Journal of Sound and Vibration (2000) **234**(3), 507–520
doi:10.1006/jsvi.2000.2866, available online at http://www.idealibrary.com on IDEAL®

VIBRATIONS OF ELASTICALLY MOUNTED MASS SUPPORTED ON SYMMETRICALLY CROSSED BEAMS

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(*Received* 20 *September* 1999, *and in final form* 17 *January* 2000)

This study presents a novel method to analyze the vibration of an elastically mounted concentrated mass supported on the joint of symmetrically crossed beams with flexible foundation. Analytical and exact solutions of the free and forced vibration responses of the system are also derived. Herein, the dynamics of the mounted mass and the crossed beams are expressed as two-way state-flow (TWSF) graph models, in which the interactions between the components are considered. Based on the proposed model, the frequency responses of the displacement of the mounted mass and every beam are derived using a topological method. Moreover, the force transmissibility from the vibrating mass to the foundation and the frequency equation are obtained. The derived results are expressed in both analytical and closed forms. Also presented herein are some special cases including identical structure properties for each beam, simply supported boundary for each beam, mass directly mounted on the beams, and their combinations.

1. INTRODUCTION

Free and forced vibrations of engines and motors mounted on structure elements is a critical problem in mechanical, aircraft and naval engineering. The design objectives focus mainly on preventing the vibrating frequency from meeting the natural frequency of the combined system and reducing the force transmitted from the vibrating source to the foundation. The vibration behavior and transmissibility of a concentrated mass mounted on single- and multiple-degree-of-freedom lumped isolator systems have received considerable interest [1–4]. Analytical and numerical methods for obtaining the fundamental frequency and mode shape of a single beam carrying a concentrated mass in free vibration condition have also been presented [1, 5–8]. While considering the effect of the isolation system between the vibrating machine and the supporting structure, related studies have examined free vibrations of a beam carrying elastically mounted masses [9–11]. Moreover, other investigators have elucidated the forced transverse vibration response and the force transmissibility subjected to excitation force on the mounted concentrated mass on a beam [12–14]. Although previous studies commonly considered the single beam structure, vibrating engines supported by multiple crossed beams are normally used in practical design. The vibration behavior of a system is affected by the dynamic interaction not only between the mounted mass and supported beams but also between one beam and another. Moreover, the flexibility of the foundation for each beam is combined into the dynamics of a system. Owing to the complexity of these dynamic interactions, using conventional methods to obtain the analytical results for the free and forced vibrations of this problem is extremely difficult.

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In the light of the above developments, this work performs free and forced vibration analysis of crossed beams with a flexible boundary carrying an elastically mounted mass by using a graph method [3, 4, 15, 16]. A flexible foundation for the supported of each beam is also considered. First, this study develops TWSF graph models for the beams with flexible boundary and carrying an elastically mounted mass. Based on the proposed models, the analytical and closed forms of the frequency responses of the mounted mass and each beam, the force transmissibility and the frequency equation of multiple crossed beams are derived. Also studied herein are some simplified cases, such as identical structure properties for each beam, simply supported boundary for each beam, and vibrating mass directly mounted on the beams structure. Two numerical examples are presented to demonstrate how the proposed method is implemented.

2. TWSF GRAPH MODELS

An elastically mounted concentrated mass (the primary system) supported on the joint of symmetrically crossed beams structure as shown in Figure 1 is considered. As assumed herein, each beam is uniform and joined at the midpoint. The ends of each beam are connected to the flexible foundation, which is modelled as the combination of the linear spring and dashpot damper. In this study, the primary system and the beams system are regarded as two coupled subsystems. The dynamics of both subsystems can be first analyzed individually and, then, the interaction effect between both subsystems is included to obtain the dynamic response of the coupled system.

If the mass of the primary system is subject to the sinusoidal varying excitation $F_e e^{jwt}$, where F_e is the amplitude of the excitation force, the displacement and force response of each beam and the primary system should also be harmonic with the same frequency. For the primary system, the relationship between the displacement and the force of the mass and the spring can be expressed as

$$W_d = \frac{F_c - F_e}{m_d \omega^2}, \qquad W_c = W_d - \frac{F_c}{j\omega c_d + k_d}, \tag{1, 2}$$

where m_d, c_d and k_d are the mass, damping, and stiffness of the primary system respectively. W_d and W_c are the complex amplitude of the displacement of the primary system $w_d(t)$ and the joint of the beams and $w_c(t)$, respectively, and F_c is the complex amplitude of the tension force $f_c(t)$ on the spring of the primary system. According to equations (1) and (2), the dynamics of the primary system can be described as a TWSF graph model, as shown in Figure 2.

For the ith beam, the beam is uniform and crossed at the midpoint. Thus, the transverse vibration of the beam is symmetric. Considering half-length of the beam in the analysis is sufficient. Since no load acts between the midpoint and the ends of each beam, the governing equation for small amplitude vibration of the beam is given by [12]

$$E_i I_i \frac{\partial^4 w_i}{\partial x_i^4}(x_i, t) + m_i \frac{\partial^2 w_i}{\partial t_i^2}(x_i, t) = 0 \quad \text{for } 0 < x_i < b_i, \tag{3}$$

where $w_i(t)$ is the beam's displacement at the cross-section x_i, E_i in Young's modulus of beam i, I_i is the moment of inertia of the beam, m_i is the mass per unit length of the beam, and b_i is the half-length of the beam. The solution of equation (3) can be calculated by separation of variables. Thus, the amplitude of the response of the beam can be expressed in

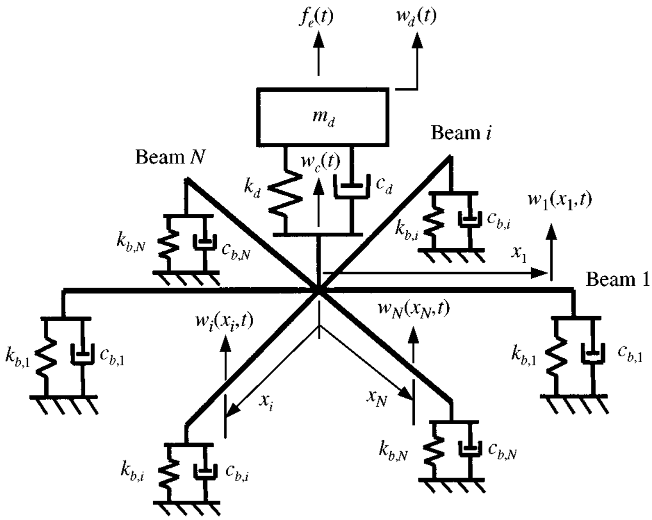

Figure 1. Crossed beams structure carrying an elastically mounted mass.

the following form:

$$W_i(x_i) = a_{i,1}\sin a_i x_i + a_{i,2}\cos a_i x_i + a_{i,3}\sinh a_i x_i + a_{i,4}\cosh a_i x_i, \tag{4}$$

where W_t is the complex amplitude of the displacement response at location x_i, $a_{i,1}$, $a_{i,2}$, $a_{i,3}$, and $a_{i,4}$ depend on the boundary conditions, and a_i is a function of the forced frequency ω given by

$$a_i = (m_i\omega^2/E_i I_i)^{1/4}. \tag{5}$$

Notably, the slope at the midpoint of the beam is zero since the vibration response of the beam is symmetric. In addition, only translational flexibility for both ends of the beam is considered, while the reaction moment at both ends is zero, such that

$$E_i I_i \frac{\mathrm{d}^2 w_i(b_i,t)}{\mathrm{d}^2 x_i} = 0, \qquad \frac{\mathrm{d}w_i(0,t)}{\mathrm{d}x_i} = 0. \tag{6, 7}$$

Moreover, the shear force around the center of the beam equals half of the summation forces acted by the primary system and other beams, denoted as $f_{c,i}(t)$. If the complex amplitude of the displacement at the ends of the beam $W_{b,i}$ is given, the constants $a_{i,1}$, $a_{i,2}$, $a_{i,3}$, and $a_{i,4}$ in equation (4) can be expressed by $W_{b,i}$ and $F_{c,i}$

$$a_{i,1} = -\frac{1}{4a_i^3 E_i I_i}F_{c,i}, \qquad a_{i,2} = \frac{1}{2c_i}W_{b,i} + \frac{s_i}{4a_i^3 E_i I_i c_i}F_{c,i}, \tag{8, 9}$$

$$a_{i,3} = \frac{1}{4a_i^3 E_i I_i}F_{c,i}, \qquad a_{i,4} = \frac{1}{2ch_i}W_{b,i} - \frac{sh_i}{4a_i^3 E_i I_i ch_i}F_{c,i}, \tag{10, 11}$$

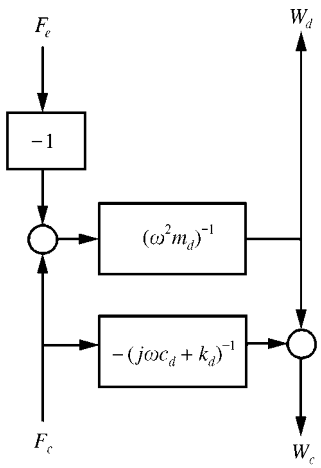

Figure 2. TWSF model of the primary system.

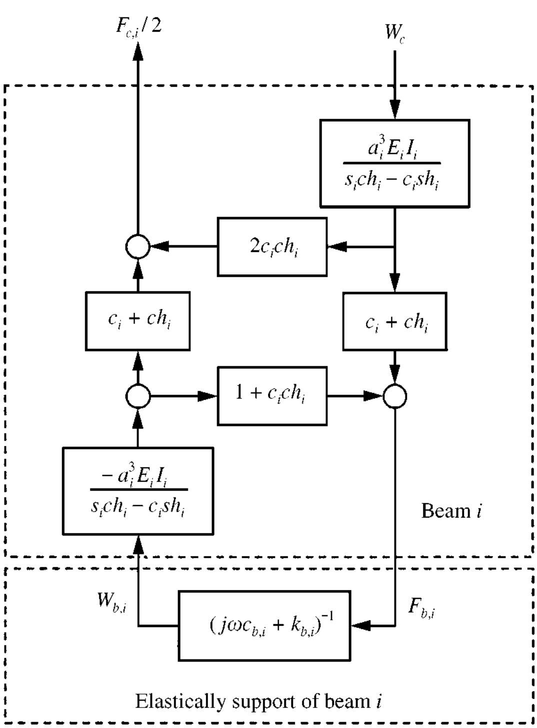

Figure 3. Graph model of beam i and its elastic support.

where c_i, s_i, ch_i, and sh_i are the symbols of $\cos(a_i b_i)$, $\sin(a_i b_i)$, $\cosh(a_i b_i)$, and $\sinh(a_i b_i)$ respectively.

Substituting equations (8)–(11) into equation (4) yields the displacement response of the beam. According to the results, the complex amplitude of the displacement response at the center of the beam can be represented by $W_{b,i}$ and $F_{c,i}$. The boundary condition at the ends of the beam reveals that the shear force at the ends of the beam equals the force acted by the flexible foundation $f_{b,i}(t)$. Thus, the complex amplitudes $F_{b,i}$ and $F_{c,i}$ are expressed as follows:

$$\frac{F_{c,i}}{2a_i^3 E_i I_i} = \frac{2c_i ch_i}{s_i ch_i - c_i sh_i} W_c - \frac{c_i + ch_i}{s_i ch_i - c_i sh_i} W_{b,i}, \tag{12}$$

$$\frac{F_{b,i}}{a_i^3 E_i I_i} = \frac{c_i + ch_i}{s_i ch_i - c_i sh_i} W_c - \frac{1 + c_i ch_i}{s_i ch_i - c_i sh_i} W_{b,i}. \tag{13}$$

According to equations (12) and (13), the relationships between $F_{b,i}$, $F_{c,i}$, $W_{b,i}$ and W_c can be described by a TWSF graph model as shown in the upper part of Figure 3.

For the flexible foundation, a combined model of massless spring and dashpot damper is assumed. The relationship between the displacement and the force response is given by

$$W_{b,i} = \frac{F_{b,i}}{\mathrm{j}\omega c_{b,i} + k_{b,i}}. \tag{14}$$

Equation (14) can also be represented as a graph model as shown in the lower portion of Figure 3.

3. FREE AND FORCED VIBRATION ANALYSIS

3.1. FORCED RESPONSE

The graph model of beam i and its supports shown in Figure 3 indicates that the paths following the state flow form a closed loop. There are two forward paths from W_c to $F_{c,i}$. Although one forward path touches the loop, the other one does not. Thus, the ratio of the complex amplitude of the displacement response W_c to the force response $F_{c,i}$, denoted as $H_{W_c,F_{c,i}}$, can be obtained as [3, 4, 16]

$$H_{W_c,F_{c,i}} = 2a_i^3 E_i I_i(-\, a_i^3 E_i I_i (s_i ch_i - c_i sh_i) + 2c_i ch_i (c_{b,i}\omega + k_{b,i}))$$

$$/(a_i^3 E_i I_i (1 + c_i ch_i) + (c_{b,i}\omega + k_{b,i})(s_i ch_i - c_i sh_i)). \tag{15}$$

When all of N crossed beams are considered, the force acting on the spring and the damper of the primary system F_c equals the summation of the force acting on all beams, which can be expressed as

$$F_c = \sum_{i=1}^{N} F_{c,i}. \tag{16}$$

According to equations (15) and (16), the graph model of the total system can be obtained by the combination of Figures 2 and 3 as shown in Figure 4.

According to Figure 4, all the transfer functions $H_{W_c,F_{c,i}}$ can be combined by direct summation. The reduced graph model contains only two loops. For the combined model, there is only one forward path from F_e to W_d. This forward path touches one of the loops.

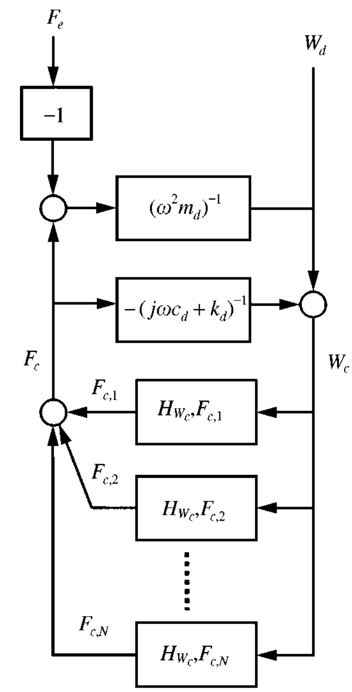

Figure 4. Graph model of the combined system.

Thus, the complex amplitude of the displacement response of the mass of the primary leads to

$$W_d = \left((j\omega c_d + k_d) + \sum_{i=1}^{N} H_{W_c,F_{e,i}}\right)F_e \Bigg/ \left(-m_d\omega^2 (j\omega c_d + k_d) + (-m_d\omega^2 + j\omega c_d + k_d)\sum_{i=1}^{N} H_{W_c,F_{e,i}}\right). \tag{17}$$

In the same manner, the response of the displacement at the joint W_c can be calculated. There is only one forward path from F_e to W_c. This forward path touches both loops in the combined graph model. Thus, the complex amplitude of the displacement W_c is

$$W_c = (j\omega c_d + k_d)F_e \Big/ \left(-m_d\omega^2 (j\omega c_d + k_d) + (-m_d\omega^2 + j\omega c_d + k_d)\sum_{i=1}^{N} H_{W_c,F_{e,i}}\right). \tag{18}$$

Hence, the transfer function H_{F_e,W_c}, defined as the ratio of the complex amplitude of the displacement W_c to the excitation force F_e, is given by

$$H_{F_e,W_c} = W_c/F_e. \tag{19}$$

In order to calculate the response of each beam, the graph model of the total system can be rearranged by combining two cascade models: the dynamic coupling of all components

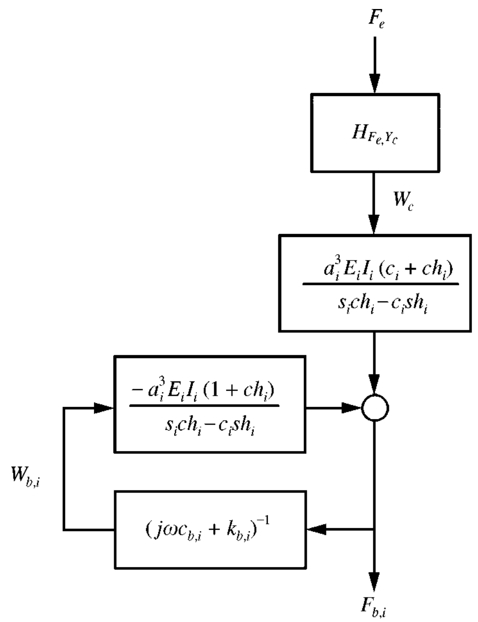

Figure 5. Rearranged graph model for forced vibration analysis.

of the system and the uncoupling dynamics of each beam and its constraints as shown in Figure 5. From the rearranged model, the displacement of the ends of each beam $W_{b,i}$ can be obtained

$$W_{b,i} = a_i^3 E_i I_i (c_i + ch_i) H_{F_e,W_c} F_e / ((\mathrm{j}\omega c_{b,i} + k_{b,i})(s_i ch_i - c_i sh_i) + a_i^3 E_i I_i (1 + c_i ch_i)). \quad (20)$$

According to the transfer function, the force transmitted from the ends of each beam to the foundation $F_{b,i}$ is given by

$$F_{b,i} = a_i^3 E_i I_i (c_i + ch_i)(\mathrm{j}\omega c_{b,i} + k_{b,i}) H_{F_e,W_c} F_e / ((\mathrm{j}\omega c_{b,i} + k_{b,i})(s_i ch_i - c_i sh_i) + a_i^3 E_i I_i (1 + c_i ch_i)). \quad (21)$$

If responses $W_{b,i}$ and $F_{b,i}$ as shown in equations (20) and (21) are substituted into equations (8)–(11), the coefficients of $a_{i,1}$, $a_{i,2}$, $a_{i,3}$, and $a_{i,4}$ can be calculated. Thus, the response at each location of the beam is known.

The force transmissibility from the excitation to the foundation T_{F_e,F_b}, defined as the absolute of F_b/F_e, can be expressed as

$$T_{F_e,F_b} = \left| \sum_{i=1}^{N} (a_i^3 E_i I_i (c_i + ch_i)(\mathrm{j}\omega c_{b,i} + k_{b,i}) H_{F_e,W_c} / ((\mathrm{j}\omega c_{b,i} + k_{b,i})(s_i ch_i - c_i sh_i) + a_i^3 E_i I_i (1 + c_i ch_i))) \right|. \quad (22)$$

3.2. FREQUENCY EQUATION

The natural frequency of a structure is normally determined by the homogeneous solution of the governing equations. The dynamic response enlarges to infinity for a harmonic excitation force when the frequency of the force meets the natural frequency of the system. Thus, the natural frequency can be obtained from the roots of the denominator of the complex frequency response as given in equation (17). Thus, the frequency equation can be obtained as

$$
\begin{aligned}
&-m_d\omega^2(\mathrm{j}\omega c_d+k_d)+(-m_d\omega^2+\mathrm{j}\omega c_d+k_d)\sum_{i=1}^{N}(2a_i^3E_iI_i(-a_i^3E_iI_i(s_ich_i-c_ish_i)\\
&\quad+2c_ich_i(\mathrm{j}\omega c_{b,i}+k_{b,i}))/(a_i^3E_iI_i(1+c_ich_i)+(\mathrm{j}\omega c_{b,i}+k_{b,i})\\
&\quad\times(s_ich_i-c_ish_i)))=0. \qquad (23)
\end{aligned}
$$

4. SPECIAL CASES

4.1. N IDENTICAL CROSSED BEAMS

If the length, cross-section, structural properties, and foundation flexibility are the same for each beam, the transfer function $H_{W_c,F_{c,i}}$ for each i from 1 to N is identical as given by

$$
\begin{aligned}
H_{W_c,F_{c,i}}&=2\bar a^3\,\overline{EI}(-\bar a^3\overline{EI}(\bar s\overline{ch}-\bar c\overline{sh})+2\bar c\overline{ch}(\mathrm{j}\omega\bar c_b+\bar k_b))/(\bar a^3\overline{EI}(1+\bar c\,\overline{ch})\\
&\quad+(\mathrm{j}\omega\bar c_b+\bar k_b)(\bar s\overline{ch}-\bar c\overline{sh})). \qquad (24)
\end{aligned}
$$

When equation (24) is substituted into equations (17) and (18), the complex amplitude of the displacement of the mounted mass W_d and the displacement at the joint of the beams W_c can be rewritten as follows:

$$
\begin{aligned}
W_d&=F_e((\mathrm{j}\omega c_d+k_d)(\bar a^3\overline{EI}(1+\bar c\,\overline{ch})+(\mathrm{j}\omega\bar c_b+\bar k_b)(\bar s\overline{ch}-\bar c\overline{sh}))+2N\bar a^3\,\overline{EI}\\
&\quad\times(-\bar a^3\overline{EI}(\bar s\,\overline{ch}-\bar c\,\overline{sh})+2\bar c\overline{ch}(\mathrm{j}\omega\bar c_b+\bar k_b)))/(-m_d\omega^2(\mathrm{j}\omega c_d+k_d)\\
&\quad\times(\bar a^3\overline{EI}(1+\bar c\overline{ch}+(\mathrm{j}\omega\bar c_b+\bar k_b)(\bar s\overline{ch}-\bar c\,\overline{sh}))+2N\bar a^3\,\overline{EI}(-m_d\omega^2\\
&\quad+\mathrm{j}\omega c_d+k_d)(-\bar a^3\,\overline{EI}(\bar s\overline{ch}-\bar c\overline{sh})+2\bar c\,\overline{ch}(\mathrm{j}\omega\bar c_b+\bar k_b))), \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
W_c&=F_e(\mathrm{j}\omega c_b+k_b)(\bar a^3\overline{EI}(1+\bar c\overline{ch})+(\mathrm{j}\omega\bar c_b+\bar k_b)(\bar s\overline{ch}-\bar c\,\overline{sh}))\\
&\quad/(-m_d\omega^2(\mathrm{j}\omega c_b+k_b)(\bar a^3\overline{EI}(1+\bar c\overline{ch})+(\mathrm{j}\omega\bar c_b+\bar k_b)(\bar s\overline{ch}-\bar c\overline{sh}))\\
&\quad+2N\bar a^3\overline{EI}(-m_d\omega^2+\mathrm{j}\omega c_b+k_b)(-\bar a^3\,\overline{EI}(\bar s\overline{ch}-\bar c\,\overline{sh})+2\bar c\overline{ch}(\mathrm{j}\omega\bar c_b+\bar k_b))). \qquad (26)
\end{aligned}
$$

In the same way, the transfer function H_{F_e,W_c} can be simplified. Since the structural properties and the boundary conditions of each beam are the same, the complex amplitude of the displacement of the ends of each beam $W_{b,i}$ will be identical and is represented as

$$W_{b,i} = F_e \bar{a}^3 \overline{EI}(\bar{c} + \overline{ch})(j\omega c_d + k_d)(\bar{a}^3 \overline{EI}(1 + \bar{c}\overline{ch}) + (j\omega \bar{c}_b + \bar{k}_b)(\bar{s}\,\overline{ch} - \bar{c}\overline{sh}))$$

$$/(\bar{a}^3 \overline{EI}(1 + \bar{c}\overline{ch}) + (j\omega \bar{c}_b + \bar{k}_b)(\bar{s}\overline{ch} - \bar{c}\overline{sh}))(-m_d\omega^2(j\omega c_d + k_d)$$

$$(\bar{a}^3 \overline{EI}(1 + \bar{c}\,\overline{ch}) + (j\omega \bar{c}_b + \bar{k}_b)(\bar{s}\,\overline{ch} - \bar{c}\overline{sh})) + 2N\bar{a}^3 \overline{EI}(-m_d\omega^2$$

$$+ j\omega c_d + k_d)(-\bar{a}^3 \overline{EI}(\bar{s}\,\overline{ch} - \bar{c}\overline{sh}) + 2\bar{c}\,\overline{ch}(j\omega \bar{c}_b + \bar{k}_b))). \quad (27)$$

Substituting the simplified transfer function into equation (22) yields

$$T_{F_e,F_b} = |2N\bar{a}^3 \overline{EI}(j\omega c_d + k_d)(-\bar{a}^3 \overline{EI}(\bar{s}\overline{ch} - \bar{c}\overline{sh}) + 2\bar{c}\,\overline{ch}(j\omega \bar{c}_b + \bar{k}_b))$$

$$/(-m_d\omega^2(j\omega c_d + k_d)(\bar{a}^3 \overline{EI}(1 + \bar{c}\,\overline{ch}) + (j\omega \bar{c}_b + \bar{k}_b)(\bar{s}\overline{ch} - \bar{c}\overline{sh}))$$

$$+ 2N\bar{a}^3 \overline{EI}(-m_d\omega^2 + j\omega c_d + k_d)(-\bar{a}^3 \overline{EI}(\bar{s}\overline{ch} - \bar{c}\,\overline{sh}) + 2\bar{c}\overline{ch}(j\omega \bar{c}_b + \bar{k}_b)))|. \quad (28)$$

The frequency equation can be determined in the same way and expressed as

$$- m_d\omega^2(j\omega c_d + k_d)(\bar{a}^3 \overline{EI}\,(1 + \bar{c}\,\overline{ch}) + (j\omega \bar{c}_b + \bar{k}_b)(\bar{s}\overline{ch} - \bar{c}\,\overline{sh}))$$

$$+ 2N\bar{a}^3 \overline{EI}(-m_d\omega^2 + j\omega c_d + k_d)(-\bar{a}^3 \overline{EI}(\bar{s}\,\overline{ch} - \bar{c}\,\overline{sh}) + 2\bar{c}\,\overline{ch}(j\omega \bar{c}_b + \bar{k}_b)) = 0. \quad (29)$$

4.2. SIMPLY SUPPORTED BEAMS

If the stiffness of the foundation is very large, the boundary conditions of each beam can be modelled as simply supported. The graph model of each beam as shown in Figure 3 can be modified to fit the assumption. In the modified model, the gain for the dynamics of the foundation is deleted. Thus, the transfer function $H_{W_c,F_{c,i}}$ can be reduced to

$$H_{W_c,F_{c,i}} = 4a_i^3 E_i I_i c_i ch_i/(s_i ch_i - c_i sh_i). \quad (30)$$

When equation (30) is substituted into equations (17) and (18), the complex amplitude of the displacement of the mounted mass W_d and the displacement at the joint of the beams W_c can be written as

$$W_d = \left(j\omega c_d + k_d + \sum_{i=1}^{N} (4a_i^3 E_i I_i c_i ch_i/(s_i ch_i - c_i sh_i))\right) F_e/(-m_d\omega^2(j\omega c_d + k_d)$$

$$+ (-m_d\omega^2 + j\omega c_d + k_d) \sum_{i=1}^{N} (4a_i^3 E_i I_i c_i ch_i/(s_i ch_i - c_i sh_i))\Big), \quad (31)$$

$$W_c = (j\omega c_d + k_d)F_e/(-m_d\omega^2(j\omega c_d + k_d)$$

$$+ (-m_d\omega^2 + j\omega c_d + k_d) \sum_{i=1}^{N} (4a_i^3 E_i I_i c_i ch_i/(s_i ch_i - c_i sh_i))). \quad (32)$$

The reduced transfer function H_{F_e,W_c} can also be obtained. Substituting H_{F_e,W_c} into equation (22) yields the force transmissibility T_{F_e,F_b} as

$$T_{F_e,F_b} = \left|(\mathrm{j}\omega c_d + k_d)\sum_{i=1}^{N}(4a_i^3 E_i I_i c_i ch_i/(s_i ch_i - c_i sh_i))F_e\Big/\Big(-m_d\omega^2(\mathrm{j}\omega c_d + k_d) + (-m_d\omega^2 + \mathrm{j}\omega c_d + k_d)\sum_{i=1}^{N}(4a_i^3 E_i I_i c_i ch_i/(s_i ch_i - c_i sh_i))\Big)\right| \quad (33)$$

and the frequency equation becomes

$$-m_d\omega^2(\mathrm{j}\omega c_d + k_d) + (-m_d\omega^2 + \mathrm{j}\omega c_d + k_d)\sum_{i=1}^{N}(4a_i^3 E_i I_i c_i ch_i/(s_i ch_i - c_i sh_i)) = 0. \quad (34)$$

When the structural properties of each simply supported beam are also identical, the transfer function $H_{W_c,F_{c,i}}$ for each i is the same and given by

$$H_{W_c,F_{c,i}} = 4\bar{a}^3\overline{EI}\,\bar{c}\,\overline{ch}/(\bar{s}\,\overline{ch} - \bar{c}\,\overline{sh}). \quad (35)$$

Substituting equation (35) into equations (17)–(19) yields the complex amplitudes W_d and W_c and transfer function H_{F_e,W_c}, in which W_d and H_{F_e,W_c} are

$$W_d = F_e((\mathrm{j}\omega c_d + k_d)(\bar{s}\,\overline{ch} - \bar{c}\,\overline{sh}) + 4N\bar{a}^3\overline{EI}\,\bar{c}\,\overline{ch})/(-m_d\omega^2(\mathrm{j}\omega c_d + k_d)(\bar{s}\,\overline{ch} - \bar{c}\,\overline{sh} + 4N\bar{a}^3\overline{EI}\,\bar{c}\,\overline{ch}(-m_d\omega^2 + \mathrm{j}\omega c_d + k_d))), \quad (36)$$

$$H_{F_e,W_c} = (\mathrm{j}\omega c_d + k_d)(\bar{s}\,\overline{ch} - \bar{c}\,\overline{sh})/(-m_d\omega^2(\mathrm{j}\omega c_d + k_d)(\bar{s}\,\overline{ch} - \bar{c}\,\overline{sh}) + 4N\bar{a}^3\dot{\overline{EI}}\,\bar{c}\,\overline{ch}(-m_d\omega^2 + \mathrm{j}\omega c_d + k_d)). \quad (37)$$

According to equation (36), the force transmissibility T_{F_e,F_b} becomes

$$T_{F_e,F_b} = |4N\bar{a}^3\overline{EI}\,\bar{c}\,\overline{ch}(\mathrm{j}\omega c_d + k_d)/(-m_d\omega^2(\mathrm{j}\omega c_d + k_d)(\bar{s}\,\overline{ch} - \bar{c}\,\overline{sh}) + 4N\bar{a}^3\overline{EI}\,\bar{c}\,\overline{ch}(-m_d\omega^2 + \mathrm{j}\omega c_d + k_d))| \quad (38)$$

and the frequency equation is given by

$$-m_d\omega^2(\mathrm{j}\omega c_d + k_d)(\bar{s}\,\overline{ch} - \bar{c}\,\overline{sh}) + 4N\bar{a}^3\overline{EI}\,\bar{c}\,\overline{ch}(-m_d\omega^2 + \mathrm{j}\omega c_d + k_d) = 0. \quad (39)$$

4.3. CONCENTRATED MASS ATTACHED TO CROSSED BEAMS

If the concentrated mass directly attached to the beams structure is considered, the state flow passing through the block of $-(\mathrm{j}\omega c_d + k_d)^{-1}$ will be neglected. Thus, the displacement responses of the concentrated mass coincide with that of the joint of each beam. This section examines four cases for the different conditions.

If the first case considers the flexibility of the foundation, the transfer function H_{F_e,W_c} can be rewritten as

$$H_{F_e,W_c} = 1 \bigg/ \left(-m_d\omega^2 + \sum_{i=1}^{N} H_{W_c,F_{c,i}} \right). \tag{40}$$

By substituting equation (40) into equations (17) and (18), the complex amplitudes W_d and W_c are the same and given by

$$W_d = F_e \bigg/ \bigg(-m_d\omega^2 + \sum_{i=1}^{N} (2a_i^3 E_i I_i(-a_i^3 E_i I_i(s_i ch_i - c_i sh_i) + 2c_i ch_i(\mathrm{j}\omega c_{b,i} + k_{b,i}))$$
$$/(a_i^3 E_i I_i(1 + c_i ch_i) + (\mathrm{j}\omega c_{b,i} + k_{b,i})(s_i ch_i - c_i sh_i)) \bigg). \tag{41}$$

The force transmissibility T_{F_e,F_b} becomes

$$T_{F_e,F_b} = \bigg| \sum_{i=1}^{N} (a_i^3 E_i I_i(c_i + ch_i)(\mathrm{j}\omega c_{b,i} + k_{b,i})/(\mathrm{j}\omega c_{b,i} + k_{b,i})(s_i ch_i - c_i sh_i)$$
$$+ a_i^3 E_i I_i(1 + c_i\, ch_i)) \bigg/ \bigg(-m_d\omega^2 + \sum_{i=1}^{N} (2a_i^3 E_i I_i(-a_i^3 E_i I_i(s_i ch_i - c_i sh_i)$$
$$+ 2c_i ch_i(\mathrm{j}\omega c_{b,i} + k_{b,i}))/(a_i^3 E_i I_i(1 + c_i ch_i) + (\mathrm{j}\omega c_{b,i} + k_{b,i})(s_i ch_i - c_i sh_i)) \bigg) \bigg| \tag{42}$$

and the frequency equation is

$$-m_d\omega^2 + \sum_{i=1}^{N} (2a_i^3 E_i I_i(-a_i^3 E_i I_i(s_i ch_i - c_i sh_i) + 2c_i ch_i(\mathrm{j}\omega c_{b,i} + k_{b,i}))$$
$$/(a_i^3 E_i I_i(1 + c_i ch_i) + (\mathrm{j}\omega c_{b,i} + k_{b,i})(s_i ch_i - c_i sh_i))) = 0. \tag{43}$$

In the second case, if the structural properties and flexibility of the foundation are also identical for each beam, W_d can be rewritten as follows:

$$W_d = F_e(\bar{a}^3\overline{EI}(1 + \bar{c}\overline{ch}) + (\mathrm{j}\omega\bar{c}_b + \bar{k}_b)(\bar{s}\overline{ch} - \bar{c}\overline{sh}))/(-m_d\omega^2(\bar{a}^3\overline{EI}(1 + \bar{c}\overline{ch})$$
$$+ (\mathrm{j}\omega\bar{c}_b + \bar{k}_b)(\bar{s}\overline{ch} - \bar{c}\overline{sh})) + 2N\bar{a}^3\overline{EI}(-\bar{a}^3\overline{EI}(\bar{s}\overline{ch} - \bar{c}\overline{sh}) + 2\bar{c}\overline{ch}(\mathrm{j}\omega\bar{c}_b + \bar{k}_b))). \tag{44}$$

Thus, the force transmissibility T_{F_e,F_b} becomes

$$T_{F_e,F_b} = |2N\bar{a}^3\overline{EI}(-\bar{a}^3\overline{EI}(\bar{s}\overline{ch} - \bar{c}\overline{sh}) + 2\bar{c}\overline{ch}(\mathrm{j}\omega\bar{c}_b + \bar{k}_b))$$
$$/(-m_d\omega^2(\bar{a}^3\overline{EI}(1 + \bar{c}\overline{ch}) + (\mathrm{j}\omega\bar{c}_b + \bar{k}_b)(\bar{s}\overline{ch} - \bar{c}\overline{sh}))$$
$$+ 2N\bar{a}^3\overline{EI}(-\bar{a}^3\overline{EI}(\bar{s}\overline{ch} - \bar{c}\overline{sh}) + 2\bar{c}\overline{ch}(\mathrm{j}\omega\bar{c}_b + \bar{k}_b)))| \tag{45}$$

and the frequency equation is

$$-m_d\omega^2(\bar{a}^3\overline{EI}(1+\bar{c}\overline{ch})+(\mathrm{j}\omega\bar{c}_b+\bar{k}_b)(\bar{s}\overline{ch}-\bar{c}\overline{sh}))+2N\bar{a}^3\overline{EI}$$
$$(-\bar{a}^3\overline{EI}(\bar{s}\overline{ch}-\bar{c}\overline{sh})+2\bar{c}\overline{ch}(\mathrm{j}\omega\bar{c}_b+\bar{k}_b))=0. \quad (46)$$

In the third case, both the flexibility of the foundation and the elasticity of the primary system are considered to be rigid. If the concentrated mass attached to the simply supported beams is considered, equation (17) can be reduced to

$$W_d = F_e \bigg/ \left(-m_d\omega^2 + \sum_{i=1}^{N} (4a_i^3 E_i I_i c_i ch_i/(s_i ch_i - c_i sh_i))\right). \quad (47)$$

Then, the force transmissibility T_{F_e,F_b} becomes

$$T_{F_e,F_b} = \left|\sum_{i=1}^{N}\left((a_i^3 E_i I_i(c_i+ch_i)/(s_i ch_i - c_i sh_i))\bigg/(-m_d\omega^2+\sum_{i=1}^{N}(4a_i^3 E_i I_i c_i ch_i/(s_i ch_i - c_i sh_i)))\right)\right| \quad (48)$$

and the frequency equation is

$$-m_d\omega^2+\sum_{i=1}^{N}(4a_i^3 E_i I_i c_i ch_i/(s_i ch_i - c_i sh_i))=0. \quad (49)$$

In the last case, when all simply supported beams of the structure are also identical, W_d and the force transmissibility T_{F_e,F_b} can be reduced to

$$W_d = (\bar{s}\overline{ch}-\bar{c}\overline{sh})/(-m_d\omega^2(\bar{s}\overline{ch}-\bar{c}\overline{sh})+4N\bar{a}^3\overline{EI}\bar{c}\overline{ch}), \quad (50)$$

$$T_{F_e,F_b} = |4N\bar{a}^3\overline{EI}\bar{c}\overline{ch}/(-m_d\omega^2(\bar{s}\overline{ch}-\bar{c}\overline{sh})+4N\bar{a}^3\overline{EI}\bar{c}\overline{ch})|. \quad (51)$$

The frequency equation becomes

$$-m_d\omega^2(\bar{s}\overline{ch}-\bar{c}\overline{sh})+4N\bar{a}^3\overline{EI}\bar{c}\overline{ch}=0. \quad (52)$$

Equations (50)–(52) for $N=1$ can be used for the vibration analysis of a single simply supported beam carrying a concentrated mass. These simplified results are identical to those derived in reference [12].

5. EXAMPLES

Two examples demonstrate the feasibility of the derived formula for analysis. In the first example, two identical crossed simply supported beams carrying an elastically mounted mass subjected to force load $F_e \sin\omega t$ on the mass are considered. The damping ratio and the natural frequency of the primary system are 0·05 and 100 rad/s respectively. The mass ratio of the mounted mass to each beam, $m_d/(2\bar{m}\bar{b})$ is 0·5. If the parameter of each beam $\bar{b}^2\sqrt{\bar{m}/\overline{EI}}$ equals $1/\sqrt{g}$, the frequency variable $\bar{a}\bar{b}$ equals $\sqrt{\omega/\sqrt{g}}$. The complex amplitude of the displacement of concentrated mass can be calculated by equation (36), in which $\bar{a}^3\overline{EI}$

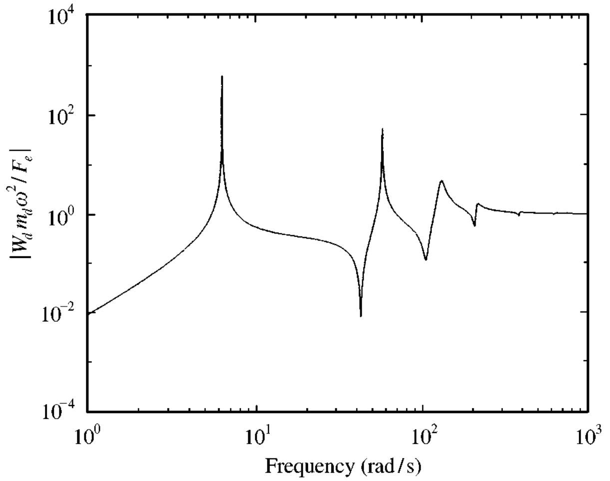

Figure 6. Magnitude of the non-dimensional dynamic response of the mounted mass with flexible supports.

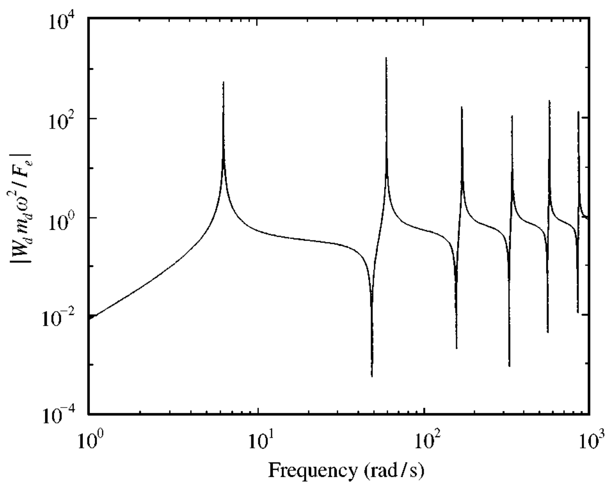

Figure 7. Magnitude of the non-dimensional dynamic response of the mounted mass directly attached on beams.

can be rewritten by $\bar{m}\bar{b}\omega^2/(\bar{a}\,\bar{b})$. Figure 6 presents the magnitude of the non-dimensional dynamic response of the mounted mass, defined as $-Y_d m_d \omega^2/F_e$.

If the concentrated mass of the first example is attached on the cross of beams, the dynamic response of the structure can be calculated by equation (50). Figure 7 illustrated the magnitude of the non-dimensional dynamic response of the mounted mass. According to Figures 6 and 7, a flexible connection between the concentrated mass and the beams can more effectively reduce the forced vibration at a high frequency than a rigid attachment.

6. CONCLUSIONS

This work presents a TWSF graphic model to represent the dynamic interaction of an elastically mounted mass supported on the beams with a flexible foundation. According to the graph model and substructure concept, analytical and closed-form results of the frequency response of the displacement of each component of the system, the force transmissibility, and the frequency equation are derived. Also presented herein are some special cases, including mass directly mounted on the beams, simply supported boundary support for crossed beams, identical structure properties for each beam, and their combination. Numerical examples reveal the ease in calculating the dynamic response using the derived formulae. Results of this study demonstrate the effectiveness of the proposed method in analyzing and designing the isolation structure with a vibrating machine.

ACKNOWLEDGMENT

The authors would like to thank the National Science Council of the Republic of China for financially supporting this research under Contract No. NSC 89-2611-E-002-027.

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