0 (170·9)	276·7	472·3
0·1	4 × 4	102·4 (101·1)	189·9	332·1
	6 × 6	101·0 (99·65)	198·6	352·2
	8 × 8	101·0 (99·62)	199·4	353·9
	10 × 10	100·9 (99·62)	199·2	353·9

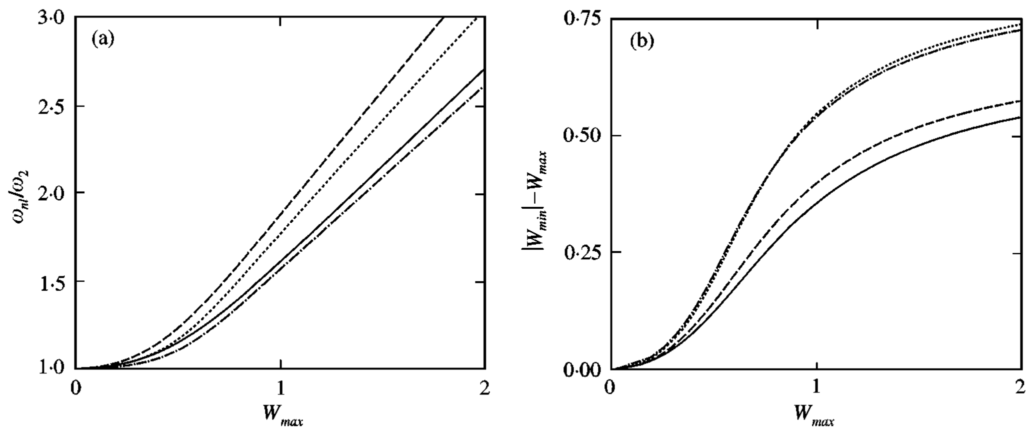

Figure 7. Effect of lamination sequence on non-linear vibration of the second mode ($\alpha = 1·25$, $r_x = 10$, $r_x/r_y = 1$, $H = 0·01$): (a) ω_{nl}/ω_2 versus W_{max}, (b) $|W_{min}| - W_{max}$ versus W_{max}: ——, $\theta = 45°/-45°/45°$; – – – – –, $\theta = 0°/90°/0°$; –·–·–·–, $\theta = 45°/-45°$; ·······, $\theta = 0°/90°$.

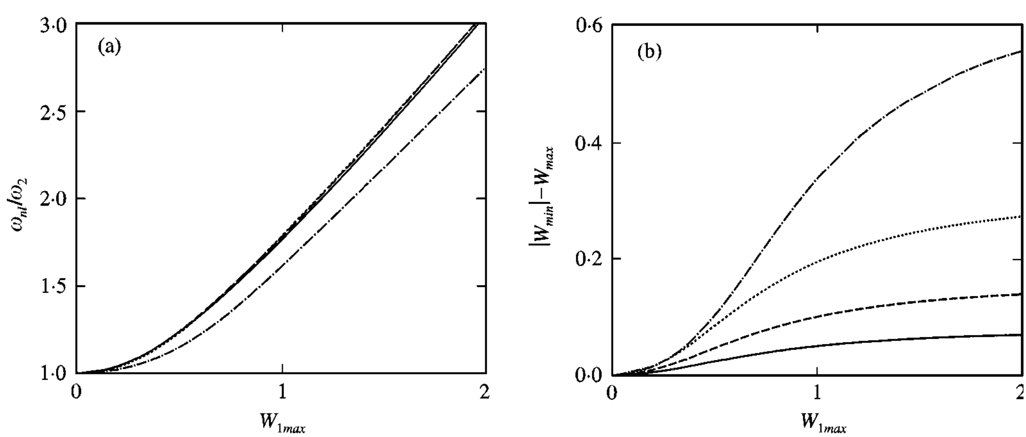

Figure 8. Effect of radius of curvature on non-linear vibration of the second mode ($\theta = 45°/-45°/45°$, $\alpha = 1$, $r_x/r_y = 1$, $H = 0·01$): (a) ω_{nl}/ω_2 versus W_{max}, (b) $|W_{min}| - W_{max}$ versus W_{max}: ——, $r_x = 100$; – – – – –, $r_x = 50$; ·······, $r_x = 25$; –·–·–·–, $r_x = 10$.

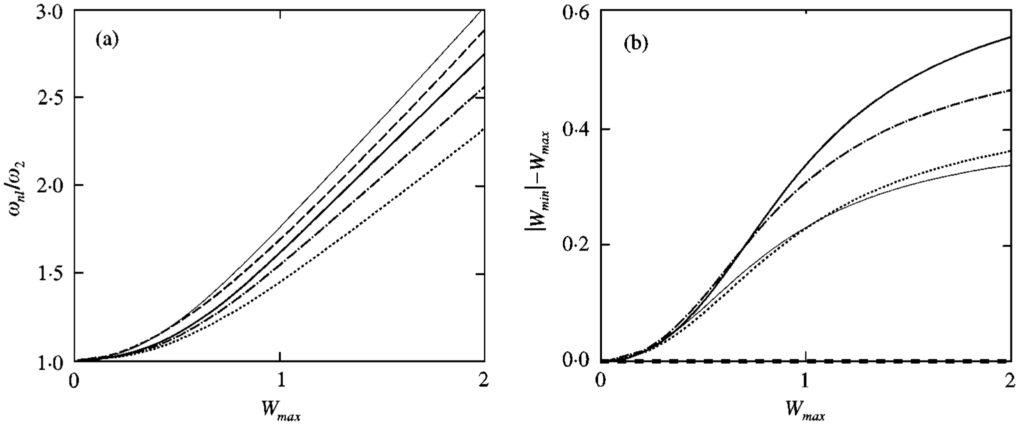

Figure 9. Effects of curvature and aspect ratios on non-linear vibration of the second mode ($\theta = 45°/-45°/45°$, $r_x = 10$, $H = 0{\cdot}01$): (a) ω_{nl}/ω_2 versus W_{max}, (b) $|W_{min}| - W_{max}$ versus W_{max}: ——, $r_x/r_y = 1$, $\alpha = 1$; —— $r_x/r_y = 0$, $\alpha = 1$; - - - - -, $r_x/r_y = -1$, $\alpha = 1$; –·–·–·–, $r_x/r_y = 1$, $\alpha = 1{\cdot}5$; ·······, $r_x/r_y = 1$, $\alpha = 2$.

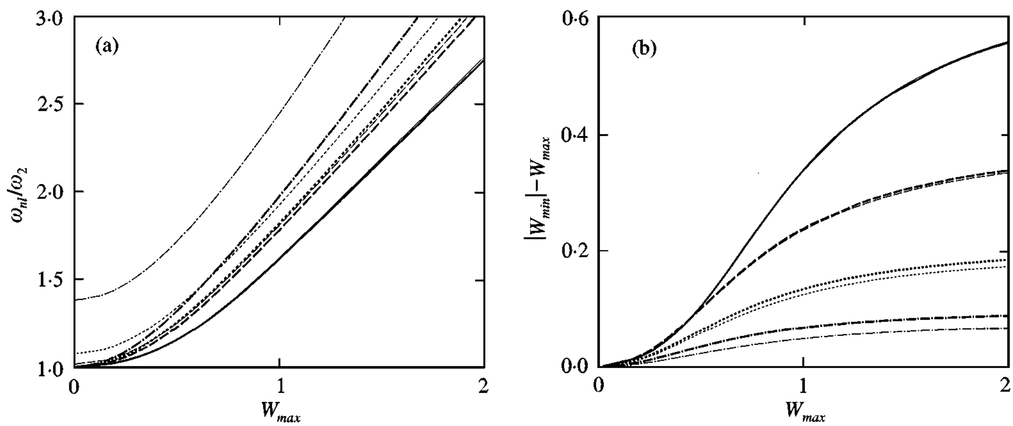

Figure 10. Effect of thickness ratio on non-linear vibration of the second mode ($\theta = 45°/-45°/45°$, $\alpha = 1$, $r_x = 10$, $r_x/r_y = 1$): (a) ω_{nl}/ω_2 versus W_{max}, (b) $|W_{min}| - W_{max}$ versus W_{max}: ——, $H = 0{\cdot}01$; - - - - -, $H = 0{\cdot}02$; ·······, $H_x = 0{\cdot}04$; –·–·–·–, $H_x = 0{\cdot}1$.

the first mode (For example, the frequency ratio at $W_{max} = 2$ of the first mode is larger in the order of ($\theta = 45°/-45°$) > ($\theta = 0°/90°$) > ($\theta = 45°/-45°/45°$) > ($\theta = 0°/90°/0°$) (Figure 2(a)), while the order for the second mode is ($\theta = 0°/90°/0°$) > ($\theta = 0°/90°$) > ($\theta = 45°/-45°/45°$) > ($\theta = 45°/-45°$) (Figure 7(a)).) Therefore, it can be concluded that the frequency ratio is also dependent upon the vibration mode.

In contrast to the results of the frequency ratio, the results of the difference $|W_{min}| - W_{max}$ of the second mode are the same tendency as that of the first mode because the difference decreases with an increase of radius of curvature (Figure 8(b)), the difference for the hypabolic shell ($r_x/r_y = -1$) becomes zero (Figures 9(b)) and so on.

Finally, we mention the effect of the transverse shear deformation on the non-linear vibration of the second mode shown in Figure 10. In Figure 10, thick and thin lines denote the results obtained by the FSDT and CST respectively. The comparison of linear natural frequencies is shown in Table 6. From Figure 10 and Table 6, it is found that the larger the

TABLE 5

Frequency parameters ω_2 of shallow shells used in Figures 7–9

θ	α	r_x	r_x/r_y	ω_2	Figures
45°/−45°/45°	1	10	1	171·0	8, 9
45°/−45°/45°	1·25	10	1	197·1	7
0°/90°/0°	1·25	10	1	186·7	7
45°/−45°	1·25	10	1	170·9	7
0°/90°	1·25	10	1	170·4	7
45°/−45°/45°	1	25	1	144·4	8
45°/−45°/45°	1	50	1	140·2	8
45°/−45°/45°	1	100	1	139·1	8
45°/−45°/45°	1	10	0	149·5	9
45°/−45°/45°	1	10	−1	148·2	9
45°/−45°/45°	1·5	10	1	224·7	9
45°/−45°/45°	2	10	1	296·7	9

TABLE 6

Comparison of frequency parameters ω_2 of symmetric angle-ply laminated shallow shells ($\theta = 45°/-45°/45°$, $r_x = r_y = 10$, $\alpha = 1$)

	ω_2	
H	CST	FSDT
0·01	171·6	171·0
0·02	148·2	145·5
0·04	141·7	131·6
0·1	139·9	101·0

thickness ratio of the shell becomes, the larger the error between the FSDT and CST solutions become. The tendency for the second mode is the same as that for the first mode.

6. SUMMARY AND CONCLUSIONS

Non-linear vibration characteristics of the first and second (asymmetric first) vibration modes of laminated shallow shells with rigidly clamped edges have been reported. In the analysis, the displacements of the shell were approximated by the power series which are eigenvectors of the first and second vibration modes calculated by the Ritz method (linear analysis). We examined the convergence characteristics of series solutions, and showed that linear and non-linear frequencies converged with an increase of the series.

It is explained clearly in the analysis that the modal interaction between the first and second modes is induced by the quadratic non-linear terms. We investigated the influence of the first mode on the non-linear vibration of the second mode, and revealed that the difference did not occur in the single-mode analysis which considers only the second mode, while the difference occurred in the two-mode analysis which considers both the first and second modes. Consequently, it can be said that the first vibration mode should be considered to examine the non-linear vibration of asymmetric vibration modes of shells.

In numerical examples, we demonstrated graphically the effects of lamination sequence, radius of curvature, curvature ratio, aspect ratio and thickness ratio on the non-linear vibration. We also compared the results obtained by the FSDT with those obtained by the CST, and showed that the larger the thickness ratio became, the stronger the effect of the transverse shear deformation became. The present paper paid attention not only to the frequency ratio but also to the difference between maximum outward and inward displacements.

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APPENDIX A

The elements in the matrices of equation (31) are as follows:

$$U^U_{aijkl} = \int_{-1}^{1}\int_{-1}^{1}\left\{A^*_{11}U_{\xi i,\xi}U_{\eta j}U_{\xi k,\xi}U_{\eta l} + A^*_{16}\alpha(U_{\xi i,\xi}U_{\eta j}U_{\xi k}U_{\eta l,\eta} + U_{\xi i}U_{\eta j,\eta}U_{\xi k,\xi}U_{\eta l})\right.$$

$$\left. + A^*_{66}\alpha^2 U_{\xi i}U_{\eta j,\eta}U_{\xi k}U_{\eta l,\eta} + \frac{\Delta S^*_{55}U_{\xi i}U_{\eta j}U_{\xi k}U_{\eta l}}{4r_x^2}\right\}\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$U^U_{bijkl} = \int_{-1}^{1}\int_{-1}^{1}\left(A^*_{12}\alpha V_{\xi i}V_{\eta j,\eta}U_{\xi k,\xi}U_{\eta l} + A^*_{16}V_{\xi i,\xi}V_{\eta j}U_{\xi k,\xi}U_{\eta l} + A^*_{26}\alpha^2 V_{\xi i}V_{\eta j,\eta}U_{\xi k}U_{\eta l,\eta}\right.$$

$$\left. + A^*_{66}\alpha V_{\xi i,\xi}V_{\eta j}U_{\xi k}U_{\eta l,\eta} + \frac{\Delta S^*_{45}V_{\xi i}V_{\eta j}U_{\xi k}U_{\eta l}}{4r_x r_y}\right)\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$U^U_{cijkl} = \int_{-1}^{1}\int_{-1}^{1}\left[\left(\frac{A^*_{11}}{2r_x} + \frac{A^*_{12}}{2r_y}\right)W_{\xi i}W_{\eta j}U_{\xi k,\xi}U_{\eta l} + \alpha\left(\frac{A^*_{16}}{2r_x} + \frac{A^*_{26}}{2r_y}\right)W_{\xi i}W_{\eta j}U_{\xi k}U_{\eta l,\eta}\right.$$

$$- \frac{\Delta}{2r_x}(S^*_{45}\alpha W_{\xi i}W_{\eta j,\eta}U_{\xi k}U_{\eta l} + S^*_{55}W_{\xi i,\xi}W_{\eta j}U_{\xi k}U_{\eta l})$$

$$- 2H(1-\Delta)\{B^*_{11}W_{\xi i,\xi\xi}W_{\eta j}U_{\xi k,\xi}U_{\eta l} + B^*_{12}\alpha^2 W_{\xi i}W_{\eta j,\eta\eta}U_{\xi k,\xi}U_{\eta l}$$

$$+ B^*_{16}\alpha(2W_{\xi i,\xi}W_{\eta j,\eta}U_{\xi k,\xi}U_{\eta l} + W_{\xi i,\xi\xi}W_{\eta j}U_{\xi k}U_{\eta l,\eta}) + B^*_{26}\alpha^3 W_{\xi i}W_{\eta j,\eta\eta}U_{\xi k}U_{\eta l,\eta}$$

$$\left. + 2B^*_{66}\alpha^2 W_{\xi i,\xi}W_{\eta j,\eta}U_{\xi k}U_{\eta l,\eta}\}\right]\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$U^U_{dijkl} = \Delta\int_{-1}^{1}\int_{-1}^{1}\left\{B^*_{11}\psi_{x\xi i,\xi}\psi_{x\eta j}U_{\xi k,\xi}U_{\eta l} + B^*_{16}\alpha(\psi_{x\xi i,\xi}\psi_{x\eta j}U_{\xi k}U_{\eta l,\eta}\right.$$

$$\left. + \psi_{x\xi i}\psi_{x\eta j,\eta}U_{\xi k,\xi}U_{\eta l}) + B^*_{66}\alpha^2\psi_{x\xi i}\psi_{x\eta j,\eta}U_{\xi k}U_{\eta l,\eta} - \frac{S^*_{55}\psi_{x\xi i}\psi_{x\eta j}U_{\xi k}U_{\eta l}}{4Hr_x}\right\}\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$U^U_{eijkl} = \Delta\int_{-1}^{1}\int_{-1}^{1}\left(B^*_{12}\alpha\psi_{y\xi i}\psi_{y\eta j,\eta}U_{\xi k,\xi}U_{\eta l} + B^*_{16}\psi_{y\xi i,\xi}\psi_{y\eta j}U_{\xi k,\xi}U_{\eta l}\right.$$

$$\left. + B^*_{26}\alpha^2\psi_{y\xi i}\psi_{y\eta j,\eta}U_{\xi k}U_{\eta l,\eta} + B^*_{66}\alpha\psi_{y\xi i,\xi}\psi_{y\eta j}U_{\xi k}U_{\eta l,\eta} - \frac{S^*_{45}\psi_{y\xi i}\psi_{y\eta j}U_{\xi k}U_{\eta l}}{4Hr_x}\right)\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$U^V_{bijkl} = \int_{-1}^{1}\int_{-1}^{1}\left\{A^*_{22}\alpha^2 V_{\xi i}V_{\eta j,\eta}V_{\xi k}V_{\eta l,\eta} + A^*_{26}\alpha(V_{\xi i,\xi}V_{\eta j}V_{\xi k}V_{\eta l,\eta} + V_{\xi i}V_{\eta j,\eta}V_{\xi k,\xi}V_{\eta l})\right.$$

$$\left. + A^*_{66}V_{\xi i,\xi}V_{\eta j}V_{\xi k,\xi}V_{\eta l} + \frac{\Delta S^*_{44}V_{\xi i}V_{\eta j}V_{\xi k}V_{\eta l}}{4r_y^2}\right\}\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$U^V_{cijkl} = \int_{-1}^{1}\int_{-1}^{1}\left[\alpha\left(\frac{A^*_{12}}{2r_x}+\frac{A^*_{22}}{2r_y}\right)W_{\xi i}W_{\eta j}V_{\xi k}V_{\eta l,\eta} + \left(\frac{A^*_{16}}{2r_x}+\frac{A^*_{26}}{2r_y}\right)W_{\xi i}W_{\eta j}U_{\xi k,\xi}U_{\eta l}\right.$$

$$- \frac{\Delta}{2r_y}(S^*_{44}\alpha W_{\xi i}W_{\eta j,\eta}V_{\xi k}V_{\eta l} + S^*_{55}W_{\xi i,\xi}W_{\eta j}V_{\xi k}V_{\eta l})$$

$$- 2H(1-\Delta)\{B^*_{12}\alpha W_{\xi i,\xi\xi}W_{\eta j}V_{\xi k}V_{\eta l,\eta} + B^*_{22}\alpha^3 W_{\xi i}W_{\eta j,\eta\eta}V_{\xi k}V_{\eta l,\eta}$$

$$+ B^*_{16}W_{\xi i,\xi\xi}W_{\eta j}U_{\xi k,\xi}U_{\eta l} + B^*_{26}\alpha^2(W_{\xi i}W_{\eta j,\eta\eta}V_{\xi k,\xi}V_{\eta l} + 2W_{\xi i,\xi}W_{\eta j,\eta}V_{\xi k}V_{\eta l,\eta})$$

$$\left. + 2B^*_{66}\alpha W_{\xi i,\xi}W_{\eta j,\eta}V_{\xi k,\xi}V_{\eta l}\}\right]\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$U^V_{dijkl} = \Delta\int_{-1}^{1}\int_{-1}^{1}\left(B^*_{12}\alpha\psi_{x\xi i,\xi}\psi_{x\eta j}V_{\xi k}V_{\eta l,\eta} + B^*_{16}\psi_{x\xi i,\xi}\psi_{x\eta j}V_{\xi k,\xi}V_{\eta l}\right.$$

$$\left. + B^*_{26}\alpha^2\psi_{x\xi i}\psi_{x\eta j,\eta}V_{\xi k}V_{\eta l,\eta} + B^*_{66}\alpha\psi_{x\xi i}\psi_{x\eta j,\eta}U_{\xi k,\xi}U_{\eta l} - \frac{S^*_{45}\psi_{x\xi i}\psi_{x\eta j}V_{\xi k}V_{\eta l}}{4Hr_y}\right)\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$U^V_{eijkl} = \Delta\int_{-1}^{1}\int_{-1}^{1}\left\{B^*_{22}\alpha^2\psi_{y\xi i}\psi_{y\eta j,\eta}V_{\xi k}V_{\eta l,\eta} + B^*_{26}\alpha(\psi_{y\xi i,\xi}\psi_{y\eta j}V_{\xi k}V_{\eta l,\eta}\right.$$

$$\left. + \psi_{y\xi i}\psi_{y\eta j,\eta}V_{\xi k,\xi}V_{\eta l}) + B^*_{66}\psi_{y\xi i,\xi}\psi_{y\eta j}V_{\xi k,\xi}V_{\eta l} - \frac{S^*_{44}\psi_{y\xi i}\psi_{y\eta j}V_{\xi k}V_{\eta l}}{4Hr_y}\right\}\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$U^W_{cijkl} = \int_{-1}^{1}\int_{-1}^{1}\left[\left(\frac{A^*_{11}}{4r_x^2}+\frac{A^*_{12}}{2r_xr_y}+\frac{A^*_{22}}{4r_y^2}\right)W_{\xi i}W_{\eta j}W_{\xi k}W_{\eta l} + \Delta\{S^*_{44}\alpha^2 W_{\xi i}W_{\eta j,\eta}W_{\xi k}W_{\eta l,\eta}\right.$$

$$+ S^*_{45}\alpha(W_{\xi i,\xi}W_{\eta j}W_{\xi k}W_{\eta l,\eta} + W_{\xi i}W_{\eta j,\eta}W_{\xi k,\xi}W_{\eta l}) + S^*_{55}W_{\xi i,\xi}W_{\eta j}W_{\xi k,\xi}W_{\eta l}\}$$

$$- H(1-\Delta)\left\{\left(\frac{B^*_{11}}{r_x}+\frac{B^*_{12}}{r_y}\right)(W_{\xi i}W_{\eta j}W_{\xi k,\xi\xi}W_{\eta l} + W_{\xi i,\xi\xi}W_{\eta j}W_{\xi k}W_{\eta l})\right.$$

$$+ 2\alpha\left(\frac{B^*_{16}}{r_x}+\frac{B^*_{26}}{r_y}\right)(W_{\xi i}W_{\eta j}W_{\xi k,\xi}W_{\eta l,\eta} + W_{\xi i,\xi}W_{\eta j,\eta}W_{\xi k}W_{\eta l})$$

$$\left. + \alpha^2\left(\frac{B^*_{12}}{r_x}+\frac{B^*_{22}}{r_y}\right)(W_{\xi i}W_{\eta j}W_{\xi k}W_{\eta l,\eta\eta} + W_{\xi i}W_{\eta j,\eta\eta}W_{\xi k}W_{\eta l})\right\}$$

$$+ 4H^2(1-\Delta)\{D^*_{11}W_{\xi i,\xi\xi}W_{\eta j}W_{\xi k,\xi\xi}W_{\eta l} + D^*_{22}\alpha^4 W_{\xi i}W_{\eta j,\eta\eta}W_{\xi k}W_{\eta l,\eta\eta}$$

$$+ D^*_{12}\alpha^2(W_{\xi i,\xi\xi}W_{\eta j}W_{\xi k}W_{\eta l,\eta\eta} + W_{\xi i}W_{\eta j,\eta\eta}W_{\xi k,\xi\xi}W_{\eta l})$$

$$+ 2D^*_{16}\alpha(W_{\xi i,\xi\xi}W_{\eta j}W_{\xi k,\xi}W_{\eta l,\eta} + W_{\xi i,\xi}W_{\eta j,\eta}W_{\xi k,\xi\xi}W_{\eta l})$$

$$+ 2D^*_{26}\alpha^3(W_{\xi i,\xi}W_{\eta j,\eta}W_{\xi k}W_{\eta l,\eta\eta} + W_{\xi i}W_{\eta j,\eta\eta}W_{\xi k,\xi}W_{\eta l,\eta})$$

$$+ 4D^*_{66}\alpha^2 W_{\xi i,\xi}W_{\eta j,\eta}W_{\xi k,\xi}W_{\eta l,\eta}]\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$U^{\psi_x}_{dijkl} = \Delta\int_{-1}^{1}\int_{-1}^{1}\Big\{D^*_{11}\psi_{x\xi i,\xi}\psi_{x\eta j}\psi_{x\xi k,\xi}\psi_{x\eta l} + D^*_{16}\alpha(\psi_{x\xi i,\xi}\psi_{x\eta j}\psi_{x\xi k}\psi_{x\eta l,\eta}$$

$$+ \psi_{x\xi i}\psi_{x\eta j,\eta}\psi_{x\xi k,\xi}\psi_{x\eta l}) + D^*_{66}\alpha^2\psi_{x\xi i}\psi_{x\eta j,\eta}\psi_{\xi k}\psi_{\eta l,\eta} - \frac{S^*_{55}\psi_{x\xi i}\psi_{x\eta j}\psi_{x\xi k}\psi_{x\eta l}}{4H^2}\Big\}\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$U^{\psi_x}_{eijkl} = \Delta\int_{-1}^{1}\int_{-1}^{1}\Big(D^*_{12}\alpha\psi_{y\xi i}\psi_{y\eta j,\eta}\psi_{x\xi k,\xi}\psi_{x\eta l} + D^*_{16}\psi_{y\xi i,\xi}\psi_{y\eta j}\psi_{x\xi k,\xi}\psi_{x\eta l}$$

$$+ D^*_{26}\alpha^2\psi_{y\xi i}\psi_{y\eta j,\eta}\psi_{x\xi k}\psi_{x\eta l,\eta} + D^*_{66}\alpha\psi_{y\xi i,\xi}\psi_{y\eta j}\psi_{x\xi k}\psi_{x\eta l,\eta} - \frac{S^*_{45}\psi_{y\xi i}\psi_{y\eta j}\psi_{x\xi k}\psi_{x\eta l}}{4H^2}\Big)\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$U^{\psi_x}_{eijkl} = \Delta\int_{-1}^{1}\int_{-1}^{1}\Big\{D^*_{22}\alpha^2\psi_{y\xi i}\psi_{y\eta j,\eta}\psi_{y\xi k}\psi_{y\eta l,\eta} + D^*_{26}\alpha(\psi_{y\xi i,\xi}\psi_{y\eta j}\psi_{y\xi k}\psi_{y\eta l,\eta}$$

$$+ \psi_{y\xi i}\psi_{y\eta j,\eta}\psi_{y\xi k,\xi}\psi_{y\eta l}) + D^*_{66}\psi_{y\xi i,\xi}\psi_{y\eta j}\psi_{y\xi k,\xi}\psi_{y\eta l} - \frac{S^*_{44}\psi_{y\xi i}\psi_{y\eta j}\psi_{y\xi k}\psi_{y\eta l}}{4H^2}\Big\}\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$T^U_{aijkl} = \int_{-1}^{1}\int_{-1}^{1} U_{\xi i}U_{\eta j}U_{\xi k}U_{\eta l}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$T^V_{bijkl} = \int_{-1}^{1}\int_{-1}^{1} V_{\xi i}V_{\eta j}V_{\xi k}V_{\eta l}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$T^W_{cijkl} = \int_{-1}^{1}\int_{-1}^{1} W_{\xi i}W_{\eta j}W_{\xi k}W_{\eta l}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$T^{\psi_x}_{dijkl} = \frac{\Delta}{12}\int_{-1}^{1}\int_{-1}^{1} \psi_{x\xi i}\psi_{x\eta j}\psi_{x\xi k}\psi_{x\eta l}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$T^{\psi_y}_{eijkl} = \frac{\Delta}{12}\int_{-1}^{1}\int_{-1}^{1} \psi_{y\xi i}\psi_{y\eta j}\psi_{y\xi k}\psi_{y\eta l}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

where

$$\Delta = \begin{cases} 1 & \text{for FSDT,} \\ 0 & \text{for CST.} \end{cases}$$