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APPLICATION OF THE CENTRE MANIFOLD THEORY IN NON-LINEAR AEROELASTICITY

L. LIU AND Y. S. WONG

Department of Mathematical Sciences, University of Alberta, Edmonton, Canada T6G 2G1

AND

B. H. K. LEE

Institute for Aerospace Research, National Research Council, Ottawa, Ontario, Canada K1A 0R6

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In this study, a frequency relation for limit cycle oscillations of a two-degree-of-freedom aeroelastic system with structural non-linearities represented by cubic restoring spring forces is derived. The centre manifold theory is applied to reduce the original system of nine-dimensional first order ordinary differential equations to a governing system in two dimensions near the bifurcation point. The principle of normal form is used to simplify the non-linear terms of the lower dimensional system. Using the frequency relation and the amplitude–frequency relationships derived from a previous study, limit cycle oscillations (LCOs) for self-excited systems can be predicted analytically. The mathematical technique proposed here has been applied to investigate LCO near a Hopf-bifurcation for an aeroelastic system with cubic restoring forces. Not only that an excellent agreement is obtained compared to the numerical results from solving the original system of eight non-linear differential equations by Runge–Kutta time integration scheme, but we also demonstrate that the use of a mathematical approach leads to a better understanding of non-linear aeroelasticity.

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1. INTRODUCTION

In dynamic response investigations of aircraft structures, classical theories assume linear aerodynamics and linear structures, so that the aeroelastic equations can be reduced to a set of linear equations that can be readily solved. However, in reality non-linearities are present in one form or the other. In many instances, linear aerodynamics give insufficiently accurate results. For example, when the airspeed approaches transonic Mach numbers, linear theory fails to detect the transonic dip and other phenomena associated with the presence of shock waves. Aircraft structures can have non-linearities that affect not only the flutter speed, but also the characteristics of the dynamical response. Hence, to obtain a better understanding of the physical and mathematical aspect of non-linear aeroelasticity, recent research [1, 17] has been directed towards the study of these two types of non-linearities.

Structural non-linearities that occur in the restoring forces can be treated as non-linear springs, such as springs with free-play, hysteresis or cubic non-linearities. These types of non-linearities have been investigated by Woolston *et al.* [2] for a two-dimensional airfoil performing pitching and plunging motions using an analog computer. There are serious drawbacks in the use of an analog computer to study non-linear flutter, and the accuracy is

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often not as high as one would desire in order to investigate the characteristics of the airfoil motion fully. Lee and LeBlanc [3] analyzed numerically a two-degree-of-freedom (d.o.f.) airfoil motion with a cubic non-linearity in the pitch degree of freedom. O'Neil *et al.* [4] performed experiments on the existence of limit cycle oscillation (LCO) of an airfoil with cubic structural non-linearities and compared their results with numerical simulations such as those given by Lee and LeBlanc. Price *et al.* [5] studied cubic non-linearity using numerical and describing function techniques. Describing function techniques [16] cannot be used to investigate the effects of initial conditions but can be used to provide good predictions of magnitudes of LCO responses. Gong *et al.* [6] investigated analytically and numerically the dynamic response of a coupled two-d.o.f. system with cubic non-linearities. They showed that harmonic, quasiperiodic and chaotic motions can exist for system parameters that correspond to those commonly used to analyze aeroelastic behavior of aircraft structures.

In this study, we concentrate on the LCOs of a two-d.o.f. aeroelastic system with structural non-linearity represented by cubic restoring spring forces. When the system is subject to an external forcing term with driving frequency ω, Lee *et al.* [7] derive analytical formulae that provide amplitude–frequency relationships for the pitch and plunge motion respectively. However, for a self-excited system (i.e., in the absence of external forcing term), the reference frequency ω is not known, and the motion cannot be determined from the amplitude–frequency relationships they derived. Several procedures were discussed in Reference [7] to estimate the frequency value ω for the self-excited system, but the results were not satisfactory except when the velocity U^* is very close to the linear flutter speed U_L^*. To overcome this limitation in Lee *et al.* [7], we apply the centre manifold theory of Carr [8] and the principle of normal form [9, 15] to derive a frequency relation for self-excited motion of a two-d.o.f. non-linear system. Using the frequency relation together with the amplitude–frequency relationships, LCOs for the self-excited system can be predicted analytically. Numerical simulations are carried out to compare the results with those obtained from the analytical analysis.

2. MODEL FORMULATION

In Figure 1, we show schematically the notations used in the analysis of a two-d.o.f. airfoil oscillating in pitch and in plunge. The plunging deflection is denoted by h, positive in the downward direction, and α is the pitch angle about the elastic axis, positive with nose up. The elastic axis is located at a distance $a_h b$ from the midchord, while the mass centre is located at a distance $x_\alpha b$ from the elastic axis. Both distances are positive when measured towards the trailing edge of the airfoil. The aeroelastic equations of motion including the structure non-linearities with subsonic aerodynamics are given as [10]

$$\xi'' + x_a\alpha'' + 2\zeta_\xi \frac{\bar{\omega}}{U^*}\xi' + \left(\frac{\bar{\omega}}{U^*}\right)^2 G(\xi) = -\frac{1}{\pi\mu}C_L(\tau) + \frac{P(\tau)b}{mU^2},$$

$$\frac{x_a}{r_\alpha^2}\xi'' + \alpha'' + 2\zeta_\alpha \frac{1}{U^*}\alpha' + \left(\frac{1}{U^*}\right)^2 M(\alpha) = \frac{2}{\pi\mu r_\alpha^2}C_M(\tau) + \frac{Q(\tau)}{mU^2 r_\alpha^2}, \tag{1}$$

where $\xi = h/b$ is the non-dimensional displacement of the elastic axis and the $'$ denotes differentiation with respect to the non-dimensional time τ defined as $\tau = Ut/b$. U^* is a non-dimensional velocity defined as $U^* = U/(b\omega_\alpha)$, and $\bar{\omega}$ is given by $\bar{\omega} = \omega_\xi/\omega_\alpha$, where ω_ξ and ω_α are the natural frequencies of the uncoupled plunging and pitching modes

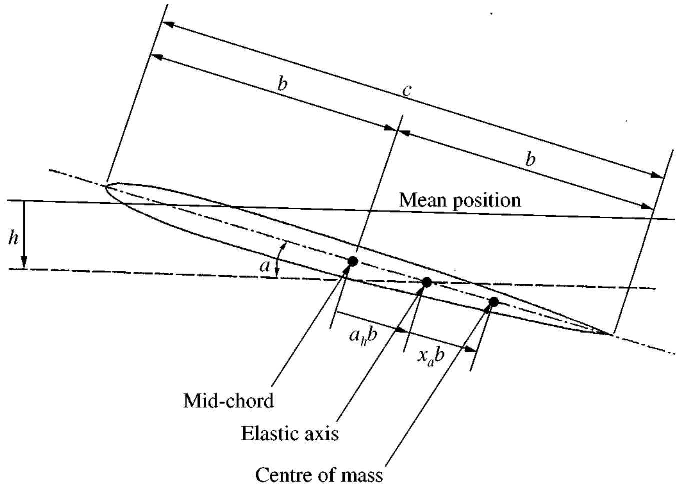

Figure 1. Two-degree-of-freedom airfoil motion.

respectively. ζ_ξ and ζ_α are the damping ratios, and r_α is the radius of gyration about the elastic axis. $G(\xi)$ and $M(\alpha)$ are the non-linear plunge and pitch stiffness terms respectively. $C_L(\tau)$ and $C_M(\tau)$ are the lift and pitching moment coefficients respectively. For incompressible flow, Fung [11] gives the following expressions for $C_L(\tau)$ and $C_M(\tau)$:

$$C_L(\tau) = \pi(\xi'' - a_h\alpha'' + \alpha') + 2\pi\{\alpha(0) + \xi'(0) + (\tfrac{1}{2} - a_h)\alpha'(0)\}\phi(\tau)$$

$$+ 2\pi \int_0^\tau \phi(\tau - \sigma)(\alpha'(\sigma) + \xi''(\sigma) + (\tfrac{1}{2} - a_h)\alpha''(\sigma))\,\mathrm{d}\sigma,$$

$$C_M(\tau) = \pi(\tfrac{1}{2} + a_h)\{\alpha(0) + \xi'(0) + (\tfrac{1}{2} - a_h)\alpha'(0)\}\phi(\tau)$$

$$+ \pi(\tfrac{1}{2} + a_h)\int_0^\tau \phi(\tau - \sigma)\{\alpha'(\sigma) + \xi''(\sigma) + (\tfrac{1}{2} - a_h)\alpha''(\sigma)\}\,\mathrm{d}\sigma$$

$$+ \frac{\pi}{2} a_h(\xi'' - a_h\alpha'') - (\tfrac{1}{2} - a_h)\frac{\pi}{2}\alpha' - \frac{\pi}{16}\alpha'',$$

where the Wagner's function $\phi(\tau)$ is given by

$$\phi(\tau) = 1 - \psi_1 \mathrm{e}^{-\varepsilon_1\tau} - \psi_2 \mathrm{e}^{-\varepsilon_2\tau}$$

and the constants $\psi_1 = 0{\cdot}165$, $\psi_2 = 0{\cdot}335$, $\varepsilon_1 = 0{\cdot}0455$, and $\varepsilon_2 = 0{\cdot}3$ are obtained from Jones [12]. $P(\tau)$ and $Q(\tau)$ are the externally applied forces and moments respectively.

Due to the existence of the integral terms in the integro-differential equations (1), it is difficult to study the dynamic behavior of the system analytically. To eliminate the integral terms, Lee *et al.* [10] introduced four new variables:

$$w_1 = \int_0^\tau \mathrm{e}^{-\varepsilon_1(\tau - \sigma)}\alpha(\sigma)\,\mathrm{d}\sigma, \qquad w_2 = \int_0^\tau \mathrm{e}^{-\varepsilon_2(\tau - \sigma)}\alpha(\sigma)\,\mathrm{d}\sigma,$$

$$w_3 = \int_0^\tau e^{-\varepsilon_1(\tau-\sigma)}\xi(\sigma)\,d\sigma, \qquad w_4 = \int_0^\tau e^{-\varepsilon_2(\tau-\sigma)}\xi(\sigma)\,d\sigma.$$

Then, the system (1) can be rewritten in a general form containing only differential operators as

$$c_0\xi'' + c_1\alpha'' + \left(c_2 + 2\zeta_\xi \frac{\bar{\omega}}{U^*}\right)\xi' + c_3\alpha' + c_4\xi + c_5\alpha + c_6 w_1 + c_7 w_2 + c_8 w_3$$

$$+ c_9 w_4 + \left(\frac{\bar{\omega}}{U^*}\right)^2 G(\xi) = f(\tau),$$

$$d_0\xi'' + d_1\alpha'' + d_2\xi' + \left(d_3 + 2\zeta_\alpha \frac{1}{U^*}\right)\alpha' + d_4\xi + d_5\alpha + d_6 w_1 + d_7 w_2 + d_8 w_3$$

$$+ d_9 w_4 + \left(\frac{1}{U^*}\right)^2 M(\alpha) = g(\tau). \tag{2}$$

The coefficients $c_0, c_1, \ldots, c_9,\ d_0,\ d_1, \ldots, d_9$ are given in Appendix A. $f(\tau)$ and $g(\tau)$ are functions depending on initial conditions, Wagner's function and the forcing terms, namely,

$$f(\tau) = \frac{2}{\mu}((\tfrac{1}{2} - a_h)\alpha(0) + \xi(0))(\psi_1\varepsilon_1 e^{-\varepsilon_1\tau} + \psi_2\varepsilon_2 e^{-\varepsilon_2\tau}) + \frac{P(\tau)b}{mU^2},$$

$$g(\tau) = -\frac{(1 + 2a_h)}{2r_\alpha^2} f(\tau) + \frac{Q(\tau)}{mU^2 r_\alpha^2}.$$

By introducing a variable vector $X = (x_1, x_2, \ldots, x_8)^T$, with $x_1 = \alpha$, $x_2 = \alpha'$, $x_3 = \xi$, $x_4 = \xi'$, $x_5 = w_1$, $x_6 = w_2$, $x_7 = w_3$, $x_8 = w_4$, the coupled equations (2) can be written as a set of eight first order ordinary differential equations:

$$X' = f(X, \tau).$$

This approach allows existing methods suitable for the study of ordinary differential equations to be used in the analysis. In this paper, we assume that there is no external forcing, i.e., $Q(\tau) = P(\tau) = 0$ in equation (1). For large values of τ when transients are damped out and steady solutions are obtained, $f(\tau) = 0$ and $g(\tau) = 0$. Thus, the system can be expressed as $X' = f(X)$. In terms of vector components, equations (2) can be expressed as

$$x_1' = x_2,$$

$$x_2' = a_{21}x_1 + \left(a_{22} - 2jc_0\zeta_\alpha \frac{1}{U^*}\right)x_2 + a_{23}x_3 + \left(a_{24} + 2jd_0\zeta_\xi \frac{\bar{\omega}}{U^*}\right)x_4 + a_{25}x_5 + a_{26}x_6$$

$$+ a_{27}x_7 + a_{28}x_8 + j\left(d_0\left(\frac{\bar{\omega}}{U^*}\right)^2 G(x_3) - c_0\left(\frac{1}{U^*}\right)^2 M(x_1)\right),$$

$$x_3' = x_4,$$

$$x_4' = a_{41}x_1 + \left(a_{42} + 2jc_1\zeta_\alpha \frac{1}{U^*}\right)x_2 + a_{43}x_3 + \left(a_{44} - 2jd_1\zeta_\xi \frac{\bar{\omega}}{U^*}\right)x_4 + a_{45}x_5 + a_{46}x_6$$

$$+ a_{47}x_7 + a_{48}x_8 + j\left(c_1\left(\frac{1}{U^*}\right)^2 M(x_1) - d_1\left(\frac{\bar{\omega}}{U^*}\right)^2 G(x_3)\right),$$

$$x_5' = x_1 - \varepsilon_1 x_5, \qquad x_6' = x_1 - \varepsilon_2 x_6, \qquad x_7' = x_3 - \varepsilon_1 x_7, \qquad x_8' = x_3 - \varepsilon_2 x_8. \tag{3}$$

The expressions for j, $a_{21}, \ldots, a_{28}$, $a_{41}, \ldots, a_{48}$ are given in Appendix B.

In this paper, the structural non-linearities are represented by cubic functions $M(\alpha)$ and $G(\xi)$, such that

$$M(\alpha) = \beta_\alpha \alpha + \beta_{\alpha^3}\alpha^3, \qquad G(\xi) = \beta_\xi \xi + \beta_{\xi^3}\xi^3, \tag{4}$$

where β_α, β_{α^3}, β_ξ and β_{ξ^3} are constants. The analysis and the results for $M(\alpha)$ or $G(\xi)$ represented by bilinear and hysteresis functions will be reported in a separate paper [13].

3. CENTRE MANIFOLD AND NORMAL FORM

Following the analysis presented by Lee *et al.* [7], the bifurcation parameter is associated with U^*, and the bifurcation value is U_L^*, which is the value of the linear flutter speed. To study the dynamic response of the system, we introduce a perturbation parameter δ such that $1/U^* = (1-\delta)/U_L^*$. Substituting this expression into equation (3), an autonomous system with the perturbation parameter is obtained, i.e., $X' = f(X;\delta)$. The equilibrium points are then evaluated from $f(X;\delta) = 0$. Without loss in generality, we assume the origin to be the equilibrium point. The original system (3) can now be rewritten as

$$X' = AX + B(\delta)X + (1-\delta)^2 F(X), \qquad \delta' = 0. \tag{5}$$

The matrix A is an 8×8 Jacobian matrix evaluated at the equilibrium point and at the bifurcation value (i.e, $\delta = 0$). The second and the third terms of equation (5) are non-linear in X and δ. The expressions for A, $B(\delta)$ and F are given in Appendix C.

The matrix A has one pair of purely imaginary eigenvalues $\lambda_1 = \mathrm{i}\omega_0$, $\bar{\lambda}_1 = -\mathrm{i}\omega_0$, one pair of complex eigenvalues with negative real parts, $\lambda_2 = b + \mathrm{i}c$, $\bar{\lambda}_2 = b - \mathrm{i}c$, and four negative real eigenvalues $\lambda_3, \lambda_4, \lambda_5, \lambda_6$. From the theory of centre manifold, it is possible to reduce the dimensionality of the system. To obtain the centre manifold, we first transform system (5) to a standard form. A transformation matrix P is obtained from the eigenspace of A, such that $P^{-1}AP = J$, where J is the Jordan canonical form of A containing all the eigenvalues of A:

$$J = \begin{pmatrix} 0 & \omega_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\omega_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & 0 & 0 & 0 & 0 \\ 0 & 0 & -c & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_6 \end{pmatrix}.$$

Introducing a new variable, $Y = P^{-1}X = (y_1, y_2, \ldots, y_8)^{\mathrm{T}}$, system (5) becomes

$$Y' = JY - P^{-1}B(\delta)PY + (1-\delta)^2 P^{-1}F(P\cdot Y), \qquad \delta' = 0. \tag{6}$$

The dynamic response of system (6), which is nine dimensional, can be investigated through an invariant two-dimensional system. Defining

$$J_A = \begin{pmatrix} 0 & \omega_0 & 0 \\ -\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad J_B = \begin{pmatrix} b & c & 0 & 0 & 0 & 0 \\ -c & b & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 \end{pmatrix}$$

and $Y_A = (y_1, y_2, \delta)^{\mathrm{T}}$, $Y_B = (y_3, y_4, y_5, y_6, y_7, y_8)^{\mathrm{T}}$, system (6) can be rewritten as

$$Y'_A = J_A Y_A + F_A(Y_A, Y_B), \qquad Y'_B = J_B Y_B + F_B(Y_A, Y_B), \tag{7}$$

where F_A and F_B are non-linear functions of Y_A and Y_B. Here, we start from the second order terms since the first order terms have already been included in the first part associated with Y_A and Y_B. J_A has one pair of purely imaginary eigenvalues and one zero eigenvalue. All eigenvalues of J_B have negative real parts. From the centre manifold theorem given by Carr [8], there exists a centre manifold H for equation (7), i.e., $Y_B = H(Y_A)$. The flow of equation (7) near the equilibrium point is governed by $Y'_A = J_A Y_A + F_A(Y_A, H(Y_A))$, which is a three-dimensional system. Theoretically, the function H can be solved from the following function equation:

$$\frac{\partial}{\partial Y_A} H(Y_A)(J_A Y_A + F_A(Y_A, H(Y_A))) = J_B H(Y_A) + F_B(Y_A, H(Y_A)). \tag{8}$$

However, the solution for the above equation with the exact expression of the function H is as difficult to obtain as the solution for the original system. Following another important result given by Carr [8], the centre manifold H can be approximated to any desired degree of accuracy. The polynomial approximation of the centre manifold H is assumed, and is denoted by $\boldsymbol{\Phi} = (\phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8)^{\mathrm{T}}$, in which

$$\phi_i(y_1, y_2, \delta) = h_{i1}y_1\delta + h_{i2}y_2\delta + h_{i3}y_1^2 + h_{i4}y_2^2 + h_{i5}\delta^2 + h_{i6}y_1y_2, \quad i = 3, 4, 5, 6, 7, 8, \tag{9}$$

where $h_{31}, h_{32}, \ldots, h_{36}, \ldots, h_{41}, \ldots, \ldots, h_{86}$ are constants to be determined from using the centre mainfold theory. Substituting equation (9) into equation (8) and replacing $H(Y_A)$ by its polynomial approximation $\boldsymbol{\Phi}(Y_A)$, we obtain

$$\frac{\partial}{\partial Y_A}\boldsymbol{\Phi}(Y_A)(J_A Y_A + F_A(Y_A, \boldsymbol{\Phi}(Y_A)) = J_B\boldsymbol{\Phi}(Y_A) + F_B(Y_A, \boldsymbol{\Phi}(Y_A)).$$

Equating the coefficients associated with $y_1\delta$, $y_2\delta$, y_1^2, y_2^2, δ^2, and y_1y_2, we obtain a system of 36 algebraic equations with $h_{31}, h_{32}, \ldots, h_{86}$ as variables. These equations can be solved by a standard computer problem such as Maple [14]. Extension to a higher order approximation of center manifold is straightforward, but the algebra becomes considerably more complex.

Once the expression of the center manifold is obtained, the original system is reduced to a three-dimensional system on the center manifold. Since the solution of the reduced system is not exactly identical to Y_A, we denote the corresponding solutions for y_1 and y_2 by u_1 and u_2 respectively. Regarding δ as a parameter, the system is reduced to two dimensions:

$$u_1' = \omega_0 u_2 + g_1(u_1, u_2, \delta), \qquad u_2' = -\omega_0 u_1 + g_2(u_1, u_2, \delta), \tag{10}$$

where g_1 and g_2 contain the non-linear terms as functions of u_1, u_2 and δ. An important result in the application of the centre manifold theorem is that the asymptotic behavior of the solutions near the equilibrium point and the bifurcation value of the original eight-dimensional system can be studied by analyzing the reduced two-dimensional system given in equation (10).

To simplify equation (10) for symbolic computations, we rewrite the system as

$$U' = BU + F(U) \tag{11}$$

with

$$B = \begin{pmatrix} b_{11}(\delta) & b_{12}(\delta) \\ b_{21}(\delta) & b_{22}(\delta) \end{pmatrix}, \qquad F(U) = \begin{pmatrix} f_1(u_1, u_2, \delta) \\ f_2(u_1, u_2, \delta) \end{pmatrix},$$

where $U = (u_1, u_2)^{\mathrm{T}}$. The first term BU is the linear part for u_1, u_2, and the second term $F(U)$ is the non-linear part of u_1, u_2.

Now define the transformation matrices

$$NP = \frac{1}{\sqrt{b_{12}^2 + \theta^2 + (\alpha - b_{11})^2}}\begin{pmatrix} 0 & b_{12} \\ \theta & \alpha - b_{11} \end{pmatrix}$$

and

$$NP^{-1} = \frac{\sqrt{b_{12} + \theta^2 + (\alpha - b_{11})^2}}{b_{12}\theta} \begin{pmatrix} -\alpha + b_{11} & b_{12} \\ \theta & 0 \end{pmatrix},$$

where $\alpha = \frac{1}{2}(b_{11} + b_{12})$ and $\theta = \sqrt{b_{11}b_{22} - b_{12}b_{21} - \alpha^2}$. By introducing a new variable $Y = NP^{-1}U = (y_1, y_2)^{\mathrm{T}}$, system (11) can be transformed into the standard form

$$Y' = JY + NP^{-1}F(NP \cdot Y) \quad \text{with } J = \begin{pmatrix} \alpha & \theta \\ -\theta & \alpha \end{pmatrix},$$

i.e.,

$$y_1' = \alpha y_1 + \theta y_2 + F_1(y_1, y_2, \delta), \qquad y_2' = -\theta y_1 + \alpha y_2 + F_2(y_1, y_2, \delta), \tag{12}$$

where F_1, F_2 are non-linear terms of y_1 and y_2, and α and θ are related to the parameter δ.

The complex form of system (12) can be written as

$$Z' = \lambda Z + h(Z, \bar{Z}),$$

where $\lambda(\delta) = \alpha(\delta) + \mathrm{i}\theta(\delta)$, and $Z = y_1 + \mathrm{i}y_2$. $h(Z, \bar{Z})$ includes non-linearities of Z and $\bar{Z}$.

By the principle of normal form, the near identity transformation is introduced:

$$Z = V + g(V, \bar{V}),$$

where V is a new variable, and g includes the second and third order non-linearities of V and $\bar{V}$. The normal form of the system (11) can be expressed as

$$V' = \lambda V + F_{21}V^2\bar{V},$$

where F_{21} is a complex number whose value is related to δ.

Taking $a(\delta) = \mathrm{Re}(F_{21})$ and $b(\delta) = \mathrm{Im}(F_{21})$, we express $V = r(\tau) * \mathrm{e}^{\omega(\tau)}$. The normal form in polar co-ordinates can be expressed as

$$r' = \alpha r + ar^3, \qquad \omega' = \theta + br^2.$$

Expanding the coefficients α, θ, a and b at $\delta = 0$, the above system becomes

$$r' = \dot{\alpha}(0)\delta r + a(0)r^3 = r(\dot{\alpha}(0)\delta + a(0)r^2),$$

$$\omega' = \theta(0) + \dot{\theta}(0)\delta + b(0)r^2 = (\theta(0) + \dot{\theta}(0)\delta) + b(0)r^2. \tag{13}$$

Note that the prime denotes derivatives with respect to τ and the dot denotes derivatives with respect to δ. The stability of the fixed point and the periodic orbit can now be analyzed. Furthermore, the frequency of the limit cycle oscillations can be predicted from a frequency relation given by

$$\omega = \omega_0 + \left(\dot{\theta}(0) - \frac{b(0)\dot{\alpha}(0)}{a(0)}\right)\delta. \tag{14}$$

The amplitude of the motion of the original system can also be predicted from the reduced system on the center manifold. However, due to errors introduced in approximating the centre manifold, the predicted amplitude value may not be sufficiently accurate.

4. AMPLITUDES OF LIMIT CYCLE OSCILLATIONS

To determine the amplitudes of LCOs associated with the pitch and plunge motions, we assume

$$\xi(\tau) = a_1(\tau)\cos(\omega\tau) + b_1(\tau)\sin(\omega\tau), \qquad \alpha(\tau) = a_2(\tau)\cos(\omega\tau) + b_2(\tau)\sin(\omega\tau),$$

$$w_i(\tau) = a_{i+2}(\tau)\cos(\omega\tau) + b_{i+2}\sin(\omega\tau), \quad i = 1, 2, 3, 4, \tag{15}$$

where $a_i(\tau)$ and $b_i(\tau)$, $i = 1, 2, \ldots, 6$ are slowly varying functions of τ, and ω is the angular frequency of the LCO. Let r and R denote the amplitude of ξ and α, respectively, where $r = \sqrt{a_1^2 + b_1^2}$ and $R = \sqrt{a_2^2 + b_2^2}$.

Substituting equation (15) into equation (2), and matching the coefficients of $\cos(\omega\tau)$ and $\sin(\omega\tau)$, leads to a system of 12 first order non-linear differential equations in a_i and b_i $i = 1, 2, \ldots, 6$. After considerable algebraic manipulations, the following amplitude equations are obtained:

$$A = \frac{(n_1^2 + s_1^2)}{m_1^2 + (p_1 + q_1 r^2)^2}, \qquad r^2 = AR^2,$$

$$R^2 = \frac{1}{q_2}(-s_2 \pm \sqrt{(p_2^2 + m_2^2)A - n_2^2}), \tag{16}$$

where $m_1, n_1, \ldots$ are functions of the system parameters and the frequency ω. The expressions of m_1, $n_1, \ldots$ in terms of system parameters and the frequency ω are given in Appendix D. The detailed derivation can be found in Lee *et al.* [7]. Now, using the amplitude–frequency relationships given in equation (16) and the frequency relation (14) derived in the previous section, the solutions of LCOs can be predicted analytically.

5. CASE STUDIES AND DISCUSSION

In order to demonstrate the accuracy of the analytical formulae given in equations (14) and (16) in predicting the frequency and amplitude of LCOs, we consider the following examples in which the aeroelastic system given in equation (1) contains cubic restoring forces. In all cases, analytical prediction are compared with solutions obtained numerically using a fourth order Runge–Kutta time-integration scheme applied to system (3).

The parameters $\mu = 100$, $a_h = -\frac{1}{2}$, $x_a = \frac{1}{4}$, $\zeta_\xi = \zeta_\alpha = 0$, $r_\alpha = 0{\cdot}5$ are used in all case studies. These system parameters are chosen from Reference [5]. The procedure discussed in the previous section does not depend on the choice of the parameters. The non-linear restoring forces $M(\alpha)$ and $G(\xi)$ are defined in formulae (4). Now, by varying the value of $\bar{\omega}$ and the coefficients β_α, β_{α^3}, β_ξ, β_{ξ^3}, we consider the following four cases shown in Table 1.

For Cases 1 and 4, structural non-linearity is applied only in the pitch degree of freedom. In Cases 2 and 3, cubic restoring forces are applied in both pitch and plunge degrees of

TABLE 1

Cases studies

Case	β_α	β_{α^3}	β_ξ	β_{ξ^3}
1	1	3	1	0
2	1	4	1	1
3	1	40	1	0·1
4	0·1	40	1	0

freedom.

In Case 1, for $\bar{\omega} = 0{\cdot}2$, the approximate centre manifold is given by

$$\phi_3 = -\,2{\cdot}278662600 y_1\delta - 2{\cdot}932984813 y_2\delta,$$

$$\phi_4 = 5{\cdot}389063673 y_1\delta - 3{\cdot}395569702 y_2\delta,$$

$$\phi_5 = -\,2{\cdot}576198739 y_1\delta + 0{\cdot}5470484684 y_2\delta,$$

$$\phi_6 = -\,0{\cdot}037592929 y_1\delta + 0{\cdot}05243761634 y_2\delta,$$

$$\phi_7 = 0{\cdot}01137223245 y_1\delta - 0{\cdot}01813461435 y_2\delta,$$

$$\phi_8 = -\,7{\cdot}328109745 y_1\delta + 0{\cdot}4092693894 y_2\delta.$$

Substituting $y_3 = \phi_3$, $y_4 = \phi_4$, $y_5 = \phi_5$, $y_6 = \phi_6$, $y_7 = \phi_7$, $y_8 = \phi_8$ into the first equation of system (7), a governing system of equations for y_1 and y_2 is obtained. Note that by replacing y_i using the above expressions given in ϕ_i, for $i = 3, 4, 5, 6, 7, 8$, the solution for system (7) can be approximated by explicit functions in terms of y_1, y_2 and δ. However, y_1, y_2 are no longer exactly identical to those defined in the original system (6), hence we denote y_1 and y_2 by u_1 and u_2. Therefore,

$$\begin{aligned} u_1' = &-\,0{\cdot}08404421373 u_2 - 0{\cdot}0050021860 45\delta u_1 + 0{\cdot}02298015261\delta u_2 \\ &+ 0{\cdot}000001060912229 u_1^3 - 0{\cdot}00001078496711 u_1^2 u_2 + 0{\cdot}07708383842\delta^2 u_1 \\ &+ 0{\cdot}00003654575512 u_1 u_2^2 - 0{\cdot}00004127944026 u_2^3 - 0{\cdot}06508466062\delta^2 u_2, \\ u_2' = &\ 0{\cdot}08404421392 u_1 - 0{\cdot}1034553702\delta u_1 + 0{\cdot}3210345363\delta u_2 \\ &+ 0{\cdot}00002847473453 u_1^3 - 0{\cdot}0002894669955 u_1^2 u_2 + 0{\cdot}3223789845\delta^2 u_1 \\ &+ 0{\cdot}0009808829109 u_1 u_2^2 - 0{\cdot}001107934352 u_2^3 - 1{\cdot}281473950\delta^2 u_2. \end{aligned}$$

Transforming this reduced system into a standard form and rewriting the standard form in complex form, we obtain the normal form after introducing the near identity transformation. Applying a Taylor expansion to the coefficients of the normal form

TABLE 2

The frequency relationship with the bifurcation parameter $\gamma = U^*/U_L^*$

$\bar{\omega}$	$\omega = \omega(\delta)$	$\omega = \omega(\gamma)$	U_L^*
0·2	$0{\cdot}0840 - 0{\cdot}0101 * \delta$	$0{\cdot}0739 + 0{\cdot}0101/\gamma^2$	6·28509
0·4	$0{\cdot}1192 - 0{\cdot}0333 * \delta$	$0{\cdot}0859 + 0{\cdot}0333/\gamma^2$	5·23376
0·6	$0{\cdot}1730 - 0{\cdot}0616 * \delta$	$0{\cdot}1114 + 0{\cdot}0616/\gamma^2$	4·40100
0·8	$0{\cdot}2264 - 0{\cdot}0823 * \delta$	$0{\cdot}1421 + 0{\cdot}0823/\gamma^2$	4·11454
1·0	$0{\cdot}2522 - 0{\cdot}0702 * \delta$	$0{\cdot}1820 + 0{\cdot}0702/\gamma^2$	4·33559

expressions in polar co-ordinates, the coefficients in equation (13) are given by

$$\omega(0) = \omega_0 = 0{\cdot}08404421382, \qquad \dot{\omega}(0) = -\,0{\cdot}06321776140,$$

$$\dot{\alpha}(0) = 0{\cdot}1580161751, \qquad a(0) = -\,0{\cdot}0002233463476, \qquad b(0) = 0{\cdot}00007505815011.$$

By analyzing system (13) with these results, we can verify that when $\delta < 0$, the equilibrium point is asymptotically stable, which means that for $U^* < U_L^*$ all motions will decay to zero amplitude. For $\delta > 0$, the equilibrium point becomes unstable. However, there is a stable periodic orbit with a frequency $\omega = 0{\cdot}0840 - 0{\cdot}0101\delta$ when $\bar{\omega} = 0{\cdot}2$.

For different values of $\bar{\omega}$, and using the same procedure, we derived the corresponding frequency relation which depends on the bifurcation parameter δ (or the ratio $\gamma = U^*/U_L^*$) as shown in Table 2.

Numerical simulations using Runge–Kutta scheme were carried out to compare with the analytical predictions. In Figures 2(a)–2(c), we display the frequency and the amplitudes for pitch and plunge motions that are predicted using the analytical formulae (14) and (16) with $\bar{\omega} = 0{\cdot}2$. Figures 2(a)–2(c) show that excellent agreement in both frequencies and amplitudes of the limit cycle oscillations is obtained.

In Case 2, we consider an aeroelastic system with cubic structural non-linearities in both pitch and plunge degrees of freedom.

For different $\bar{\omega}$, with the corresponding bifurcation value U_L^*, the frequency relations with the bifurcation parameter $\delta = 1 - (U_L^*/U^*)^2$ are shown in Table 3.

Furthermore, in Figures 3(a) and 3(b), we display the frequencies and the amplitudes for pitch motions that are predicted using the analytical formulae (14) and (16) when $\bar{\omega} = 0{\cdot}2$. These results are compared with numerical simulations, and it is shown that excellent agreement in both frequency and amplitude is obtained. We see some variations of the frequencies and the amplitudes when the ratio U^*/U_L^* increases from the bifurcation point ($U^*/U_L^* = 1$). This is expected due to limitations of the centre manifold theory.

In Case 3, we investigate the aeroelastic system with a stronger non-linear term in $M(\alpha)$ such that $\beta_{\alpha^3}/\beta_\alpha = 40$. At $\bar{\omega} = 0{\cdot}2$, the frequency equation is given by $\omega = 0{\cdot}0840 - 0{\cdot}0101(1 - (U_L^*/U^*)^2)$. The variation of frequency ω with U^*/U_L^* is plotted in Figure 4(a). Notice that the results of Figure 4(a) are almost identical to those displayed in Figure 2(a) for Case 1. Recall that the linear coefficients β_α equal to one for both Cases 3 and 1, but the non-linear coefficients β_{α^3} equal to 40 and 3 for Case 3 and 1, respectively. Although it may seem rather surprising to observe that the frequency relation is not sensitive to the non-linear coefficient, a satisfactory explanation will be provided shortly. The corresponding amplitudes of pitch and plunge motions when $\bar{\omega} = 0{\cdot}2$, as shown in Figures

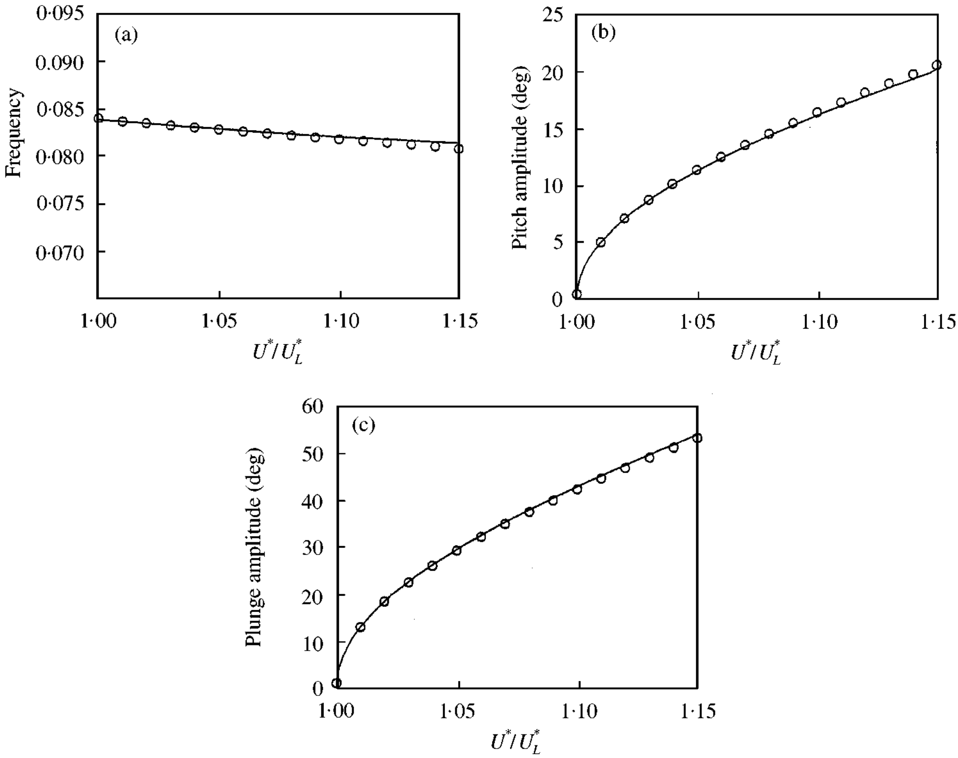

Figure 2. Dynamical response for Case 1. (a) Frequency; (b) amplitude of pitch motion; (c) amplitude of plunge motion: —, analytical prediction; ∘∘∘, numerical result.

TABLE 3

The frequency relationship with the bifurcation parameter $\gamma = U^*/U_L^*$

$\bar{\omega}$	$\omega = \omega(\delta)$	$\omega = \omega(\gamma)$	U_L^*
0·2	$0{\cdot}0840 + 0{\cdot}0082 * \delta$	$0{\cdot}0922 - 0{\cdot}0082/\gamma^2$	6·28509
0·4	$0{\cdot}1192 - 0{\cdot}0158 * \delta$	$0{\cdot}1034 + 0{\cdot}0158/\gamma^2$	5·23376
0·6	$0{\cdot}1730 - 0{\cdot}0554 * \delta$	$0{\cdot}1176 + 0{\cdot}0554/\gamma^2$	4·40100
0·8	$0{\cdot}2244 - 0{\cdot}0812 * \delta$	$0{\cdot}1432 + 0{\cdot}0812/\gamma^2$	4·11454
1·0	$0{\cdot}2522 - 0{\cdot}0683 * \delta$	$0{\cdot}1839 + 0{\cdot}0683/\gamma^2$	4·33559

4(b) and 4(c), however, are not the same as those reported in Case 1 (see Figures 2(b) and 2(c)).

In Case 4, we consider a very strong non-linear case in the pitch degree of freedom where $\beta_{\alpha^3}/\beta_\alpha = 400$. Our proposed procedure is applied. The frequency equation at $\bar{\omega} = 0{\cdot}2$ is given by $\omega = 0{\cdot}1822 - 0{\cdot}0659(1 - U_L^*/U^*)^2)$. Comparisons with numerical simulations are shown in Figures 5(a) and 5(b).

From the results reported here, it is evident that our analytical analysis gives an accurate prediction of the frequency and amplitudes of pitch and plunge motions of LCOs.

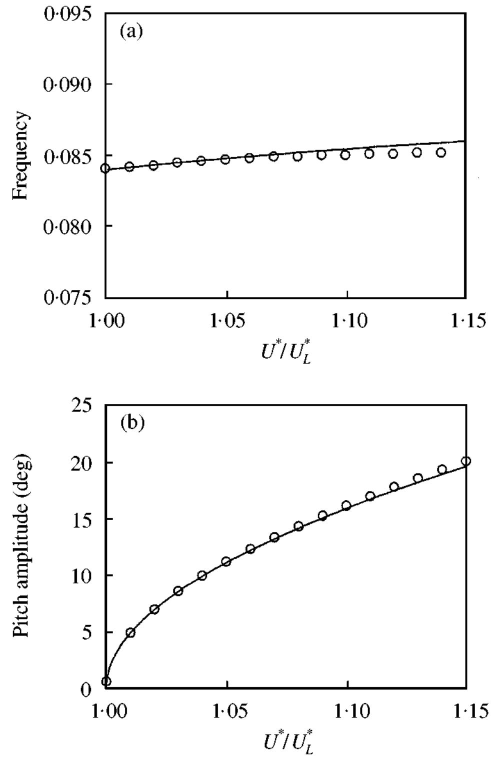

Figure 3. Dynamical response for Case 2. (a) Frequency; (b) amplitude of pitch motion: —, analytical prediction; ∘∘∘, numerical result.

Moreover, while numerical simulations show that the frequency variation with U^*/U_L^* is almost the same for Cases 1 and 3, an explanation is provided by using the analytical analysis. Since both linear coefficients β_α and β_ξ are identical for Case 1 and 3, the linear flutter speed, $U_L^* = 6.28509$, is identical at $\bar{\omega} = 0{\cdot}2$ for both cases. Now, by applying the centre manifold theory and normal form method to system (3), we obtain coefficients of formula (14) in terms of β_{α^3} and β_{ξ^3}:

$$\omega_0 = 0{\cdot}08404421382, \qquad \omega(0) = -\,0{\cdot}06321776140, \qquad \alpha(0) = 0{\cdot}1580161751,$$

$$a(0) = -\,0{\cdot}00007444878252\beta_{\alpha^3} + 0{\cdot}00006278101583\beta_{\xi^3},$$

$$b(0) = 0{\cdot}00002501938337\beta_{\alpha^3} + 0{\cdot}000006148895571\beta_{\xi^3}.$$

Notice that the non-linear coefficients β_{α^3} and β_{ξ^3} only affect the coefficients $a(0)$ and $b(0)$. The frequency relation in terms of β_{α^3} and β_{ξ^3} is given by

$$\omega = 0{\cdot}08404421382 + \left(-\,0{\cdot}063217761450 - 0{\cdot}1580161751\,\frac{0{\cdot}00002501938337\beta_{\alpha^3} + 0{\cdot}00006148895571\beta_{\xi^3}}{-\,0{\cdot}00007444878252\beta_{\alpha^3} + 0{\cdot}00006278101583\beta_{\xi^3}} \right)\delta. \quad (17)$$

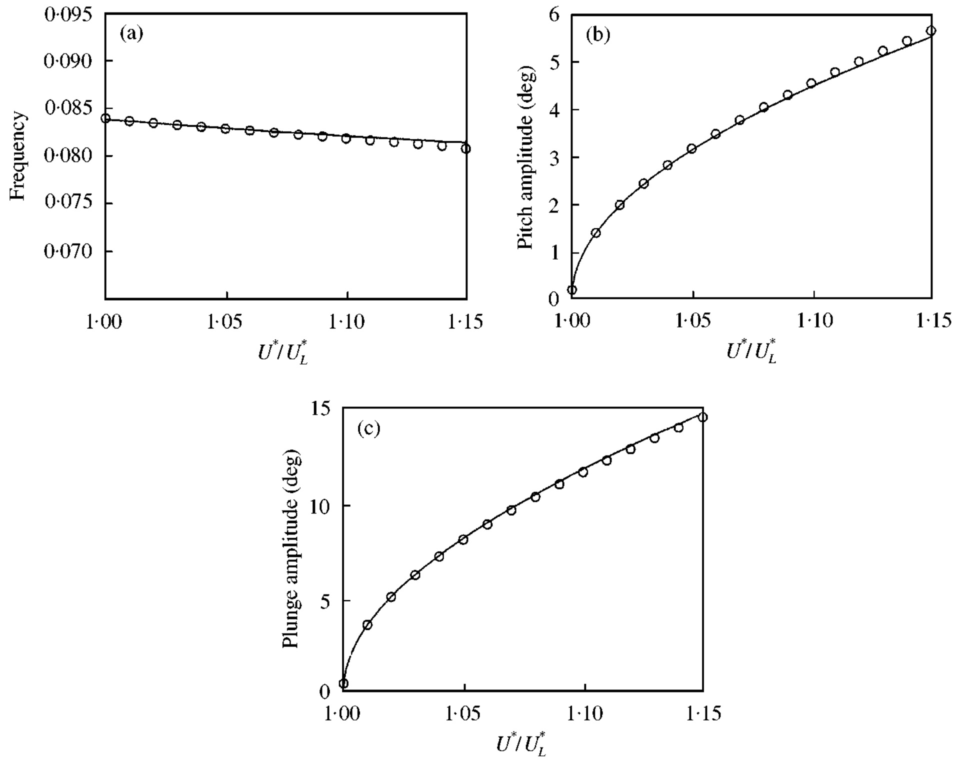

Figure 4. Dynamical response for Case 3. (a) Frequency; (b) amplitude of pitch motion; (c) amplitude of plunge motion: —, analytical prediction; ◦ ◦ ◦, numerical result.

From the above formula, it is clear that when either β_{α^3} or β_{ξ^3} is zero, the other non-linear coefficient β_{ξ^3} or β_{α^3} will not affect the resulting frequency. Hence, the frequency ω depends only on δ and is independent of β_{ξ^3} or β_{α^3}. In Case 1, $\beta_{\xi^3} = 0$, and the frequency relation is independent of the value of β_{α^3}. When both coefficients are present but with $\beta_{\alpha^3} \gg \beta_{\xi^3}$ as in Case 3, it is easy to verify that the effect due to non-linear coefficient β_{ξ^3} can be neglected. Therefore, Case 3 can be considered to be similar to Case 1.

Unlike the frequency relation, the amplitude equations given in equation (16) indicate that the amplitude of LCOs is more sensitive with the variation in the non-linear coefficients β_{α^3} and β_{ξ^3}. This observation is indeed confirmed by the results reported in Figures 2 and 4.

6. CONCLUDING REMARKS

In this paper, we derived a frequency relation for the self-excited two-degree-of-freedom aeroelastic system with structural non-linearities represented by cubic springs. Together with the amplitude equations derived in our previous study, the limit cycle oscillations for the self-excited system can be predicted analytically. Our study shows that the frequency and amplitude of LCOs do not depend on the choice of initial conditions. Moreover, it has been shown that when the structural non-linearity is applied only in one-degree-of-freedom, or when non-linearities appear in both pitch and plunge degrees of freedom but with one of the non-linear coefficient much greater than the other non-linear term, then the frequency relation is not affected by the non-linear coefficients β_{α^3} or β_{ξ^3}. However, the corresponding

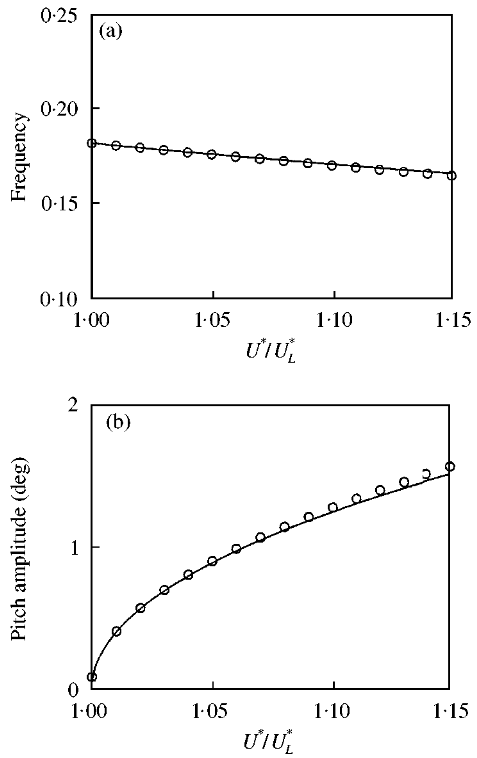

Figure 5. Dynamical response for Case 4. (a) Frequency; (b) amplitude of pitch motion: —, analytical prediction; ○ ○ ○, numerical result.

amplitude of the LCO is sensitive with the variation in β_{α^3} and β_{ξ^3}. The mathematical approach presented here not only provides an accurate agreement with numerical results obtained by using a fourth order Runge–Kutta time-integration scheme, but it also leads to a better understanding of non-linear aeroelasticity especially near the bifurcation points. In the present work, we focus on the study of LCO through a Hopf-bifurcation. The period of doubling phenomenon in which an LCO subsequently gives rise to a two-period orbit by means of a flip-bifurcation has been detected in aeroelastic system with cubic structural non-linearities [1]. The phenomenon is interesting and important, since it may provide a route leading to the investigation of the period-two and chaotic motions. However, since the bifurcation analysis will now depend upon periodically varying parameters instead of fixed points in a Hopf-bifurcation, a general procedure based on time-dependent centre manifold theory and time-dependent normal form will be required.

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APPENDIX A: COEFFICIENTS IN EQUATIONS (2)

$$c_0 = 1 + \frac{1}{\mu}, \qquad c_1 = x_\alpha - \frac{a_h}{\mu}, \qquad c_2 = \frac{2}{\mu}(1 - \psi_1 - \psi_2),$$

$$c_2 = \frac{2}{\mu}(1 + (1 - 2a_h)(1 - \psi_1 - \psi_2)), \qquad c_4 = \frac{2}{\mu}(\epsilon_1\psi_1 + \epsilon_2\psi_2),$$

$$c_5 = \frac{2}{\mu}(1 - \psi_1 - \psi_2 + (\tfrac{1}{2} - a_h)(\epsilon_1\psi_1 + \epsilon_2\psi_2))$$

$$c_6 = \frac{2}{\mu}\epsilon_1\psi_1(1 - \epsilon_1(\tfrac{1}{2} - a_h)), \qquad c_7 = \frac{2}{\mu}\epsilon_2\psi_2(1 - \epsilon_2(\tfrac{1}{2} - a_h)),$$

$$c_8 = -\frac{2}{\mu}\epsilon_1^2\psi_1, \qquad c_9 = -\frac{2}{\mu}\epsilon_2^2\psi_2, \qquad d_0 = \frac{x_\alpha}{r_\alpha^2} - \frac{a_h}{\mu r_\alpha^2}, \qquad d_1 = 1 + \frac{1 + 8a_h^2}{8\mu r_\alpha^2},$$

$$d_2 = -\frac{1+2a_h}{\mu r_\alpha^2}(1-\psi_1-\psi_2),$$

$$d_3 = \frac{1-2a_h}{2\mu r_\alpha^2} - \frac{(1+2a_h)(1-2a_h)(1-\psi_1-\psi_2)}{2\mu r_\alpha^2},$$

$$d_4 = -\frac{1+2a_h}{\mu r_\alpha^2}(\epsilon_1\psi_1+\epsilon_2\psi_2),$$

$$d_5 = -\frac{1+2a_h}{\mu r_\alpha^2}(1-\psi_1-\psi_2) - \frac{(1+2a_h)(1-2a_h)(\psi_1\epsilon_1-\psi_2\epsilon_2)}{2\mu r_\alpha^2},$$

$$d_6 = -\frac{(1+2a_h)\psi_1\epsilon_1}{\mu r_\alpha^2}(1-\epsilon_1(\tfrac{1}{2}-a_h)),$$

$$d_7 = -\frac{(1+2a_h)\psi_2\epsilon_2}{\mu r_\alpha^2}(1-\epsilon_2(\tfrac{1}{2}-a_h)),$$

$$d_8 = \frac{(1+2a_h)\psi_1\epsilon_1^2}{\mu r_\alpha^2}, \qquad d_9 = \frac{(1+2a_h)\psi_2\epsilon_2^2}{\mu r_\alpha^2},$$

APPENDIX B: COEFFICIENTS IN EQUATIONS (3)

$$j = \frac{1}{c_0 d_1 - c_1 d_0},$$

$$a_{21} = j(-d_5c_0 + c_5d_0), \qquad a_{22} = j(-d_3c_0 + c_3d_0), \qquad a_{23} = j(-d_4c_0 + c_4d_0),$$

$$a_{24} = j(-d_2c_0 + c_2d_0), \qquad a_{25} = j(-d_6c_0 + c_6d_0), \qquad a_{26} = j(-d_7c_0 + c_7d_0),$$

$$a_{27} = j(-d_8c_0 + c_8d_0), \qquad a_{28} = j(-d_9c_0 + c_9d_0),$$

$$a_{41} = j(d_5c_1 - c_5d_1), \qquad a_{42} = j(d_3c_1 - c_3d_1), \qquad a_{43} = j(d_4c_1 - c_4d_1),$$

$$a_{44} = j(d_2c_1 - c_2d_1), \qquad a_{45} = j(d_6c_1 - c_6d_1), \qquad a_{46} = j(d_7c_1 - c_7d_1),$$

$$a_{47} = j(d_8c_1 - c_8d_1), \qquad a_{48} = j(d_9c_1 - c_9d_1).$$

APPENDIX C: EXPRESSIONS FOR THE MATRICES *A*, *B*, AND *F* IN EQUATIONS (5)

$$b_{21} = jc_0\left(\frac{1}{U_L^*}\right)^2\beta_\alpha, \quad b_{22} = jc_0\left(\frac{1}{U_L^*}\right)2\zeta_\alpha, \quad b_{23} = jd_0\left(\frac{\bar{\omega}}{U_L^*}\right)^2\beta_\xi, \quad b_{24} = jd_0\left(\frac{\bar{\omega}}{U_L^*}\right)2\zeta_\xi,$$

$$b_{41} = jc_1\left(\frac{1}{U_L^*}\right)^2\beta_\alpha, \quad b_{42} = jc_1\left(\frac{1}{U_L^*}\right)2\zeta_\alpha, \quad b_{43} = jd_1\left(\frac{\bar{\omega}}{U_L^*}\right)^2\beta_\xi, \quad b_{44} = jd_1\left(\frac{\bar{\omega}}{U_L^*}\right)2\zeta_\xi,$$

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}+b_{23} & a_{24}+b_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ a_{41}+b_{41} & a_{42}+b_{42} & a_{43}-b_{43} & a_{44}-b_{44} & a_{45} & a_{46} & a_{47} & a_{48} \\ 0 & 0 & 0 & 0 & -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\epsilon_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_2 \end{pmatrix}$$

$$B=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{21}(\delta^2-2\delta) & b_{22}\delta & b_{23}(\delta^2-2\delta) & -b_{24}\delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{41}(\delta^2-2\delta) & -b_{22}\delta & -b_{43}(\delta^2-2\delta) & b_{44}\delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F=\begin{pmatrix} 0 \\ -b_{21}\dfrac{\beta_{\alpha^3}}{\beta_\alpha}x_1^3+b_{23}\dfrac{\beta_{\xi^3}}{\beta_\xi}x_3^3 \\ 0 \\ b_{41}\dfrac{\beta_{\alpha^3}}{\beta_\alpha}x_1^3-b_{43}\dfrac{\beta_{\xi^3}}{\beta_\xi}x_3^3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

APPENDIX D: COEFFICIENTS OF EQUATION (16)

$$m_1=\left(c_2+2\zeta_\xi\frac{\bar{\omega}}{U^*}\right)\omega-\frac{c_8\omega}{\epsilon_1^2+\omega^2}-\frac{c_9\omega}{\epsilon_2^2+\omega^2}, \qquad n_1=c_3\omega-\frac{c_6\omega}{\epsilon_1^2+\omega^2},-\frac{c_7\omega}{\epsilon_2^2+\omega^2},$$

$$p_1=\frac{c_8\epsilon_1}{\epsilon_1^2+\omega^2}+\frac{c_9\epsilon_2}{\epsilon_2^2+\omega^2}+c_4+\beta_\xi\left(\frac{\bar{\omega}}{U^*}\right)^2-c_0\omega^2, \qquad q_1=\frac{3}{4}\beta_{\xi^3}\left(\frac{\bar{\omega}}{U^*}\right)^2,$$

$$s_1 = c_5 - c_1\omega^2 + \frac{c_6\epsilon_1}{\epsilon_1^2 + \omega^2} + \frac{c_7\epsilon_2}{\epsilon_2^2 + \omega^2}, \qquad m_2 = d_2\omega - \frac{d_8\omega}{\epsilon_1^2 + \omega^2} - \frac{d_9\omega}{\epsilon_2^2 + \omega^2},$$

$$n_2 = \left(d_3 + 2\zeta_\alpha \frac{1}{U^*}\right)\omega - \frac{d_6\omega}{\epsilon_1^2 + \omega^2} - \frac{d_7\omega}{\epsilon_2^2 + \omega^2}, \qquad p_2 = \frac{c_8\epsilon_1}{\epsilon_1^2 + \omega^2} + \frac{d_9\epsilon_2}{\epsilon_2^2 + \omega^2} + d_4 - d_0\omega^2,$$

$$q_2 = \frac{3}{4}\beta_{\alpha^3}\left(\frac{1}{U^*}\right)^2, \qquad s_2 = d_5 + \beta_\alpha\left(\frac{1}{U^*}\right)^2 - d_1\omega^2 + \frac{d_6\epsilon_1}{\epsilon_1^2 + \omega^2} + \frac{d_7\epsilon_2}{\epsilon_2^2 + \omega^2}.$$

APPENDIX E: NOMENCLATURE

a_h	non-dimensional distance from airfoil mid-chord to elastic axis
b	airfoil semi-chord
h	plunge displacement
m	airfoil mass
r	amplitude of ξ
r_α	radius of gyration about the elastic axis
t	time
x_α	non-dimensional distance from the elastic axis to the centre of mass
$C_L(\tau)$, $C_M(\tau)$	aerodynamic lift and pitching moment coefficients
$G(\xi)$, $M(\alpha)$	non-linear plunge and pitch stiffness terms
$P(\tau)$, $Q(\tau)$	externally applied forces and moments
R	amplitude of α
U	free stream velocity
U^*	non-dimensional velocity, $U^* = U(b\omega_\alpha)$
U_L^*	linear flutter speed
X, Y	system variable vectors
V, Z	complex variables
ξ	non-dimensional plunge displacement, $\xi = h/b$
α	pitch angle of airfoil
ω	frequency of the motion
μ	airfoil/air mass ratio, $\mu = m(\pi\rho b^2)$
τ	non-dimensional time, $\tau = Ut/b$
δ	perturbation parameter
ψ_1, ψ_2	constants in Wagner's function
ϵ_1, ϵ_2	constants in Wagner's function
β_α, β_{α^3}	constants in non-linear pitch stiffness term $M(\alpha)$
β_ξ, β_{ξ^3}	constants in non-linear plunge stiffness term $G(\xi)$
ζ_ξ, ζ_α	viscous damping ratios in plunge and in pitch
$\bar{\omega}$	frequency ratio, $\bar{\omega} = \omega_\xi/\omega_\alpha$
ω_ξ, ω_α	natural frequencies in plunge and in pitch
$\phi(\tau)$	Wagner's function