Journal of Sound and Vibration (2000) **234**(3), 521–535
doi:10.1006/jsvi.2000.2890, available online at http://www.idealibrary.com on IDEAL®

VIBRATIONS OF AN AXIALLY ACCELERATING BEAM WITH SMALL FLEXURAL STIFFNESS

E. ÖZKAYA AND M. PAKDEMİRLİ

Department of Mechanical Engineering, Celal Bayar University, 45140, *Muradiye, Manisa, Turkey.*
E-mail: *mpak@nil.com.tr*

(*Received* 31 *August* 1999, *and in final form* 18 *January* 2000)

Transverse vibrations of an axially moving beam are considered. The axial velocity is harmonically varying about a mean velocity. The equation of motion is expressed in terms of dimensionless quantities. The beam effects are assumed to be small. Since, in this case, the fourth order spatial derivative multiplies a small parameter, the mathematical model becomes a boundary layer type of problem. Approximate solutions are searched using the method of multiple scales and the method of matched asymptotic expansions. Results of both methods are contrasted with the outer solution.

1. INTRODUCTION

Band-saws, fiber textiles, paper sheets, aerial cable tramways, oil pipelines, magnetic tapes and power transmission belts are all classified as axially moving continua. A vast amount of literature exists on the topic which is reviewed by Ulsoy *et al.* [1] and Wickert and Mote [2]. These review papers cover the literature up to 1988. Wickert and Mote [3] studied both the second order and fourth order models separately. They developed a formalism in which the equations were cast in a suitable form for which the travelling string eigenfunctions are orthogonal. By using complex forms, a more compact representation of the solutions were obtained by the same authors [4]. Pakdemirli and Ulsoy [5] showed that, if a direct-perturbation method is used instead of a discretization-perturbation method, there is no need to express the equation of motion in a convenient form as was done in references [3, 4]. A non-linear analysis including stretching effects were performed by Wickert [6]. Recently, a stability analysis was done by Öz and Pakdemirli [7] for a travelling beam with harmonically varying axial velocity.

Many of the systems such as power-transmission belts, band-saws and pipes transporting fluids may either be modelled as a string or a beam. Therefore, the transition behavior from a string to a beam becomes significant. Since, in the transition phase, the flexural rigidity term is small compared to other terms, the highest order derivative is multiplied by a small parameter which makes it necessary to construct a boundary layer type of solution. Boundary layer solutions consists of two parts: (1) an outer solution which is valid for the whole domain except in a very small region near the boundaries. This solution does not in general satisfy the boundary conditions imposed by the boundaries, (2) an inner solution which is valid near the boundaries. This solution has to satisfy the boundary conditions. Inner and outer solutions are then matched and a composite expansion valid for all parts of the domain are constructed. For the problem considered, an outer solution which is valid everywhere except at the ends was constructed by Öz *et al.* [8]. The velocity was harmonically varying about a constant mean velocity in that study. Pellicano and Zirilli [9] found a composite expansion including the inner and outer solutions of the constant velocity case. Their analysis include both linear and non-linear terms. Using the method of multiple scales, Pakdemirli and Özkaya [10] constructed a composite expansion for the constant velocity case.

Most of the work on axially moving continua dealt with constant axial velocity. Real systems however are subject to accelerations and decelerations. In band-saws, belts and wire-saw manufacturing small speed fluctuations do occur. In this work, the harmonically varying velocity case is investigated. Boundary layer solutions are constructed using the method of multiple scales and the method of matched asymptotic expansions. Results of those methods are contrasted with the outer solution. For a simply supported beam, the improvement in the solutions by using a boundary layer approach is the satisfaction of moment conditions at the ends. A solution corresponding to the fixed – fixed case is also presented.

2. EQUATION OF MOTION

The dimensionless equation of motion for a travelling beam with time-dependent velocity is [8] (see Figure 1)

$$\frac{\partial^2 y}{\partial t^2} + \frac{\mathrm{d}v}{\mathrm{d}t}\frac{\partial y}{\partial x} + 2v\,\frac{\partial^2 y}{\partial x\,\partial t} + (v^2 - 1)\,\frac{\partial^2 y}{\partial x^2} + \bar{v}_f^2\,\frac{\partial^4 y}{\partial x^4} = 0, \tag{1}$$

where y is the vertical displacement, $v(t)$ is the time-dependent axial velocity. For a detailed derivation of equation of motion for constant velocity case, see reference [6]. $\bar{v}_f^2$ is a dimensionless parameter defined as

$$\bar{v}_f^2 = \frac{EI}{PL^2}, \tag{2}$$

where EI is the flexural rigidity, P is the axial tension force and L is the length of the beam. The dimensionless quantities are defined from the corresponding dimensional ones (denoted by asterisk) as follows:

$$x = x^*/L, \qquad y = y^*/L, \qquad t = t^*\,(1/L)\,\sqrt{P/\rho A}, \qquad v = v^*/\sqrt{P/\rho A}, \tag{3}$$

where ρ is the density and A is the cross-sectional area of the beam. Now assume that the velocity is harmonically varying about a constant mean velocity

$$v = v_0 + \varepsilon v_1 \sin \Omega t, \tag{4}$$

Figure 1. Schematics of an axially moving beam.

where ε is a small parameter. The dimensional velocity variation frequency (Ω^*) is related to the dimensionless one (Ω) through the relation.

$$\Omega^* = \Omega(1/L)\sqrt{P/\rho A}. \tag{5}$$

If EI is small compared to PL^2, $\bar{v}_f^2$ may be chosen as

$$\bar{v}_f^2 = \varepsilon\, v_f^2. \tag{6}$$

Using conditions (4) and (6), the equation of motion (1) will be solved approximately in the following sections. In section 3, an approximate solution is presented using the method of matched asymptotic expansions. In section 4, the same problem is solved using the method of multiple scales. Solutions presented in sections 3 and 4 are for simply supported beams. In section 5, a boundary layer solution for a fixed–fixed beam is also presented.

3. METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

In this section, the method of matched asymptotic expansions (MMAE) [11] will be used to construct a uniform expansion valid for all ranges of the spatial variable. Since the equation treated is a partial differential equation and elimination of secularities from the time variable is needed, this method is combined with the method of multiple time scales by introducing two time variables $T_0 = t$ and $T_1 = \varepsilon t$. First the outer solution and then the inner solutions at both ends will be found. All solutions will be matched and a composite final solution will be constructed. The end conditions for simply supported beam are

$$y(0, t) = y(1, t) = 0. \qquad y''(0, t) = y''(1, t) = 0. \tag{7}$$

3.1. OUTER SOLUTION

First, an outer solution valid for all ranges of spatial variable except at the ends will be constructed. The outer expansion is

$$y^o(x, t; \varepsilon) = y_0^o(x, T_0, T_1) + \varepsilon y_1^o(x, T_0, T_1) + \cdots \tag{8}$$

Time derivatives are

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{D}_0 + \varepsilon \mathrm{D}_1, \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \mathrm{D}_0^2 + 2\varepsilon \mathrm{D}_0 \mathrm{D}_1 + \cdots. \tag{9}$$

Substituting (4), (6), (8) and (9) into the equation of motion and separating terms of different orders, one obtains

$$O(1)\text{:} \quad \mathrm{D}_0^2 y_0^o + 2v_0\, \mathrm{D}_0\, y_0^{o\prime} + (v_0^2 - 1) y_0^{o\prime\prime} = 0, \tag{10}$$

$$O(\varepsilon)\text{:} \quad \mathrm{D}_0^2 y_1^o + 2v_0 \mathrm{D}_0 y_1^{o\prime} + (v_0^2 - 1) y_1^{o\prime\prime} = -v_f^2 y_0^{o\prime\prime} - 2\mathrm{D}_0 \mathrm{D}_1 y_0^o - 2v_0 \mathrm{D}_1\, y_0^{o\prime}$$
$$- 2v_1 \sin \Omega T_0 D_0 y_0^{o\prime} - \Omega v_1 \cos \Omega T_0 y_0^{o\prime} - 2v_0 v_1 \sin \Omega T_0 y_0^{o\prime\prime}. \tag{11}$$

The solution of order 1 is

$$y_0^o(x, T_0, T_1) = A_n(T_1)\, \mathrm{e}^{\mathrm{i}\omega_n T_0}\ \ Y_n(x) + \bar{A}_n(T_1)\, \mathrm{e}^{-\mathrm{i}\omega_n T_0}\ \ \bar{Y}_n(x), \tag{12}$$

where

$$\omega_n = n\pi(1 - v_0^2), \qquad Y_n(x) = C_n \mathrm{e}^{\mathrm{i}\alpha_n x} \sin n\pi x, \qquad \alpha_n = n\pi v_0, \qquad n = 1, 2, 3, \dots . \tag{13}$$

Inserting equation (12) into equation (11) and eliminating secular terms, one has

$$\mathrm{D}_1 A_n - \mathrm{i} k_0 A_n = 0, \tag{14}$$

where

$$k_0 = \tfrac{1}{2}\, v_f^2 n^3 \pi^3 (v_0^4 + 6v_0^2 + 1). \tag{15}$$

Solution of equation (14) yields

$$A_n = A_0 \mathrm{e}^{\mathrm{i}k_0 T_1}. \tag{16}$$

Inserting equations (16) and (13) into equation (12) and rearranging using original variables, one has

$$y_0^o(x, t) = C_n \cos[(\omega + \varepsilon k_0) t + n\pi v_0 x + \theta] \sin n\pi x, \tag{17}$$

where C_n and θ are arbitrary constants.

The solution of order ε is

$$y_1^o(x, t) = C_n \left\{ -\frac{n\pi v_0 v_1}{\Omega} \cos \Omega t \sin n\pi x \sin[(\omega + \varepsilon k_0)t + n\pi v_0 x + \theta] + \frac{n\pi v_1}{\Omega} \cos \Omega t \cos n\pi x \cos[(\omega + \varepsilon k_0)t + n\pi v_0 x + \theta] \right\}. \tag{18}$$

Hence, the outer solution is

$$y^o(x, t) = C_n \left\{ \cos[(\omega + \varepsilon k_0)t + n\pi v_0 x + \theta] \sin n\pi x + \varepsilon \left(-\frac{n\pi v_0 v_1}{\Omega} \cos \Omega t \sin n\pi x \sin[(\omega + \varepsilon k_0)t + n\pi v_0 x + \theta] + \frac{n\pi v_1}{\Omega} \cos \Omega t \cos n\pi x \times \cos[(\omega + \varepsilon k_0)t + n\pi v_0 x + \theta] \right) \right\} + \cdots . \tag{19}$$

It is not expected for the outer solution to satisfy the end conditions. This solution satisfies $y(0, t) = y(1, t) = 0$ conditions with $O(\varepsilon)$ error but does not satisfy at all the moment conditions $y''(0, t) = y''(1, t) = 0$.

3.2. INNER SOLUTIONS

For each end of the beam, separate inner solutions should be constructed.

(i) *Inner solution at the left-hand side.*

One stretches the spatial variable as follows:

$$\xi = \frac{x}{\varepsilon^\gamma}. \tag{20}$$

Substituting this variable into the original equation, in the distinguished limit $\gamma = \frac{1}{2}$ and hence one obtains

$$\xi = \frac{x}{\sqrt{\varepsilon}}. \tag{21}$$

Assuming now an inner expansion of the form

$$y^i = y_0^i + \sqrt{\varepsilon}\, y_1^i + \varepsilon\, y_2^i + \cdots \tag{22}$$

and inserting equations (21) and (22) into the equation of motion, one finally obtains the following set of equations:

$$O(1): \quad v_f^2 \frac{\partial^4 y_0^i}{\partial \xi^4} - (1 - v_0^2)\frac{\partial^2 y_0^i}{\partial \xi^2} = 0, \tag{23}$$

$$O(\sqrt{\varepsilon}): \quad v_f^2 \frac{\partial^4 y_1^i}{\partial \xi^4} - (1 - v_0^2)\frac{\partial^2 y_1^i}{\partial \xi^2} = -2v_0 \frac{\partial^2 y_0^i}{\partial \xi \partial t}, \tag{24}$$

$$O(\varepsilon): \quad v_f^2 \frac{\partial^4 y_2^i}{\partial \xi^4} - (1 - v_0^2)\frac{\partial^2 y_2^i}{\partial \xi^2} = -2v_0 \frac{\partial^2 y_1^i}{\partial \xi \partial t} - \frac{\partial^2 y_0^i}{\partial t^2} - 2v_0 v_1 \sin \Omega t \frac{\partial^2 y_0^i}{\partial \xi^2}. \tag{25}$$

The conditions to be satisfied are

$$\frac{\partial^2 y_0^i}{\partial \xi^2}(0, t) = 0, \qquad \frac{\partial^2 y_1^i}{\partial \xi^2}(0, t) = 0, \qquad \frac{\partial^2 y_0^i}{\partial \xi^2}(0, t) = -\frac{\partial^2 y_0^o}{\partial x^2}(0, t). \tag{26}$$

The last condition is the matching condition with the outer solution so that the error for moment condition can be eliminated from the first term of approximation. If equations (23)–(25) are solved subject to the boundary conditions (26), the inner solution at the left-hand side is

$$y^i = C_n \varepsilon \frac{v_f^2}{1 - v_0^2} 2n^2 \pi^2 v_0 \sin[(\omega + \varepsilon k_0)t + \theta]\, e^{(-\sqrt{1 - v_0^2}/v_f)(x/\sqrt{\varepsilon})}. \tag{27}$$

(ii) *Inner solution at the right-hand side.*
For the right-hand side, the inner variable is

$$\eta = \frac{1 - x}{\sqrt{\varepsilon}}. \tag{28}$$

A similar analysis yields

$$y^I = C_n \varepsilon \frac{v_f^2}{1 - v_0^2} 2n^2 \pi^2 v_0 \cos n\pi \sin[(\omega + \varepsilon k_0)t + \theta]\, e^{(-\sqrt{1 - v_0^2}/v_f)(1 - x)/\sqrt{\varepsilon}}. \tag{29}$$

Combining all solutions (the left, right and outer expansions), the composite expansion valid for all ranges of x is

$$y(x, t) = C_n \Bigg\{ \sin n\pi x \cos[(\omega + \varepsilon k_0)t + n\pi v_0 x + \theta] + \varepsilon \Bigg(-\frac{n\pi v_0 v_1}{\Omega} \cos \Omega t \sin n\pi x$$

$$\sin[(\omega + \varepsilon k_0)t + n\pi v_0 x + \theta] + \frac{n\pi v_1}{\Omega} \cos\Omega t \cos n\pi x \cos[(\omega + \varepsilon k_0)t + n\pi v_0 x + \theta]$$

$$+\frac{v_f^2}{1-v_0^2}\,2n^2\pi^2 v_0\sin\left[(\omega+\varepsilon k_0)t+\theta\right]\,\mathrm{e}^{(-\sqrt{1-v_0^2}/v_f)(x/\sqrt{\varepsilon})}$$

$$\left.\left.+\frac{v_f^2}{1-v_0^2}\,2n^2\pi^2 v_0\cos n\pi\sin\left[(\omega+\varepsilon k_0)t+n\pi v_0+\theta\right]\mathrm{e}^{-(\sqrt{1-v_0^2}/v_f)(1-x)/\sqrt{\varepsilon}}\right)\right\}+\cdots. \tag{30}$$

Note that this solution as well as the solution presented in the next section are valid in the absence of principal parametric resonances ($\Omega \cong 2\omega_n$) or combination resonances ($\Omega \cong \omega_n \pm \omega_m$). Such resonant solutions have already been investigated analytically [5, 8] and numerically [12, 13] for a string.

4. METHOD OF MULTIPLE SCALES

In this section, the method of multiple scales [11] will be used to construct a uniform solution valid for all ranges of the spatial variable. Since the algebra is much involved in constructing a composite solution for both ends and for the middle part, we choose for simplicity to construct first a solution valid for the left-hand side and middle and then a solution valid for the right-hand side and middle. Finally, both solutions will be combined.

For spatial and time variation representing different scales, one may use the following variables:

$$x_0 = x, \qquad x_1 = \frac{x}{\sqrt{\varepsilon}}, \qquad T_0 = t, \qquad T_1 = \varepsilon t, \tag{31}$$

where x_0 is the outer spatial variable and x_1 is the inner stretched spatial variable at the left-hand side. Two time scales are used to eliminate the secularities. With respect to the new variables, the derivatives are defined as follows:

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial x_0} + \frac{1}{\sqrt{\varepsilon}}\frac{\partial}{\partial x_1},\\
\frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial x_0^2} + \frac{2}{\sqrt{\varepsilon}}\frac{\partial^2}{\partial x_0\,\partial x_1} + \frac{1}{\varepsilon}\frac{\partial^2}{\partial x_1^2},\\
\frac{\partial^4}{\partial x^4} &= \frac{\partial^4}{\partial x_0^4} + \frac{4}{\sqrt{\varepsilon}}\frac{\partial^4}{\partial x_0^3\partial x_1} + \frac{6}{\varepsilon}\frac{\partial^4}{\partial x_0^2\,\partial x_1^2} + \frac{4}{\varepsilon\sqrt{\varepsilon}}\frac{\partial^4}{\partial x_0\,\partial x_1^3} + \frac{1}{\varepsilon^2}\frac{\partial^4}{\partial x_1^4},\\
\frac{\partial}{\partial t} &= \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1},\\
\frac{\partial^2}{\partial t^2} &= \frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2}{\partial T_0\partial T_1}.
\end{aligned} \tag{32}$$

A suitable expansion for $y(x, t)$ would be

$$\begin{aligned}
y(x,t;\varepsilon) &= y_0(x_0, x_1, T_0, T_1) + \sqrt{\varepsilon}\,y_1(x_0, x_1, T_0, T_1) + \varepsilon\,y_2(x_0, x_1\, T_0, T_1)\\
&\quad + \varepsilon\sqrt{\varepsilon}\,y_3(x_0, x_1, T_0, T_1) + \varepsilon^2 y_4(x_0, x_1, T_0, T_1) + \cdots.
\end{aligned} \tag{33}$$

Substituting all into the original equation of motion, using the harmonically varying velocity function defined in equation (4), separating at each order of ε, one has

$$O\left(\frac{1}{\varepsilon}\right): \quad (v_0^2 - 1)\frac{\partial^2 y_0}{\partial x_1^2} + v_f^2 \frac{\partial^4 y_0}{\partial x_1^4} = 0, \tag{34}$$

$$O\left(\frac{1}{\sqrt{\varepsilon}}\right): \quad (v_0^2 - 1)\frac{\partial^2 y_1}{\partial x_1^2} + v_f^2 \frac{\partial^4 y_1}{\partial x_1^4} = -2v_0 \frac{\partial^2 y_0}{\partial x_1 \partial T_0} - 2(v_0^2 - 1)\frac{\partial^2 y_0}{\partial x_0 \partial x_1} - 4v_f^2 \frac{\partial^4 y_0}{\partial x_0 \partial x_1^3}, \tag{35}$$

$$\begin{aligned} O(1): \quad & (v_0^2 - 1)\frac{\partial^2 y_2}{\partial x_1^2} + v_f^2 \frac{\partial^4 y_2}{\partial x_1^4} = -\frac{\partial^2 y_0}{\partial T_0^2} - 2v_0 \frac{\partial^2 y_0}{\partial x_0 \partial T_0} - 2v_0 \frac{\partial^2 y_1}{\partial x_1 \partial T_0} \\ & - (v_0^2 - 1)\left(\frac{\partial^2 y_0}{\partial x_0^2} + 2\frac{\partial^2 y_1}{\partial x_0 \partial x_1}\right) - 2v_0 v_1 \sin \Omega T_0 \frac{\partial^2 y_0}{\partial x_1^2} - v_f^2\left(6\frac{\partial^4 y_0}{\partial x_0^2 \partial x_1^2} + 4\frac{\partial^4 y_1}{\partial x_0 \partial x_1^3}\right), \end{aligned} \tag{36}$$

$$\begin{aligned} O(\sqrt{\varepsilon}): \quad & (v_0^2 - 1)\frac{\partial^2 y_3}{\partial x_1^2} + v_f^2 \frac{\partial^4 y_3}{\partial x_1^4} = -\frac{\partial^2 y_1}{\partial T_0^2} - v_1 \Omega \cos \Omega T_0 \frac{\partial y_0}{\partial x_1} - 2v_0 \frac{\partial^2 y_0}{\partial x_1 \partial T_1} \\ & - 2v_0\left(\frac{\partial^2 y_1}{\partial x_0 \partial T_0} + \frac{\partial^2 y_2}{\partial x_1 \partial T_0}\right) - 2v_1 \sin \Omega T_0 \frac{\partial^2 y_0}{\partial x_1 \partial T_0} - (v_0^2 - 1)\left(\frac{\partial^2 y_1}{\partial x_0^2} + 2\frac{\partial^2 y_2}{\partial x_0 \partial x_1}\right) \\ & - 2v_0 v_1 \sin \Omega T_0\left(2\frac{\partial^2 y_0}{\partial x_0 \partial x_1} + \frac{\partial^2 y_1}{\partial x_1^2}\right) - v_f^2\left(4\frac{\partial^4 y_0}{\partial x_0^3 \partial x_1} + 6\frac{\partial^4 y_1}{\partial x_0^2 \partial x_1^2} + 4\frac{\partial^4 y_2}{\partial x_0 \partial x_1^3}\right) \end{aligned} \tag{37}$$

$$\begin{aligned} O(\varepsilon): \quad & (v_0^2 - 1)\frac{\partial^2 y_4}{\partial x_1^2} + v_f^2 \frac{\partial^4 y_4}{\partial x_1^4} = -\frac{\partial^2 y_2}{\partial T_0^2} - v_1 \Omega \cos \Omega T_0\left(\frac{\partial y_0}{\partial x_0} + \frac{\partial y_1}{\partial x_1}\right) \\ & - 2v_0\left(\frac{\partial^2 y_2}{\partial x_0 \partial T_0} + \frac{\partial^2 y_3}{\partial x_1 \partial T_0}\right) - 2v_1 \sin \Omega T_0\left(\frac{\partial^2 y_0}{\partial x_0 \partial T_0} + \frac{\partial^2 y_1}{\partial x_1 \partial T_0}\right) \\ & - (v_0^2 - 1)\left(\frac{\partial^2 y_2}{\partial x_0^2} + 2\frac{\partial^2 y_3}{\partial x_0 \partial x_1}\right) - 2v_0 v_1 \sin \Omega T_0\left(\frac{\partial^2 y_0}{\partial x_0^2} + 2\frac{\partial^2 y_1}{\partial x_0 \partial x_1} + \frac{\partial^2 y_2}{\partial x_1^2}\right) \\ & - v_1^2 \sin^2 \Omega T_0 \frac{\partial^2 y_0}{\partial x_1^2} - v_f^2\left(\frac{\partial^4 y_0}{\partial x_0^4} + 4\frac{\partial^4 y_1}{\partial x_0^3 \partial x_1} + 6\frac{\partial^4 y_2}{\partial x_0^2 \partial x_1^2} + 4\frac{\partial^4 y_3}{\partial x_0 \partial x_1^3}\right) \\ & - 2\frac{\partial^2 y_0}{\partial T_0 \partial T_1} - 2v_0\left(\frac{\partial^2 y_0}{\partial x_0 \partial T_1} + \frac{\partial^2 y_1}{\partial x_1 \partial T_1}\right). \end{aligned} \tag{38}$$

A solution of equation (34) is

$$\begin{aligned} y_0 = {} & A(x_0, T_0, T_1)\, \mathrm{e}^{(\sqrt{1 - v_0^2}/v_f)x_1} + B(x_0, T_0, T_1)\, \mathrm{e}^{-(\sqrt{1 - v_0^2}/v_f)x_1} + C(x_0, T_0, T_1)\, x_1 \\ & + D(x_0, T_0, T_1). \end{aligned} \tag{39}$$

For decaying solutions, one chooses $A = C = 0$. The term with B is a part of the inner solution. One may require $B = 0$ for not allowing the inner solution to appear at the first order. Hence

$$y_0 = D(x_0, T_0, T_1) \tag{40}$$

is the solution at this order.

For order $(1/\sqrt{\varepsilon})$, one substitutes the above solution into equation (35):

$$(v_0^2-1)\frac{\partial^2 y_1}{\partial x_1^2}+v_f^2\frac{\partial^4 y_1}{\partial x_1^4}=0. \tag{41}$$

One may now choose

$$y_1=0 \tag{42}$$

for simplicity. Inserting y_0 and y_1 to the right-hand side of equation (36), one obtains the following equation:

$$(v_0^2-1)\frac{\partial^2 y_2}{\partial x_1^2}+v_f^2\frac{\partial^4 y_2}{\partial x_1^4}=-\frac{\partial^2 D}{\partial T_0^2}-2v_0\frac{\partial^2 D}{\partial x_0\,\partial T_0}-(v_0^2-1)\frac{\partial^2 D}{\partial x_0^2}. \tag{43}$$

In order not to introduce secular terms, D should be selected such that

$$\frac{\partial^2 D}{\partial T_0^2}+2v_0\frac{\partial^2 D}{\partial x_0\partial T_0}+(v_0^2-1)\frac{\partial^2 D}{\partial x_0^2}=0. \tag{44}$$

Note that this equation is the equation for a strip moving with constant velocity. A decaying type solution is selected for y_2:

$$y_2=E_1(x_0,T_0\ T_1)\,\mathrm{e}^{(-\sqrt{1-v_0^2}/v_f)x_1}+F_1(x_0,T_0,T_1). \tag{45}$$

At order $\sqrt{\varepsilon}$, the solvability condition is

$$v_0\frac{\partial E_1}{\partial T_0}+(1-v_0^2)\frac{\partial E_1}{\partial x_0}=0 \tag{46}$$

and a decaying solution is selected as

$$y_3=G_1(x_0,T_0,T_1)\,\mathrm{e}^{-(\sqrt{1-v_0^2}/v_f)x_1} \tag{47}$$

Finally at order ε, the elimination of secularities yield

$$\frac{\partial^2 F_1}{\partial T_0^2}+2v_0\frac{\partial^2 F_1}{\partial x_0\partial T_0}+(v_0^2-1)\frac{\partial^2 F_1}{\partial x_0^2}=-v_f^2\frac{\partial^4 D}{\partial x_0^4}-2\frac{\partial^2 D}{\partial T_0\partial T_1}-2v_0\frac{\partial^2 D}{\partial x_0\partial T_1}$$
$$-v_1\Omega\cos\Omega T_0\frac{\partial D}{\partial x_0}-2v_1\sin\Omega T_0\frac{\partial^2 D}{\partial x_0\partial T_0}-2v_0v_1\sin\Omega T_0\frac{\partial^2 D}{\partial x_0^2}, \tag{48}$$

$$\frac{\partial^2 E_1}{\partial T_0^2}+2v_0\frac{\partial^2 E_1}{\partial x_0\partial T_0}+5(1-v_0^2)\frac{\partial^2 E_1}{\partial x_0^2}+2v_0v_1\sin\Omega T_0\frac{1-v_0^2}{v_f^2}E_1-2v_0\frac{\sqrt{1-v_0^2}}{v_f}\frac{\partial G_1}{\partial T_0}$$
$$-\frac{2(1-v_0^2)^{3/2}}{v_f}\frac{\partial G_1}{\partial x_0}=0. \tag{49}$$

Substituting the solutions obtained for the expansion, up to order ε, one has the approximate boundary layer solution

$$y=D(x_0,T_0,T_1)+\varepsilon(E_1(x_0,T_0,T_1)\ \mathrm{e}^{-(\sqrt{1-v_0^2}/v_f)x_1}+F_1(x_0,T_0,T_1))+\cdots. \tag{50}$$

The above solution contains the inner expansion at the left-hand side and the outer expansion. One may now calculate the outer expansion and the right-hand side boundary layer solution by defining the inner variable at the right-hand side:

$$x_2 = \frac{(1-x)}{\sqrt{\varepsilon}}. \tag{51}$$

A similar calculation with only inserting x_2 instead of x_1 makes some sign changes in the equations. The final solution of this case is

$$y = D(x_0, T_0, T_1) + \varepsilon (E_2(x_0, T_0, T_1)\ \mathrm{e}^{-(\sqrt{1-v_0^2}/v_f)x_2} + F_2(x_0, T_0, T_1)) + \cdots. \tag{52}$$

To obtain the composite expansion valid for all parts of the domain, one has to add solution (50) to solution (52) and subtract the outer solution which is common. Hence, the final solution is

$$y = D(x_0, T_0, T_1) + \varepsilon (E_1(x_0, T_0, T_1)\,\mathrm{e}^{-(\sqrt{1-v_0^2}/v_f)x_1} + E_2(x_0, T_0, T_1)\,\mathrm{e}^{-(\sqrt{1-v_0^2}/v_f)x_2}$$
$$+ F_1(x_0, T_0, T_1)) + \cdots . \tag{53}$$

4.1. BOUNDARY CONDITIONS AND DETERMINATION OF FUNCTIONS

The arbitrary functions given in the composite expansion (53) will be determined using the solvability conditions and boundary conditions. In an analogous manner given in section 3, the function $D(x_0, T_0, T_1)$, which is the first term in the outer solution, is found to be

$$D(x_0, T_0, T_1) = C_n \cos[(\omega + \varepsilon k_0)t + n\pi v_0 x + \theta] \sin n\pi x. \tag{54}$$

The remaining functions are found as follows:

$$E_1 = C_n \alpha_1 \sin[(\omega + \varepsilon k_0)t - n\pi v_0 x + \theta_1], \tag{55}$$

$$E_2 = C_n \alpha_2 \sin[(\omega + \varepsilon k_0)t - n\pi v_0 x + \theta_2], \tag{56}$$

$$F_1 = C_n \left\{ -\frac{n\pi v_0 v_1}{\Omega} \cos \Omega t \sin[(\omega + \varepsilon k_0)t + n\pi v_0 x + \theta] \sin n\pi x \right.$$
$$\left. + \frac{n\pi v_1}{\Omega} \cos \Omega t \cos[(\omega + \varepsilon k_0)t + n\pi v_0 x + \theta] \cos n\pi x \right\}. \tag{57}$$

To eliminate the error in satisfying the boundary conditions $y''(0, t) = y''(1, t) = 0$ for the first order of approximation, one has to select

$$\alpha_1 = \frac{2v_f^2 n^2 \pi^2 v_0}{(1-v_0^2)}, \quad \theta_1 = \theta, \tag{58}$$

$$\alpha_2 = \frac{2v_f^2 n^2 \pi^2 v_0}{(1-v_0^2)} \cos n\pi, \quad \theta_2 = \theta + 2n\pi v_0. \tag{59}$$

The final solution may be expressed by substituting all the functions found into the composite expansion (53):

$$y(x, t) = C_n \Bigg\{ \cos[(\omega + \varepsilon k_0)t + n\pi v_0 x + \theta] \sin n\pi x + \varepsilon$$

$$\times \Bigg[\frac{2v_f^2 n^2 \pi^2 v_0}{(1 - v_0^2)} \sin[(\omega + \varepsilon k_0)t - n\pi v_0 x + \theta]\, \mathrm{e}^{-(\sqrt{1 - v_0^2}/v_f)(x)/\sqrt{\varepsilon}}$$

$$+ \frac{2v_f^2 n^2 \pi^2 v_0}{(1 - v_0^2)} \cos n\pi \sin[(\omega + \varepsilon k_0)t - n\pi v_0 x + 2n\pi v_0 + \theta]\, \mathrm{e}^{-(\sqrt{1 - v_0^2}/v_f)(1 - x)/\sqrt{\varepsilon}}$$

$$- \frac{n\pi v_0 v_1}{\Omega} \cos \Omega t \sin[(\omega + \varepsilon k_0)t + n\pi v_0 x + \theta] \sin n\pi x$$

$$+ \frac{n\pi v_1}{\Omega} \cos \Omega t \cos[(\omega + \varepsilon k_0)t + n\pi v_0 x + \theta] \cos n\pi x \Bigg] \Bigg\} + \cdots . \qquad (60)$$

Solution (60) can be contrasted to the solution (30) obtained by matched asymptotic expansions. Both solutions are similar except that additional $(-n\pi v_0 x)$ terms appear in the coefficient functions of inner solutions in the method of multiple scales. Due to this difference, method of matched asymptotic expansions satisfy the boundary conditions with an $O(\varepsilon)$ error, there is an $O(\sqrt{\varepsilon})$ error for the moment conditions $(y''(0, t) = y''(1, t) = 0)$ in the method of multiple scales solutions. The error introduced in the deflection conditions is the same for both methods, namely $O(\varepsilon)$.

In Figure 2(a), MMAE, MMS and the outer solutions are compared for deflections. Figure 2(b) is a plot of three solutions for the second derivative of deflections. It can be seen that while there is no improvement in the outer solution for deflections using boundary layer solution, a substantial improvement compared to the outer solution is achieved in satisfying the moment conditions at the ends in both MMS and MMAE solutions. Note that MMS and MMAE solutions are indistinguishable for the special parameters selected. To distinguish both methods, another set of parameters are chosen and plots of deflection and moment curves are given in Figures 3(a) and 3(b) respectively. Note that MMAE solutions are better in satisfying the moment conditions. This is primarily due to the fact that, there is an $O(\varepsilon)$ error in satisfying moment conditions for MMAE solutions whereas the error introduced is $O(\sqrt{\varepsilon})$ for MMS.

Finally, both methods may yield better approximations if additional terms are considered in the expansions.

5. THE CASE OF FIXED–FIXED SUPPORTS

For fixed–fixed supports, the boundary conditions are

$$y(0, t) = y(1, t) = 0, \qquad y'(0, t) = y'(1, t) = 0. \qquad (61)$$

Since it is observed that the method of matched asymptotic expansions yielded slightly better results than the method of multiple scales, calculations are performed using MMAE

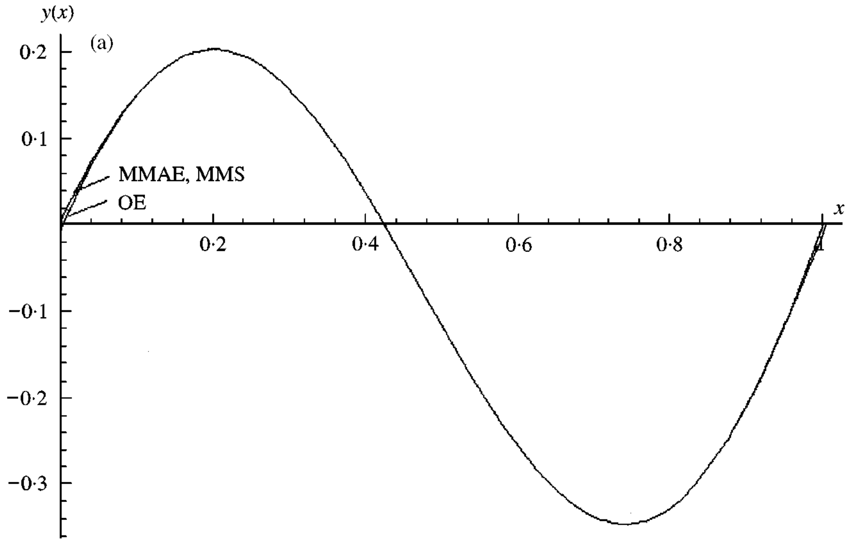

Figure 2 (a). Comparison of deflection curves for outer expansion, method of multiple scales and matched asymptotic expansion ($n = 1$, $v_0 = 0{\cdot}5$, $v_1 = 0{\cdot}1$, $\Omega = 5$, $v_f = 0{\cdot}1$, $t = 3$, $\varepsilon = 0{\cdot}1$, $C_n = 1$, $\theta = 0$).

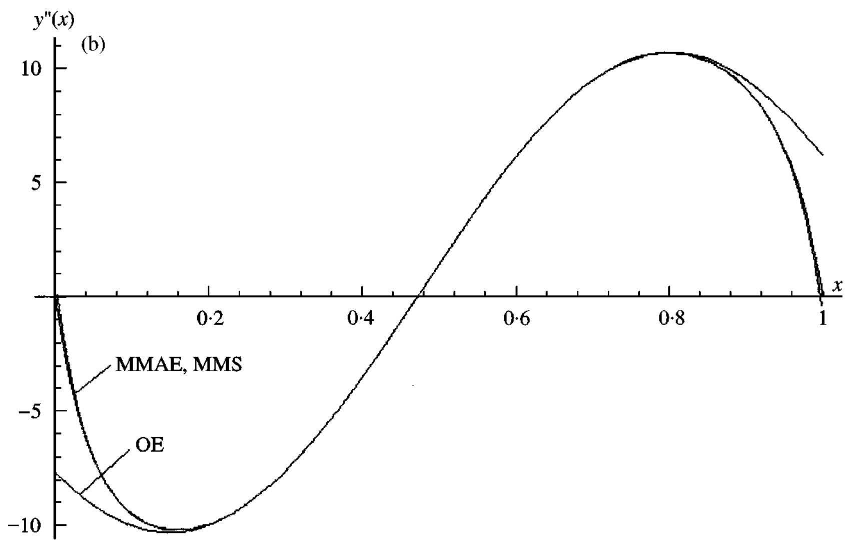

Figure 2 (b). Comparison of second derivative of deflection curves for outer expansion, method of multiple scales and matched asymptotic expansion ($n = 1$, $v_0 = 0{\cdot}5$, $v_1 = 0{\cdot}1$, $\Omega = 5$, $v_f = 0{\cdot}1$, $t = 3$, $\varepsilon = 0{\cdot}1$, $C_n = 1$, $\theta = 0$).

only. Carrying out the algebra similar to those given in section 3, one finally obtains the approximate solution as follows:

$$y(x, t) = C_n \Bigg\{ \sin n\pi x \cos[(\omega + \varepsilon k_0) t + n\pi v_0 x + \theta]$$

$$+ \sqrt{\varepsilon} \left(\frac{v_f}{\sqrt{1 - v_0^2}}\, n\pi \cos[(\omega + \varepsilon k_0) t + \theta]\, \mathrm{e}^{-(\sqrt{1 - v_0^2}/v_f)(x/\sqrt{\varepsilon})} \right)$$

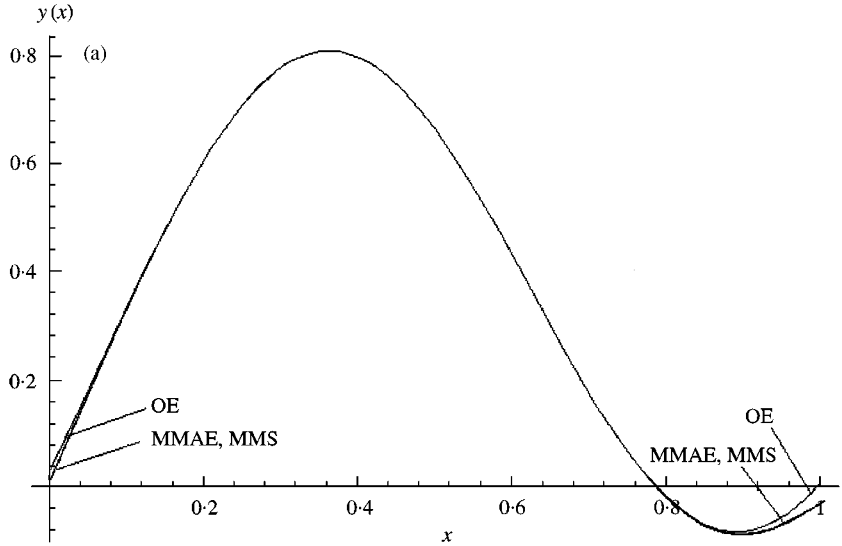

Figure 3 (a). Comparison of deflection curves for outer expansion, method of multiple scales and matched asymptotic expansion ($n = 1$, $v_0 = 0{\cdot}8$, $v_1 = 0{\cdot}5$, $\Omega = 5$, $\theta = 0$, $v_f = 0{\cdot}1$, $t = 10$, $\varepsilon = 0{\cdot}1$, $C_n = 1$).

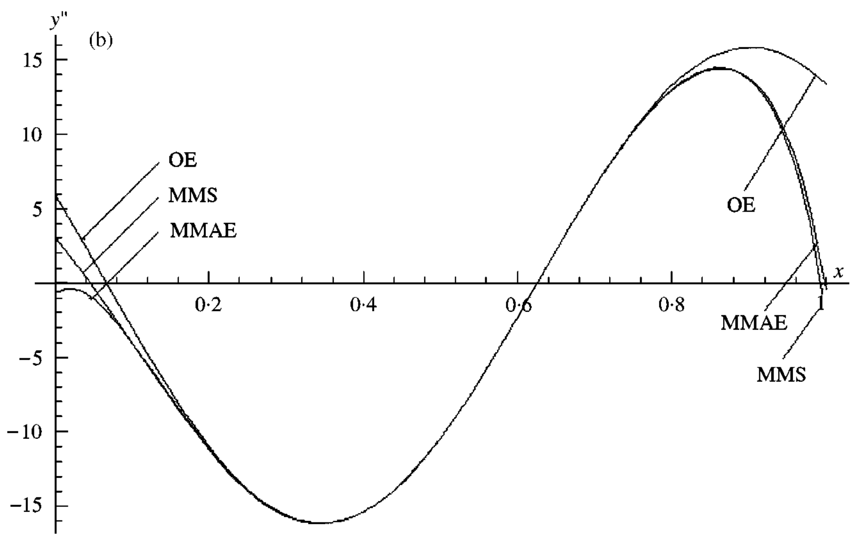

Figure 3 (b). Comparison of second derivative of deflection curves for outer expansion, method of multiple scales and matched asymptotic expansion ($n = 1$, $v_0 = 0{\cdot}8$, $v_1 = 0{\cdot}5$, $\Omega = 5$, $\theta = 0$, $v_f = 0{\cdot}1$, $t = 10$, $\varepsilon = 0{\cdot}1$, $C_n = 1$).

$$- \frac{v_f}{\sqrt{1 - v_0^2}}\, n\pi \cos n\pi \cos\left[(\omega + \varepsilon k_0) t + n\pi v_0 + \theta\right] \mathrm{e}^{-(\sqrt{1-v_0^2}/v_f)((1-x)/\sqrt{\varepsilon})}$$

$$+ \varepsilon\Bigg(- \frac{n\pi v_0 v_1}{\Omega} \cos \Omega t \sin n\pi x \sin\left[(\omega + \varepsilon k_0) t + n\pi v_0 x + \theta\right]$$

$$+ \frac{n\pi v_1}{\Omega} \cos \Omega t \cos n\pi x \cos\left[(\omega + \varepsilon k_0) t + n\pi v_0 x + \theta\right]\Bigg)\Bigg\} + \cdots . \tag{62}$$

In Figures 4(a) and 4(b), plots of deflection and slope curves of MMAE are contrasted to the outer solutions. For deflections, the outer solution has an $O(\varepsilon)$ error at the ends but the

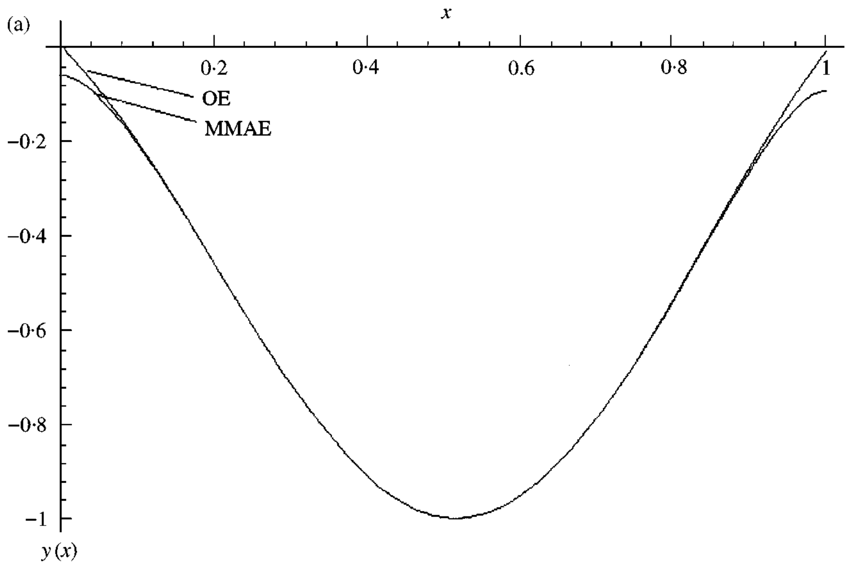

Figure 4 (a). Comparison of deflection curves for outer expansion, and matched asymptotic expansion for fixed–fixed end conditions ($n = 1$, $v_0 = 0{\cdot}6$, $v_1 = 0{\cdot}1$, $\Omega = 2$, $\theta = 0$, $v_f = 0{\cdot}1$, $t = 1$, $\varepsilon = 0{\cdot}1$, $C_n = 1$).

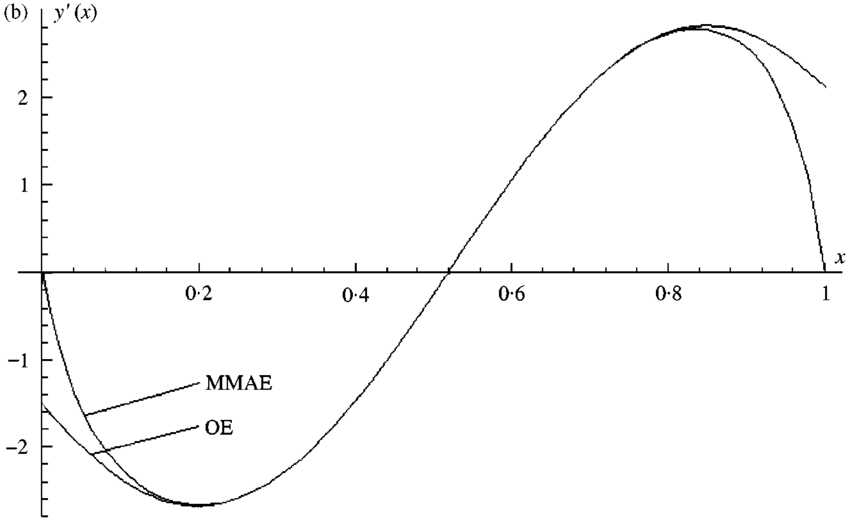

Figure 4 (b). Comparison of second derivative of deflection curves for outer expansion and matched asymptotic expansion for fixed–fixed end conditions. ($n = 1$, $v_0 = 0{\cdot}6$, $v_1 = 0{\cdot}1$, $\Omega = 2$, $\theta = 0$, $v_f = 0{\cdot}1$, $t = 1$, $\varepsilon = 0{\cdot}1$, $C_n = 1$).

MMAE solution has an $O(\sqrt{\varepsilon})$ error. This explains the coarse match of the boundary layer solution at the ends. For the slopes however, there is much improvement in employing boundary layer type solutions. The boundary layer solutions may be improved by adding an additional term in the perturbation expansion. This will require however extensive algebra.

6. CONCLUDING REMARKS

Approximate boundary layer solutions are presented for an axially accelerating beam with small beam effects. The method of matched asymptotic expansions and the method of multiple scales are applied to the problem and composite expansions including two inner solutions and one outer solution are found. Since exact solutions in closed-form functions are not available, approximate analytical solutions might be useful to check numerical work which may appear in the future.

By utilizing boundary-layer-type solutions, substantial improvements are achieved especially at the ends compared to the outer solution. It is found that method of matched asymptotic expansions solution is slightly better in satisfying the boundary conditions than the method of multiple scales solution. While three expansions are needed in finding method of matched asymptotic expansions solution, only two are sufficient in the case of the method of multiple scales solution. In MMS, matching conditions are not needed also. These advantages bring another disadvantage: constructing the solutions at each order of approximation is not as straightforward in the method of multiple scales as in the method of matched asymptotic expansions and requires some experience.

For finding boundary-layer-type solutions of axially moving materials, MMAE is recommended. Note however that different time scales are also used in eliminating secularities in this method. To be more precise, multiple scales in both spatial and time variables (MMS) is not recommended compared to the combination of multiple time scales and matched asymptotic expansions (MMAE).

Finally, all solutions presented here are non-resonant solutions. It is well known that principal and combination resonances occur for specific choices of the speed fluctuation frequency [5, 8, 12, 13]. Here, the speed fluctuation frequencies are assumed to be away from those critical values.

ACKNOWLEDGMENTS

This work is supported by the Scientific and Technical Research Council of Turkey (TÜBİTAK) under project no: MISAG-119.

REFERENCES

1. A. G. Ulsoy, C. D. Mote Jr. and R. Syzmani 1978 *Holz als Roh-und Werkstoff* **36**, 273–280. Principal developments in band saw vibration and stability research.
2. J. A. Wickert and C. D. Mote Jr. 1988 *Shock and Vibration Digest* **20**, 3–13. Current research on the vibration and stability of axially moving materials.
3. J. A. Wickert and C. D. Mote Jr. 1990 *ASME Journal of Applied Mechanics* **57**, 738–744. Classical vibration analysis of axially moving continua.
4. J. A. Wickert and C. D. Mote Jr. 1991 *Applied Mechanics Reviews* **44**, 279–284. Response and discretization methods for axially moving materials.
5. M. Pakdemirli and A. G. Ulsoy 1997 *Journal of Sound and Vibration* **203**, 815–832. Stability of an axially accelerating string.
6. J. A. Wickert 1992 *International Journal of Non-Linear Mechanics* **27**, 503–517. Non-linear vibration of a traveling tensioned beam.
7. H. R. Öz and M. Pakdemirli 1999 *Journal of Sound and Vibration* **227**, 239–257. Vibrations of an axially moving beam with time dependant velocity.
8. H. R. Öz, M. Pakdemirli and E. Özkaya 1998 *Journal of Sound and Vibration* **215**, 571–576. Transition behavior from string to beam for an axially accelerating material.
9. F. Pellicano and F. Zirilli 1998 *International Journal of Non-Linear Mechanics* **33**, 691–711. Boundary layers and non-linear vibrations in axially moving beam.
10. M. Pakdemirli and E. Özkaya 1998 *Mathematical and Computational Applications* **3**, 93–100. Approximate boundary layer solution of a moving beam problem.
11. A. H. Nayfeh 1981 *Introduction to Perturbation Techniques*. New York: John Wiley.

12. M. PAKDEMİRLİ, A. G. ULSOY and A. CERANOĞLU 1994 *Journal of Sound and Vibration* **169**, 179–196. Transverse vibration of an axially accelerating string.
13. M. PAKDEMİRLİ and H. BATAN 1993 *Journal of Sound and Vibration* **168**, 371–378. Dynamic stability of a constantly accelerating string.