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THE ELLIPTIC MULTIPLE SCALES METHOD FOR A CLASS OF AUTONOMOUS STRONGLY NON-LINEAR OSCILLATORS

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1. INTRODUCTION

Perturbation methods have been very successful in accurately predicting dynamical motion for weakly non-linear systems in the form

$$\ddot{x} + x = \varepsilon g(x, \dot{x}), \tag{1}$$

where ε is a small positive parameter and g is a polynomial function of its arguments. A number of methods are applicable in seeking approximate steady state, periodic solutions to equation (1). These include the methods of harmonic balance (HB) [1], Lindstedt–Poincaré (LP) [2], Krylov–Bogolioubov–Mitropolski (KBM) [3], averaging [4] and multiple scales (MS) [1]. In the former two methods, one seeks directly a periodic steady state solution, which is assumed *a priori* to occur. On the other hand, the latter three methods yield a set of first order differential equations which describe the slow time evolution of the amplitude and phase of the response. The periodic steady state solution is obtained by setting these amplitude and phase time derivatives to zero. The advantage of these latter methods is that they allow one in a single analysis to study both the steady state responses and their stability. All these methods are now considered to be classical standard tools for the analytical investigations of weakly non-linear systems.

The extension to strongly non-linear systems, where the unperturbed system is already non-linear, has not received the same attention for at least two reasons [5]. First, analytical solutions for non-linear systems are generally unknown, so that an analytical investigation cannot be carried out. Second, the perturbation schemes themselves become much more difficult to implement.

However, Mickens and Oyedeji [6] investigated a class of non-linear oscillator with the equation

$$\ddot{x} + x^3 = \varepsilon g(x, \dot{x}) \tag{2}$$

by using the HB method and the slowly varying amplitude and phase method with circular functions. Following this technique, Yuste and Bejarano [7] also investigated equation (2) but adopted Jacobian elliptic functions instead of circular ones. The accuracy of the elliptic functions method is obviously higher than that of the circular functions method.

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Recently, some researchers have presented several techniques for the more general case of the form

$$\ddot{x} + c_1 x + c_2 x^3 = \varepsilon g(\mu, x, \dot{x}), \quad (3)$$

where c_1, c_2 are constants and μ is a control parameter. For instance, Margallo *et al.* [8, 9] presented an elliptic HB method using generalized Fourier series and elliptic functions. Yuste and Bejarano [10] developed an elliptic KBM method and Coppola and Rand [11] used symbolic computation to implement an averaging method with elliptic functions. All the methods mentioned above have their own advantages to obtain approximate analytical solutions. However, most of them were only implemented to give first order approximate solutions. To obtain a second order approximation, Coppola [5] formulated an averaging method using the Lie transform method. In a series of papers, Chen and Cheung [12–14] performed an elliptic LP method and derived an elliptic perturbation method based on expanding the amplitude in a power series of ε.

Recently, the elliptic LP and averaging methods were conducted, respectively, by Belhaq *et al.* [15] and Belhaq and Lakrad [16] to derive a criterion of homoclinic bifurcation to an autonomous planar system aiming on the collision of the approximate periodic orbit with the saddle instead of considering as usual, the distance between the separatrix. The elliptic HB were also used to study mixed parity non-linear oscillators [17].

In this paper, we formulate the multiple scales method for studying the oscillators of type (3), in which the Jacobian elliptic functions are employed instead of the usual circular functions.

2. THE ELLIPTIC MULTIPLE SCALES METHOD

Consider the strongly non-linear oscillator (3), the solution and the operation of time differentiation of this equation are expressed in a power series of ε:

$$x(t; \varepsilon) = \sum_{m=0}^{\infty} \varepsilon^m x_m(t), \quad (4)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \sum_{m=0}^{\infty} \varepsilon^m D_m. \quad (5)$$

where $D_m = \partial/\partial T_m$ and $T_m = \varepsilon^m t$. Here T_m are independent scales of time which get slower and slower as m increases. Thus, $T_0 = \tau$ is a fast time scale on which the main oscillatory behaviour occurs and $T_i = \varepsilon^i \tau$ are slow time scales characterizing modulations of amplitudes and phases.

Expanding the function $g(\mu, x, \dot{x})$ in power series of ε as follows

$$g(\mu, x, \dot{x}) = \sum_{m=0}^{\infty} \varepsilon^m g_m(\mu, T_0, T_1, \ldots), \quad (6)$$

substituting equations (4)–(6) into equation (3), and equating coefficients of like powers of ε, lead to the following equations

- order $O(\varepsilon^0)$:

$$D_0^2 x_0 + c_1 x_0 + c_2 x_0^3 = 0, \quad (7)$$

- order $O(\varepsilon^1)$:

$$D_0^2 x_1 + c_1 x_1 + 3c_2 x_0^2 x_1 = -2D_0 D_1 x_0 + g_0(\mu, x_0, D_0 x_0), \tag{8}$$

- order $O(\varepsilon^i)$:

$$D_0^2 x_i + c_1 x_i + 3c_2 x_0^2 x_i = -2D_0 D_i x_0 + G_{i-1}(\mu, g_{i-1}), \tag{9}$$

where G_{i-1} depends on g_{i-1} and all solutions and derivatives previous to order i. Equation (7) has an exact analytical solution which can be expressed by Jacobian elliptic functions in the form

$$x_0 = A(T_1, T_2, \ldots, T_i) ep(\omega T_0 + \phi(T_1, T_2, \ldots, T_i), k^2). \tag{10}$$

Here $ep(.,k^2)$ is one convenient Jacobian elliptic function (i.e. (sn, cn) or (dn)) with modulus k; the quantities A, ω and ϕ are respectively the amplitude, the frequency and the phase. A survey of elliptic function properties is given in Appendix A.

Let the prime denote the derivatives of elliptic function with respect to its argument $u = \omega T_0 + \phi(T_1, T_2, \ldots, T_i)$. For a given (ep), $(ep)''$ can be written as

$$(ep)'' = [\alpha(k)(ep) + \beta(k)(ep)^3]. \tag{11}$$

Here $\alpha(k)$ and $\beta(k)$ are functions of the modulus k.

On the other hand, $(ep)'$ can be written as

$$(ep)' = \gamma(k)(ep_1)(ep_2), \tag{12}$$

where $\gamma(k)$ is a function of k, and (ep_1) and (ep_2) are other two elliptic functions which are different from (ep) (e.g., if $ep = \mathrm{dn}$, then $\gamma(k) = -k^2$, $ep_1 = \mathrm{sn}$ and $ep_2 = \mathrm{cn}$).

The frequency ω and the modulus k are hence expressed as functions of A, c_1 and c_2:

$$\omega^2 = -\frac{c_1}{\alpha(k)}, \tag{13}$$

$$\frac{\beta(k)}{\alpha(k)} = \frac{c_2}{c_1} A^2. \tag{14}$$

Equation (8) can be rewritten as follows

$$\omega^2 x_1'' + c_1 x_1 + 3c_2 x_0^2 x_1 = -2\omega(D_1 A)\cdot ep'(u, k^2) - 2\omega A(D_1\phi) ep''(u, k^2) + g_0(\mu, x_0, x_0'). \tag{15}$$

It is worth noting that the homogeneous equation of (8) has x_0' as a solution. Multiplying both sides of equation (15) by x_0' and then integrating the equation, we obtain

$$[\omega^2(x_0'\cdot x_1' - x_1\cdot x_0'')]_0^\tau + \int_0^\tau x_1[\omega^2 x_0''' + c_1 x_0' + 3c_2 x_0^2 x_0']\,\mathrm{d}u = \int_0^\tau [-2\omega(D_1 x_0') + g_0(x_0, x_0', \mu)]\, x_0'\,\mathrm{d}u \tag{16}$$

Differentiating equation (7) with respect to u leads to

$$\omega^2 x_0''' + c_1 x_0' + 3c_2 x_0^2 x_0' = 0. \tag{17}$$

Note that x_0 is a periodic function with period T (T is $4\mathrm{K}[k]$ for sn and cn or $2\mathrm{K}[k]$ for dn, $\mathrm{K}[k]$ is the first kind complete elliptic integral). The functions x_0' and x_0'' are also periodic functions with the period $4\mathrm{K}[k]$.

Assume that x_1 is also a periodic function with the period $4\mathrm{K}[k]$. Then letting $\tau = 4\mathrm{K}[k]$ in equation (16) gives

$$\int_0^{4\mathrm{K}[k]} [-2\omega(D_1 A)\cdot ep'(u, k^2) - 2\omega A(D_1\phi)\cdot ep''(u, k^2) + g_0(x_0, x_0', \mu)]\cdot x_0'\,\mathrm{d}u = 0. \tag{18}$$

Due to the oddness of the term related to the modulation of the phase and the necessary condition for having a periodic solution, i.e., the vanishing of the amplitude modulation $(D_1 A) = 0$, equation (18) becomes

$$\int_0^{4\mathrm{K}[k]} g_0(x_0, x_0', \mu)\cdot x_0'\,\mathrm{d}u = 0, \tag{19}$$

which is the periodicity condition in the case where equation (19) has a non-zero solution. This condition arises in a mathematically rigorous way from Melnilov's approach for bifurcation of periodic or homoclinic orbits. Thus, a particular solution of equation (16) with initial conditions $x_0'(0) = 0$, $x_1(0) = 0$, $x_1'(0) = 0$, respectively can be expressed as

$$x_1(u) = x_0'(u)\int_0^u \frac{1}{\omega^2 x_0'^2(\sigma_1)}\left\{\int_0^{\sigma_1} x_0'[-2\omega(D_1 A)\cdot ep'(u, k^2) + g_0(x_0, x_0', \mu) - 2\omega(D_1\phi)x_0'']\,\mathrm{d}\sigma_2\right\}\mathrm{d}\sigma_1. \tag{20}$$

Secular terms are produced by x_0'' in the bracket on the right-hand side of equation (20). Indeed, integrating the last term of equation (20) leads to

$$x_0'(u)\int_0^u \frac{1}{x_0'^2(\sigma_1)}\left[\int_0^{\sigma_1}\frac{2(D_1\phi)}{\omega}x_0'x_0''\,\mathrm{d}\sigma_2\right]\mathrm{d}\sigma_1 = \frac{(D_1\phi)}{\omega}x_0'(u)u. \tag{21}$$

Here the term $x_0'u$ tends to infinity as $u\to\infty$. However, in order that equation (4) remains a uniformly valid expansion, x_1/x_0 should be bounded for all u. To kill secular terms, $(D_1\phi)$ is chosen to eliminate the coefficient of x_0'' in the bracket on the right-hand side of equation (20). In the present case, where $g(\mu, x_0, x_0')$ does not contain the term x_0'' explicitly or implicitly, secular terms force one to choose $(D_1\phi) = 0$. Therefore, equation (20) becomes

$$x_1 = x_0'\int_0^u \frac{1}{\omega^2 x_0'^2}\left\{\int_0^{\sigma_1} x_0'[g_0(x_0, x_0', \mu)]\,\mathrm{d}\sigma_2\right\}\mathrm{d}\sigma_1. \tag{22}$$

This expression of x_1 is the same as the one given by the elliptic LP method [14]. For details for calculating x_1 in the case of the oscillator (3) see reference [14].

Taking into account the assumption that the scales of time are independent, the modulation equations of amplitude and phase are given by

$$(D_1 A) = \frac{\int_0^{4K[k]} g_0(\mu, x_0, x_0') \cdot x_0' \, du}{2\omega \int_0^{4K[k]} (ep') \cdot x_0' \, du}, \tag{23}$$

$$(D_1 \phi) = \frac{\int_0^{4K[k]} g_0(\mu, x_0, x_0') x_0 \, du - \int_0^{4K[k]} 2c_2 x_1 x_0^3 \, du}{2\omega \int_0^{4K[k]} x_0'' \cdot x_0 \, du}. \tag{24}$$

Equation (9) can be written as

$$\omega^2 x_i'' + c_1 x_i + 3c_2 x_0^2 x_i = -2\omega (D_i A) \cdot ep'(u, k^2) - 2\omega (D_i \phi) x_0'' + G_{i-1}(\mu, g_{i-1}), \tag{25}$$

A particular solution of equation (25) is given by

$$x_i = x_0' \int \frac{1}{\omega^2 x_0'^2} \left\{ \int x_0' [G_{i-1}(\mu, g_{i-1})] \, du \right\} du. \tag{26}$$

Hence, one obtains the modulation equations of amplitude A and phase ϕ with respect to the time scale T_i:

$$(D_i A) = \frac{\int_0^{4K[k]} G_{i-1}(\mu, g_{i-1}) \cdot x_0' \, du}{2\omega \int_0^{4K[k]} (ep') \cdot x_0' \, du}, \tag{27}$$

$$(D_i \phi) = \frac{\int_0^{4K[k]} G_{i-1}(\mu, g_{i-1}) \cdot x_0 \, du - \int_0^{4K[k]} 2c_2 x_1 x_0^3 \, du}{2\omega \int_0^{4K[k]} x_0'' \cdot x_0 \, du}. \tag{28}$$

3. CONCLUSION

The proposed multiple scales method using Jacobi elliptic functions offers the following advantages. It provides the second approximate term x_1 to correct the approximation x_0, in agreement with the LP method [14]. It gives also the modulation equations of amplitude and phase, in agreement with the elliptic KBM method [10]. Finally, it offers the possibility for deriving higher order approximations in an interactive way.

REFERENCES

1. A. H. NAYFEH and D. T. MOOK 1979 *Non-linear Oscillations*. New York: John Wiley.
2. A. H. NAYFEH 1981 *Introduction to Perturbation Techniques*. New York: Wiley.
3. N. N. BOGOLIOUBOV and Y. A. MITROPOLSKY 1961 *Asymptotic Methods in the Theory of Nonlinear Oscillations*. New York: Gordon and Breach.
4. A. H. NAYFEH 1973 *Perturbation Methods*. New York: John Wiley.

5. V. T. COPPOLA 1997 Nonlinear Dynamics: the Richard Rand 50th Anniversary Volume (A. Guran, editor). Singapore: World Scientific. Perturbation methods for strongly nonlinear oscillators using Lie transforms.
6. R. E. MICKENS and K. OYEDEJI 1985 *Journal of Sound and Vibration* **102**, 579–582. Construction of approximate analytical solutions to a new class of non-linear oscillator equation.
7. S. B. YUSTE and J. D. BEJARANO 1986 *Journal of Sound and Vibration* **110**, 347–350. Construction of approximate analytical solutions to a new class of non-linear oscillator equations.
8. J. G. MARGALLO, J. D. BEJARANO and S. B. YUSTE 1988 *Journal of Sound and Vibration* **125**, 13–21. Generalized Fourier series for the study of limit cycles.
9. J. G. MARGALLO and J. D. BEJARANO 1990 *Journal of Sound and Vibration* **136**, 453–466. Generalized Fourier series and limit cycles of generalized Van der Pol oscillators.
10. S. B. YUSTE and J. D. BEJARANO 1989 *International Journal of Control* **49**, 1127–1141. Extension and improvement to the Krylov–Bogolioubov methods using elliptic functions.
11. V. T. COPPOLA and R. H. RAND 1990 *Acta Mechanica* **81**, 125–142. Averaging using elliptic functions: approximations of limit cycle.
12. Y. K. CHEUNG, S. H. CHEN and S. L. LAU 1991 *International Journal of Non-Linear Mechanics* **26**, 367–378. A modified Lindstedt–Poincaré method for certain strongly non-linear oscillators.
13. S. H. CHEN and Y. K. CHEUNG 1996 *Journal of Sound and Vibration* **192**, 453–464. An elliptic-perturbation method for certain strongly non-linear oscillators.
14. S. H. CHEN and Y. K. CHEUNG 1997 *Nonlinear Dynamics* **12**, 199–213. An elliptic Lindstedt-Poincaré method for certain strongly non-linear oscillators.
15. M. BELHAQ, B. FIEDLER and F. LAKRAD 2000 *Nonlinear Dynamics*. Homoclinic connections in strongly self-excited nonlinear oscillators: the Melnikov function and the elliptic Lindstedt–Poincaré method (to be published).
16. M. BELHAQ and F. LAKRAD 2000 *Chaos Solitons and Fractals*. Prediction of homoclinic bifurcation: the elliptic averaging method (to be published).
17. M. BELHAQ and F. LAKRAD 2000 *Journal of Sound and Vibration*. On the elliptic harmonic balance method for mixed parity non-linear oscillators (to be published).
18. P. F. BYRD and M. D. FRIEDMAN 1971 *Handbook of Elliptic Integrals for Engineers and Scientists*. Berlin: Springer-Verlag.

APPENDIX A

For the convenience of readers, we collect some facts on Jacobian elliptic functions [18]. Jacobian elliptic functions are doubly periodic functions defined over the complex plane. They satisfy algebraic relations which are analogous to those for trigonometric functions. The fundamental three elliptic functions are $\mathrm{cn}(u, k)$, $\mathrm{sn}(u, k)$ and $\mathrm{dn}(u, k)$. Each of the elliptic functions depends on the square of the modulus k as well as the argument u. Note that the elliptic functions sn and cn may be thought of as generalizations of sin and cos where their period depends on the modulus k.

TABLE 1

Properties of Jacobi elliptic functions

Property	$\mathrm{sn}(\cdot, k)$	$\sin(\cdot)$	$\mathrm{cn}(\cdot, k)$	$\cos(\cdot)$	$\mathrm{dn}(\cdot, k)$
Max. value	1	1	1	1	1
Min. value	-1	-1	-1	-1	$\sqrt{1-k^2}$
Period	$4\mathrm{K}(k)$	2π	$4\mathrm{K}(k)$	2π	$2\mathrm{K}(k)$
Parity	odd	odd	even	even	even
$\mathrm{d}f/\mathrm{d}u$	$\mathrm{cn}\cdot\mathrm{dn}$	cos	$-\mathrm{sn}\cdot\mathrm{dn}$	$-\sin$	$-k^2\cdot\mathrm{sn}\cdot\mathrm{cn}$
$f_k = 0$	sin	sin	cos	cos	1

The elliptic functions satisfy the following identities, which are analogous to $\sin^2 + \cos^2 = 1$:

$$\mathrm{sn}^2 + \mathrm{cn}^2 = 1,$$

$$k^2\mathrm{sn}^2 + \mathrm{dn}^2 = 1, \tag{A.1}$$

$$1 - k^2 + k^2\mathrm{cn}^2 = \mathrm{dn}^2.$$

Only two of these three relations are algebraically independent. In Table 1, additional properties of Jacobi elliptic functions are summarized.

Here $\mathrm{K}(k)$ is the complete elliptic integral of the first kind,

$$\mathrm{K}(0) = \pi/2, \quad \mathrm{K}(1) = +\infty. \tag{A.2}$$

The complete elliptic integral of the second kind is denoted by $\mathrm{E}(k)$. When k increases from 0 to 1, then $\mathrm{E}(k)$ decreases from $\pi/2$ to 1.