Journal of Sound and Vibration (2000) **234**(5), 881–894
doi:10.1006/jsvi.2000.2905, available online at http://www.idealibrary.com on IDEAL®

INFLUENCE OF THE POISSON RATIO ON THE NATURAL FREQUENCIES OF STEPPED-THICKNESS CIRCULAR PLATE

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(*Received* 23 *June* 1999, *and in final form* 21 *January* 2000)

The natural frequencies of simply supported and clamped, stepped-thickness plates are determined using classical plate solutions with exact continuity conditions at the step. The effect of incorporating the Poisson ratio in the continuity conditions on the natural frequencies for nodal diameters, 0, 1 and nodal interior circle numbers 0, 1, 2 is thoroughly investigated. For engineering applications, a design criterion is proposed for simply supported and clamped plates based on an approximate linear model for the natural frequencies. The literature lacks experimental results on this type of plates. Hence, in this paper experimental results are presented for four models with two Poisson's ratios and prove their consistency with the proposed criterion.

1. INTRODUCTION

Stepped-thickness circular plates are of practical and academic interest in many fields of engineering and science [1, 2]. Classical as well as non-classical outer boundaries have been dealt with in different parts of the literature using exact or approximate continuity conditions along the step variation. The latter approximation is normally made by disregarding the Poisson ratio in the moment and shear force continuity equations to generate simpler mathematical formulation. Along the same line, it has been proved that considering the Poisson ratio in the edge moment boundary condition for a simply supported uniform and homogeneous circular plate has a significant effect on the fundamental frequency [3, 4]. On the other hand, reference [5] gives a comparison of the Kirchhoff and Mindlin formulation of a circular disk by showing the variation of the frequency parameter as a function of the Poisson ratio for different shape ratios and particular mode shapes. Experimental natural frequencies of a disk held at its centre by a doubly local axial connection are presented in the latter reference. Further, references [6, 7] have only commented on the inaccuracy resulted from using approximate continuity condition for stepped-thickness rectangular plates. In conclusion, this parameter contributes to the physical nature of the problem and affects the values of the natural frequencies, and should not be disregarded for the sake of mathematical simplicity.

The Poisson ratio has been tastefully considered or ignored in the continuity conditions to generate simplified version of the frequency equation. The fundamental frequencies for free-edge stepped-thickness plates are determined by using approximate method [1] or exact formulation [2] without considering the Poisson ratio in the continuity conditions. On the other hand, the vibration characteristics of simply supported and clamped stepped

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plates are analyzed in references [8–12] using various theoretical approaches and as special cases of the general methods used for plates with elastic supports. These references presented results for the natural frequency either disregarding or considering the Poisson ratio at the continuity conditions without any quantitative measure. Further, to the best of the authors' knowledge, no experimental results were recorded for stepped thickness circular plates except for the fundamental frequency of free boundary [1, 12].

This paper presents a theoretical and experimental investigation of the effect of incorporating the Poisson ratio in the continuity conditions on the natural frequencies of simply supported and clamped stepped-thickness plates. It is aimed at obtaining a quantitative measure and a design criterion of this effect and justifying it experimentally.

2. THEORETICAL NATURAL FREQUENCIES

Figure 1 shows a circular plate with one change in thickness. The solution of the equation of motion for free vibration may be written as

$$w(r, \theta, t) = w(r, \theta)\mathrm{e}^{\mathrm{i}\omega t}, \tag{1}$$

where w is the displacement as a function of the radius r and the angular co-ordinate θ, and ω is the natural frequency. The continuity conditions at $r = r_2$ may be written as

$$w_1 = w_2, \qquad \frac{\partial w_1}{\partial r} = \frac{\partial w_2}{\partial r}, \qquad M_1 = M_2, \qquad V_1 = V_2, \tag{2}$$

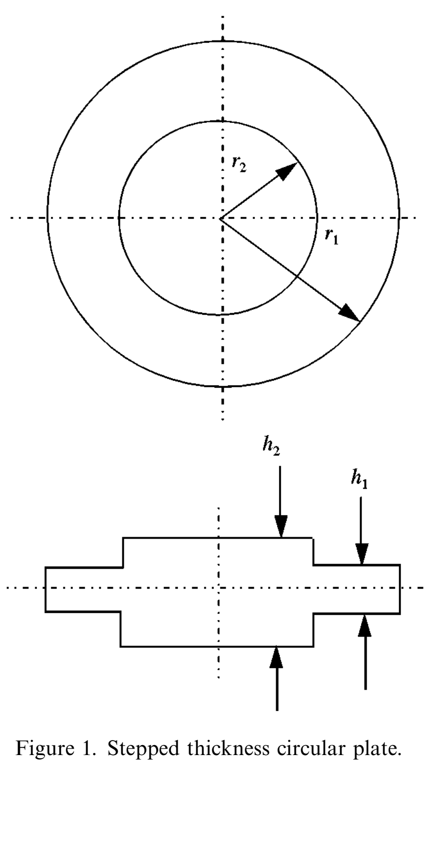

Figure 1. Stepped thickness circular plate.

with the moment M and the Kirchhoff's equivalent shearing force V of the form [13]

$$M_i = D_i\left\{\nabla^2 w_i - (1-v)\left(\frac{1}{r_i}\frac{\partial w_i}{\partial r} + \frac{\partial^2 w_i}{r_i^2\partial\theta^2}\right)\right\},$$

$$V_i = D_i\left\{\frac{\partial}{\partial r}(\nabla^2 w_i) + (1-v)\frac{\partial}{r_i\partial\theta}\left(\frac{1}{r_i}\frac{\partial^2 w_i}{\partial r\,\partial\theta} - \frac{1}{r_i^2}\frac{\partial w_i}{\partial\theta}\right)\right\}, \tag{3}$$

where

$$D_i = \frac{Eh_i^3}{12(1-v^2)} \tag{4}$$

with $i = 1$ and 2 referring to sections 1 and 2 of the plate, respectively, E is the modulus of elasticity, h the plate thickness and v the Poisson ratio.

For simply supported plate the assumed solution must satisfy the boundary conditions at $r = r_1$, namely

$$w_1 = 0, \qquad D_1\left\{\frac{\partial^2 w_1}{\partial r^2} + v\left(\frac{1}{r_1}\frac{\partial w_1}{\partial r} + \frac{1}{r_1^2}\frac{\partial^2 w_1}{\partial\theta^2}\right)\right\} = 0. \tag{5}$$

However, for clamped boundaries it must satisfy

$$w_1 = 0, \qquad \frac{\partial w_1}{\partial r} = 0. \tag{6}$$

The general solution is assumed as

$$w_i(r,\theta) = \sum_{n=0}^{\infty}[A_{in}\mathrm{J}_n(kr) + B_{in}\mathrm{Y}_n(kr) + C_{in}\mathrm{I}_n(kr) + G_{in}\mathrm{K}_n(kr)]\cos n\theta, \tag{7}$$

where

$$k^4 = \frac{\rho\omega^2 h}{D}. \tag{8}$$

J_n and Y_n are the Bessel functions of the first and second kinds of order n, and I_n and K_n are the modified Bessel functions of the first and second kinds, of order n respectively. The values of the constants A_{in}, B_{in}, C_{in} and G_{in} are determined from the boundary conditions. For plate 2, however, B_{in} and G_{in} are set to zero for finite solution at $r = 0$.

Substitute equation (7) in the boundary conditions (5) or (6), accordingly, then solve for A_{1n}, and B_{1n} in terms of C_{1n} and G_{1n}, and after some mathematical manipulation plug the results in the continuity conditions (2), to obtain

$$[E_{ij}]\begin{Bmatrix} C_{1n} \\ G_{1n} \\ A_{2n} \\ C_{2n} \end{Bmatrix} = 0. \tag{9}$$

The matrix E_{ij} is given in Appendix A for simply supported and for clamped outer boundaries. Natural frequencies are determined from the vanishing determinant of the coefficient matrix $[E_{ij}]$.

3. PROPOSED APPROXIMATE MODEL

Contemplating the mathematical structure of the current problem indicates that it is very difficult, if not impossible, to derive a closed-form formulation for the natural frequencies in terms of ν as it is implemented in the continuity conditions. Therefore, it is appropriate to generate a large range of values of these frequencies for different values of h_1/h_2 and ν and thereafter study the general trend to propose an appropriate empirical model.

Using the theoretical procedure described above, frequencies for $N = 0$, 1 and $S = 0$, 1, 2 are obtained, where N denotes the number of nodal diameters and S the number of interior nodal circles. For a specified value of step radius $(1 - r_2/r_1)$, the Poisson ratio is allowed to vary within the practical range of 0–0·5 and the thickness ratio h_1/h_2 of 1 to 0·5 in steps of 0·1. Step radii of 0·5 and 0·4 are considered for the simply supported conditions and 0·35 and 0·4 for the clamped boundaries. The results are expressed in terms of a non-dimensional natural frequency parameter of free vibration λ, where

$$\lambda^2 = \omega r^2 \sqrt{\frac{\rho h}{D}}. \qquad (10)$$

To study the effect of the Poisson ratio on the natural frequencies a correction factor is defined as follows:

$$\Delta = \frac{\lambda_{NS}}{(\lambda_{NS})_0} - 1, \qquad (11)$$

where λ_{NS} is the non-dimensional frequency parameter at a given ν and $(\lambda_{NS})_0$ is the equivalent parameter at $\nu = 0$.

A total of 120 graphs (not shown here for brevity) similar to Figure 2 were constructed for Δ in terms of ν and various values of (h_1/h_2). Inspecting these graphs reveals that the computed frequencies take a well-defined trend. Various curves were fitted to the data using logarithmic, exponential and polynomial trends and it was concluded that the best appropriate fit would be a straight line of the form

$$\Delta = m\nu, \qquad (12)$$

where m is the slope (variation of Δ with respect to ν) which is a function of thickness ratio, S and N. Although the mathematical structure of the problem indicates that the relation between Δ and ν is non-linear, nevertheless, within the range of ν investigated in this research and for practical purpose, straight-line approximation seems to work well. To assess the efficiency of the curve fitting process an R-squared value was determined for all curves. This value is defined as

$$R^2 = 1 - \frac{SSE}{SST}, \qquad (13)$$

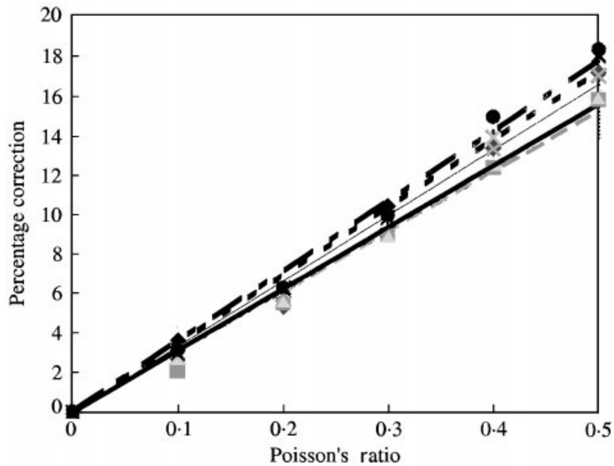

Figure 2. Percentage correction factor Δ versus v for a simply supported stepped thickness circular plate with step-radius of 0·4 and h_1/h_2 between 0·5 and 1: ◆ 1; ■ 0·9; ▲ 0·8; × 0·7; ✱ 0·6; ● 0·5.

where

$$SSE = \sum (Y_i - \hat{Y}_i)^2 \tag{14}$$

and

$$SST = \left(\sum Y_i^2\right) - \frac{(\sum Y_i)^2}{n}. \tag{15}$$

In order to have a good picture of understanding of the variation in Δ, the slopes for these curves are summarized in a column chart (Figure 3). All these slopes were obtained with R^2-value varies between 0·94 and 0·999, which indicates the adequacy of the curve-fitting procedure for practical applications. Therefore, a design criterion is proposed which states that *for a given set of* (h_1/h_2), *S and N values it is appropriate to use straight-line approximation empirically between* Δ and v *for values of* v *between* 0 *and* 0·5.

4. EXPERIMENTAL WORK

No experimental results are available in the open literature for stepped-thickness circular plates except those for free outer boundaries [1, 2]. One of the main objectives of this work is to investigate the effect of stepped-thickness variation and the Poisson ratio experimentally and to justify the accuracy of the proposed model. In this work two steel models ($E = 207$ GPa and $\rho = 7800\ \mathrm{kg/m^3}$) and two aluminium models ($E = 66$ GPa and $\rho = 2670\ \mathrm{kg/m^3}$) were cut and milled to the dimensions given in Table 1. The values for the non-dimensional step-radius $(1 - r_2/r_1)$ given in the table are approximated to two digits and are used as reference values only.

For simply supported tests the boundary conditions were simulated by using a ring with an internal diameter of 0·5 cm smaller than the external diameter of the model. The plate was seated on the ring. Holes were drilled and taped on the ring at 18° interval such that the perimeters just touch the circumference of the plate from outside. Eight millimeter screws were fitted in these holes. The screws were selected such that when a screw is fastened, 2 mm

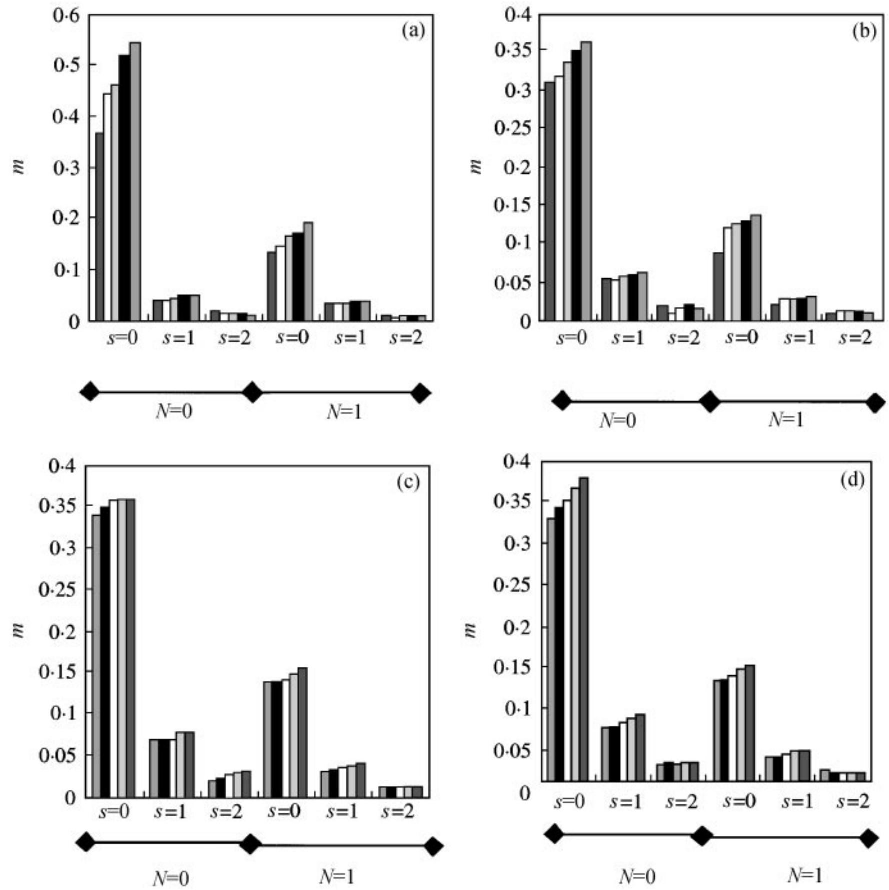

Figure 3. Slopes of Δ versus ν for h_1/h_2 between 0·5 (extreme left) and 0·9 (extreme right) in steps of 0·1 for each s value; simply supported plate with step radius: (a) 0·5 and (b) 0·4; and clamped plate with step radius: (c) 0·4 and (d) 0·35: ■ 0·9; ■ 0·8; □ 0·7; □ 0·6; □ 0·5.

TABLE 1

Dimensions of tested models (mm)

Metal	Simply supported			Clamped edge			$\frac{h_1}{h_2}$		
	r_2	r_1	$\left(1-\frac{r_2}{r_1}\right)$	r_2	r_1	$\left(1-\frac{r_2}{r_1}\right)$			
Steel	96·5	196	0·5	96·5	161	0·4	0·5	0·6	0·9
	96·5	164	0·4	96·5	148·5	0·35			
Aluminium	94	191	0·5	94	156·5	0·4	0·5	0·7	0·8
	94	160	0·4	94	144·5	0·35			

of its head would be clipping the plate edge. It is believed that this set-up will simulate the simply supported conditions. Clamped boundary conditions, on the other hand, were simulated by gluing the plate along the boundaries between two concentric rings of equal dimensions. These two rings and the plate were then clamped together with 8 mm screws

TABLE 2

Non-dimensional natural frequencies for a simply supported plate with step radius of 0·5 (values in second row of each h_1/h_2-ratio are experimental frequencies and those in the third row are by linear model)

	Aluminium $v = 0·25$						Steel $v = 0·3$					
	$N = 0$			$N = 1$			$N = 0$			$N = 1$		
$\frac{h_1}{h_2}$	$S = 0$	$S = 1$	$S = 2$	$S = 0$	$S = 1$	$S = 2$	$S = 0$	$S = 1$	$S = 2$	$S = 0$	$S = 1$	$S = 2$
1·0	4·861	29·432	74·030	13·770	48·094	102·374	4·906	29·557	74·062	13·920	48·198	102·514
	—	—	—	—	—	—	—	—	—	—	—	—
	4·824	29·527	73·996	13·813	48·187	102·388	—	—	—	—	—	—
0·9	5·128	29·794	74·333	14·243	48·774	103·184	5·299	29·833	74·483	14·360	48·852	103·608
	—	—	—	—	—	—	5·326	30·008	75·504	14·520	49·714	102·608
	5·208	29·839	74·437	14·257	48·754	103·158	—	—	—	—	—	—
0·8	5·370	30·106	75·180	14·606	49·335	103·895	5·487	30·186	75·273	14·695	49·414	103·531
	5·350	30·388	75·114	14·124	49·006	104·646	—	—	—	—	—	—
	5·336	30·152	75·167	14·622	49·301	103·508	—	—	—	—	—	—
0·7	5·596	30·542	76·570	14·836	49·604	104·622	5·745	30·608	75·857	14·518	49·686	103·944
	5·483	30·728	76·755	15·045	51·638	104·550	—	—	—	—	—	—
	5·571	30·583	75·770	14·908	49·577	103·882	—	—	—	—	—	—
0·6	6·182	31·274	77·988	15·167	49·999	103·928	6·363	31·472	76·617	15·259	50·077	103·956
	—	—	—	—	—	—	6·381	31·892	75·415	14·961	52·184	105·525
	6·172	31·278	76·551	15·129	49·968	104·001	—	—	—	—	—	—
0·5	6·447	32·108	77·283	15·550	50·554	104·420	6·668	32·188	77·304	15·617	50·652	104·447
	6·584	32·261	78·003	15·254	49·616	104·692	6·756	32·169	78·278	15·333	50·184	104·613
	6·392	32·092	77·252	15·471	50·525	104.378	—	—	—	—	—	—

TABLE 3

Non-dimensional natural frequencies for a simply supported stepped plate with step radius of 0·4 (values in second row of each h_1/h_2-ratio are experimental frequencies and those in the third row are by linear model)

	Aluminium $\nu = 0{\cdot}25$						Steel $\nu = 0{\cdot}3$					
	$N = 0$			$N = 1$			$N = 0$			$N = 1$		
$\frac{h_1}{h_2}$	$S = 0$	$S = 1$	$S = 2$	$S = 0$	$S = 1$	$S = 2$	$S = 0$	$S = 1$	$S = 2$	$S = 0$	$S = 1$	$S = 2$
1·0	4·861	29·431	74·029	13·770	48·094	102·374	4·906	29·557	74·062	13·920	48·198	102·514
	—	—	—	—	—	—	—	—	—	—	—	—
	4·816	29·506	73·996	13·813	48·187	102·388	—	—	—	—	—	—
0·9	6·176	31·014	74·753	14·686	49·449	103·958	6·276	31·089	74·822	14·720	49·481	104·030
	—	—	—	—	—	—	6·133	30·599	73·668	14·289	49·876	103·459
	6·202	31·033	74·709	14·664	49·407	103·934	—	—	—	—	—	—
0·8	6·350	31·281	75·195	14·817	50·211	104·499	6·452	31·340	75·160	14·938	50·230	104·602
	—	—	—	—	—	—	—	—	—	—	—	—
	6·385	31·276	75·147	14·910	50·236	104·534	—	—	—	—	—	—
0·7	6·542	31·509	75·793	15·025	50·878	105·471	6·690	31·613	75·850	15·146	50·907	105·553
	6·729	31·876	76·410	14·875	49·937	106·200	—	—	—	—	—	—
	6·598	31·507	75·805	15·127	50·909	105·442	—	—	—	—	—	—
0·6	6·722	32·062	77·071	15·339	51·212	105·054	6·874	32·166	77·132	15·447	51·237	106·028
	—	—	—	—	—	—	6.857	32.977	78.360	15·019	52·240	104·807
	6·799	32·067	76·992	15·405	51·229	105·939	—	—	—	—	—	—
0·5	6·866	32·587	77·626	15·433	51·603	106·428	7·025	32·638	77·660	15·581	51·691	106·479
	7·024	33·115	77·437	14·941	51·736	107·152	7·087	32·302	77·994	15·666	51·996	105·186
	6·961	32·567	77·566	15·524	51·647	106·442	—	—	—	—	—	—

TABLE 4

Non-dimensional natural frequencies for a clamped stepped plate with step radius of 0·4 (values in second row of each h_1/h_2-ratio are experimental frequencies and those in the third row are by linear model)

	Aluminium $\nu = 0·25$						Steel $\nu = 0·3$					
	$N = 0$			$N = 1$			$N = 0$			$N = 1$		
$\frac{h_1}{h_2}$	$S = 0$	$S = 1$	$S = 2$	$S = 0$	$S = 1$	$S = 2$	$S = 0$	$S = 1$	$S = 2$	$S = 0$	$S = 1$	$S = 2$
1·0	10·211	39·769	89·125	21·261	60·813	120·034	10·211	39·769	89·125	21·261	60·813	120·034
	—	—	—	—	—	—	—	—	—	—	—	—
	10·216	39·711	89·104	21·260	60·820	120·080	—	—	—	—	—	—
0·9	11·310	40·535	89·681	22·054	61·321	120·453	11·499	40·628	89·760	22·188	61·419	120·528
	—	—	—	—	—	—	11·458	40·440	90·720	22·107	62·210	119·972
	11·318	40·297	89·649	22·053	61·355	120·467	—	—	—	—	—	—
0·8	11·477	40·692	89·759	22·198	61·494	120·592	11·691	40·851	89·864	22·335	61·591	120·692
	11·567	40·783	89·647	21·874	60·655	122·348	—	—	—	—	—	—
	11·501	40·664	89·758	22·195	61·496	120·607	—	—	—	—	—	—
0·7	11·731	40·860	89·955	22·401	61·616	120·681	11·951	41·005	90·103	22·544	61·727	120·761
	11·758	40·599	89·184	22·629	62·384	119·799	—	—	—	—	—	—
	11·757	40·828	89·958	22·401	61·613	120·703	—	—	—	—	—	—
0·6	11·968	40·984	90·229	22·537	61·754	120·832	12·173	41·137	90·321	22·683	61·854	120·934
	—	—	—	—	—	—	12·234	41·179	90·639	22·977	61·694	120·703
	11·977	40·998	90·161	22·532	61·747	120·845	—	—	—	—	—	—
0·5	12·096	41·319	90·389	22·926	61·893	120·955	12·292	41·480	90·488	23·082	61·200	121·031
	12·017	41.615	90.796	22·254	61·755	121·284	12·160	41·200	90·639	22·380	61·918	120·554
	12·095	41·331	90·287	22·946	61·916	120·946	—	—	—	—	—	—

TABLE 5

Non-dimensional natural frequencies for a clamped stepped plate with step radius of 0·35 (values in second row of each h_1/h_2-ratio are experimental frequencies and those in the third row are by linear model)

	Aluminium $\nu = 0{\cdot}25$						Steel $\nu = 0{\cdot}3$					
	$N = 0$			$N = 1$			$N = 0$			$N = 1$		
$\frac{h_1}{h_2}$	$S = 0$	$S = 1$	$S = 2$	$S = 0$	$S = 1$	$S = 2$	$S = 0$	$S = 1$	$S = 2$	$S = 0$	$S = 1$	$S = 2$
1·0	10·211	39·769	89·126	21·261	60·813	120·034	10·211	39·769	89·125	21·261	60·813	120·034
	—	—	—	—	—	—	—	—	—	—	—	—
	10·215	39·771	89·104	21·260	61·820	120·080	—	—	—	—	—	—
0·9	11·423	40·660	89·854	22·047	61·446	120·687	11·625	40·811	89·965	22·205	61·521	120·780
	—	—	—	—	—	—	11·755	39·737	89·278	22·133	61·469	12·414
	11·443	40·707	89·863	22·083	61·511	120·667	—	—	—	—	—	—
0·8	11·679	41·018	90·047	22·299	61·727	120·734	11·877	41·151	90·152	22·449	61·795	120·813
	—	—	—	—	—	—	—	—	—	—	—	—
	11·713	41·039	90·038	22·335	61·767	120·747	—	—	—	—	—	—
0·7	11·799	41·184	90·078	22·532	61·858	120·946	11·999	41·327	90·169	22·678	61·950	121·026
	11·521	41·114	89·381	21·988	61·746	120·329	—	—	—	—	—	—
	11·851	41·229	90·086	22·593	61·911	120·946	—	—	—	—	—	—
0·6	12.042	41·422	90·383	22·661	62·118	121·083	12·255	41·544	90·459	22·808	62·230	121·167
	—	—	—	—	—	—	12.221	40·718	90·388	22·664	61·938	120·600
	12·059	41·442	90·340	22·700	62·193	121·075	—	—	—	—	—	—
0·5	12·217	41·842	90·591	23·144	62·299	121·147	12·440	41·990	90·700	23·307	62·416	121·22
	11·873	41·986	90·620	22·893	62·883	120·637	12·125	42·151	90·459	22·853	62·210	120·61
	12·237	41·795	90·536	23·167	62·403	121·152	—	—	—	—	—	—

positioned at 18° intervals. The internal diameters of the rings are considered as the external diameter of the plate model. Eventually, different rings were used for different plates. It was believed that this would simulate the clamped conditions assumed in theory.

The frequency response for each plate model was investigated by slowly increasing the driving frequency of an electromagnetic shaker by means of a sine generator, while holding the shaker in a stationary position. A natural frequency was distinguished by observing the sharp increase in the amplitude of a vibration transducer output, which was displayed on an oscilloscope, and the intensity of the tone emitted. Different points of excitation and measurement were attempted to avoid any mode disappearance that may be caused by mispositioning the shaker and vibration pickup. All results are summarized in Tables 2–5 and discussed in the following section.

5. RESULTS AND DISCUSSION

The main results of this work are summarized in Tables 2–5 and Figure 3. These tables are of three-fold purpose. The first is to show some of the numerical results obtained from the frequency equation (9), in particular for the experimental models tested in this work. These values are given in the first row for each (h_1/h_2) value. The second is to summarize the experimental data as shown in the second row of each (h_1/h_2) value. These two rows well indicate the consistency and agreement between the two sets of values. Also it indicates that the experimental boundary conditions serve well the theoretical assumptions adopted in this work. The third fold, however, is to show the values of the non-dimensional frequency as determined by using the proposed linear approximation stated above. It should be noticed from the data recorded in Figure 3 that the frequencies for $\nu = 0{\cdot}25$ were deliberately not used in the curve-fitting procedure for a matter of checking the proposed approximation. From these values it is clear that the linear model serves its purpose for practical approximation.

From Figure 3 it can be seen that for a given thickness ratio, the trend is well distinguished for $S = 0, 1$ and $N = 0, 1$ and within marginal computation error for $S, N > 1$. Evidently, for a given set of values of thickness ratio, S and N stiffer plate is obtained with larger values of ν. It can also be seen that the frequency can differ by 30% for $\nu = 0{\cdot}5$ as compared with $\nu = 0$. Also it is clear that moving from $S = 0$ to 1 within the same N has more effect than moving from $N = 0$ to 1 for the same S. In general, it can be stated that larger Poisson's ratio indicates stiffer plates and larger natural frequencies. Also as the step size increases the natural frequencies are expected to be larger.

In conclusion, the Poisson ratio should not be disregarded in the continuity equation in particular for the fundamental frequency. The trend of variation of the correction factor for $S = 0$, 1 and $N = 0$, 1 could be approximated by a straight-line model for practical applications. For other value the correction factor is insignificant to be considered.

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APPENDIX A

Frequency equations:
Assuming that

$$\lambda_1 = k_1.r_1 \qquad \lambda_2 = k_1.r_2, \qquad \lambda_3 = k_3.r_2,$$

(1) for simply supported plate the frequency equation is

$$\begin{bmatrix} EE1 & BB1 & -\mathrm{J}_n(\lambda_3) & -\mathrm{I}_n(\lambda_3) \\ EE1' & BB1' & -\left(\dfrac{h_1}{h_2}\right)^{0\cdot5}\mathrm{J}'_n(\lambda_3) & -\left(\dfrac{h_1}{h_2}\right)^{0\cdot5}\mathrm{I}'_n(\lambda_3) \\ \left(EE1'' + \dfrac{v}{\lambda_2}EE1'\right) & \left(BB1'' + \dfrac{v}{\lambda_2}BB1'\right) & -\left(\dfrac{h_2}{h_1}\right)^{2}\mathrm{J}''_n(\lambda_3) & -\left(\dfrac{h_2}{h_1}\right)^{2}\mathrm{I}''_n(\lambda_3) \\ EE1''' & BB1''' & -\left(\dfrac{h_2}{h_1}\right)^{1\cdot5}\mathrm{J}'''_n(\lambda_3) & -\left(\dfrac{h_2}{h_1}\right)^{1\cdot5}\mathrm{I}'''_n(\lambda_3) \end{bmatrix} = 0,$$

where

$$EE1 = \mathrm{J}_n(\lambda_2).F + \mathrm{Y}_n(\lambda_2).E + \mathrm{I}_n(\lambda_2),$$

$$BB1 = \mathrm{J}_n(\lambda_2).F1 + \mathrm{Y}_n(\lambda_2).E1 + \mathrm{K}_n(\lambda_2),$$

$$EE1' = \mathrm{J}'_n(\lambda_2).F + \mathrm{Y}'_n(\lambda_2).E + \mathrm{I}'_n(\lambda_2),$$

$$BB1' = \mathrm{J}'_n(\lambda_2).F1 + \mathrm{Y}'_n(\lambda_2).E1 + \mathrm{K}'_n(\lambda_2),$$

$$EE1'' = \mathrm{J}''_n(\lambda_2)F + \mathrm{Y}''_n(\lambda_2).E + \mathrm{I}''_n(\lambda_2),$$

$$EE1''' = \mathrm{J}'''_n(\lambda_2).F + \mathrm{Y}'''_n(\lambda_2).E + \mathrm{I}'''_n(\lambda_2),$$

$$BB1'' = \mathrm{J}_n''(\lambda_2).F1 + \mathrm{Y}_n''(\lambda_2).E1 + \mathrm{K}_n''(\lambda_2),$$

$$BB1''' = \mathrm{J}_n'''(\lambda_2).F1 + \mathrm{Y}_n'''(\lambda_2).E1 + \mathrm{K}_n'''(\lambda_2),$$

$$E = \left(\frac{(\mathrm{I}_n''(\lambda_1) + (v/\lambda_1)\mathrm{I}_n'(\lambda_1))}{(\mathrm{Y}_n''(\lambda_1) + (v/\lambda_1)\mathrm{Y}_n'(\lambda_1))}\right)\left\{\frac{\dfrac{\mathrm{J}_n(\lambda_1)}{(\mathrm{J}_n''(\lambda_1) + v/\lambda_1\,\mathrm{J}_n'(\lambda_1))} - \dfrac{\mathrm{I}_n.(\lambda_1)}{(\mathrm{I}_n''(\lambda_1) + v/\lambda_1\,\mathrm{I}_n'(\lambda_1))}}{\dfrac{\mathrm{Y}_n(\lambda_1)}{(\mathrm{Y}_n''(\lambda_1) + v/\lambda_1\,\mathrm{Y}_n'(\lambda_1))} - \dfrac{\mathrm{J}_n.(\lambda_1)}{(\mathrm{J}_n''(\lambda_1) + v/\lambda_1\,\mathrm{J}_n'(\lambda_1))}}\right\},$$

$$E1 = \left(\frac{(\mathrm{K}_n''(\lambda_1) + (v/\lambda_1)\mathrm{K}_n'(\lambda_1))}{(\mathrm{Y}_n''(\lambda_1) + (v/\lambda_1)\mathrm{Y}_n'(\lambda_1))}\right)\left\{\frac{\dfrac{\mathrm{J}_n(\lambda_1)}{(\mathrm{J}_n''(\lambda_1) + (v/\lambda_1)\mathrm{J}_n'(\lambda_1))} - \dfrac{\mathrm{K}_n(\lambda_1)}{(\mathrm{K}_n''(\lambda_1) + (v/\lambda_1)\mathrm{K}_n'(\lambda_1))}}{\dfrac{\mathrm{Y}_n(\lambda_1)}{(\mathrm{Y}_n''(\lambda_1) + (v/\lambda_1)\mathrm{Y}_n'(\lambda_1))} - \dfrac{\mathrm{J}_n(\lambda_1)}{(\mathrm{J}_n''(\lambda_1) + (v/\lambda_1)\mathrm{J}_n'(\lambda_1))}}\right\},$$

$$F = \left(\frac{(\mathrm{I}_n''(\lambda_1) + (v/\lambda_1)\mathrm{I}_n'(\lambda_1))}{(\mathrm{J}_n''(\lambda_1) + (v/\lambda_1)\mathrm{J}_n'(\lambda_1))}\right)\left\{\frac{\dfrac{\mathrm{Y}_n(\lambda_1)}{(\mathrm{Y}_n''(\lambda_1) + (v/\lambda_1)\mathrm{Y}_n'(\lambda_1))} - \dfrac{\mathrm{I}_n(\lambda_1)}{(\mathrm{I}_n''(\lambda_1) + (v/\lambda_1)\mathrm{I}_n'(\lambda_1))}}{\dfrac{\mathrm{Y}_n(\lambda_1)}{(\mathrm{J}_n''(\lambda_1) + (v/\lambda_1)\mathrm{J}_n'(\lambda_1))} - \dfrac{\mathrm{Y}_n(\lambda_1)}{(\mathrm{Y}_n''(\lambda_1) + (v/\lambda_1)\mathrm{Y}_n'(\lambda_1))}}\right\},$$

$$F1 = \left(\frac{(\mathrm{I}_n''(\lambda_1) + (v/\lambda_1)\mathrm{I}_n'(\lambda_1))}{(\mathrm{Y}_n''(\lambda_1) + (v/\lambda_1)\mathrm{Y}_n'(\lambda_1))}\right)\left\{\frac{\dfrac{\mathrm{Y}_n(\lambda_1)}{(\mathrm{Y}_n''(\lambda_1) + (v/\lambda_1)\mathrm{Y}_n'(\lambda_1))} - \dfrac{\mathrm{K}_n(\lambda_1)}{(\mathrm{K}_n''(\lambda_1) + (v/\lambda_1)\mathrm{K}_n'(\lambda_1))}}{\dfrac{\mathrm{J}_n(\lambda_1)}{(\mathrm{J}_n''(\lambda_1) + (v/\lambda_1)\mathrm{J}_n'(\lambda_1))} - \dfrac{\mathrm{Y}_n(\lambda_1)}{(\mathrm{Y}_n''(\lambda_1) + (v/\lambda_1)\mathrm{Y}_n'(\lambda_1))}}\right\},$$

(2) for clamped conditions the characteristic equation is

$$\begin{bmatrix} EE2 & BB2 & -\mathrm{J}_n(\lambda_3) & -\mathrm{I}_n(\lambda_3) \\ EE2' & BB2' & -\left(\dfrac{h_1}{h_2}\right)^{0\cdot 5}\mathrm{J}_n'(\lambda_3) & -\left(\dfrac{h_1}{h_2}\right)^{0\cdot 5}\mathrm{I}_n'(\lambda_3) \\ \left(EE2'' + \dfrac{v}{\lambda_2}EE2'\right) & \left(BB2'' + \dfrac{v}{\lambda_2}BB2'\right) & -\left(\dfrac{h_2}{h_1}\right)^2(\mathrm{J}_n''(\lambda_3) + \mathrm{J}_n'(\lambda_3)) & -\left(\dfrac{h_2}{h_1}\right)^2.\left(\mathrm{I}_n''(\lambda_3) + \dfrac{v}{\lambda_3}\mathrm{I}_n'(\lambda_3)\right) \\ EE2''' & BB2''' & -\left(\dfrac{h_2}{h_1}\right)^{1\cdot 5}\mathrm{J}_n'''(\lambda_3) & -\left(\dfrac{h_2}{h_1}\right)^{1\cdot 5}\mathrm{I}_n'''(\lambda_3) \end{bmatrix} = 0,$$

where

$$EE2 = \mathrm{J}_n(\lambda_2).F2 + \mathrm{Y}_n(\lambda_2).F4 + \mathrm{I}_n(\lambda_2),$$

$$EE2' = \mathrm{J}'_n(\lambda_2).F2 + \mathrm{Y}'_n(\lambda_2).F4 + \mathrm{I}'_n(\lambda_2),$$

$$EE2'' = \mathrm{J}''_n(\lambda_2).F2 + \mathrm{Y}''_n(\lambda_2).F4 + \mathrm{I}''_n(\lambda_2),$$

$$EE3''' = \mathrm{J}'''_n(\lambda_2).F2 + \mathrm{Y}'''_n(\lambda_2).F4 + \mathrm{I}'''_n(\lambda_2),$$

$$BB2 = \mathrm{J}_n(\lambda_2).F3 + \mathrm{Y}_n(\lambda_2).F5 + \mathrm{K}_n(\lambda_2),$$

$$BB2' = \mathrm{J}'_n(\lambda_2).F3 + \mathrm{Y}'_n(\lambda_2).F5 + \mathrm{K}'_n(\lambda_2),$$

$$BB2'' = \mathrm{J}''_n(\lambda_2).F3 + \mathrm{Y}''_n(\lambda_2).F5 + \mathrm{K}''_n(\lambda_2),$$

$$BB2''' = \mathrm{J}'''_n(\lambda_2).F3 + \mathrm{Y}'''_n(\lambda_2).F5 + \mathrm{K}'''_n(\lambda_2),$$

$$F2 = \left(\frac{\mathrm{I}'_n(\lambda_1)}{\mathrm{J}'_n(\lambda_1)}\right)\left\{\frac{\dfrac{\mathrm{Y}_n(\lambda_1)}{\mathrm{Y}'_n(\lambda_1)} - \dfrac{\mathrm{I}_n(\lambda_1)}{\mathrm{I}'_n(\lambda_1)}}{\dfrac{\mathrm{J}_n(\lambda_1)}{\mathrm{J}'_n(\lambda_1)} - \dfrac{\mathrm{Y}_n(\lambda_1)}{\mathrm{Y}'_n(\lambda_1)}}\right\},$$

$$F3 = \left(\frac{\mathrm{K}'_n(\lambda_1)}{\mathrm{J}'_n(\lambda_1)}\right)\left\{\frac{\dfrac{\mathrm{Y}_n(\lambda_1)}{\mathrm{Y}'_n(\lambda_1)} - \dfrac{\mathrm{K}_n(\lambda_1)}{\mathrm{K}'_n(\lambda_1)}}{\dfrac{\mathrm{J}'_n(\lambda_1)}{\mathrm{J}'_n(\lambda_1)} - \dfrac{\mathrm{Y}'_n(\lambda_1)}{\mathrm{Y}'_n(\lambda_1)}}\right\},$$

$$F4 = \left(\frac{\mathrm{I}'_n(\lambda_1)}{\mathrm{Y}'_n(\lambda_1)}\right)\left\{\frac{\dfrac{\mathrm{J}_n(\lambda_1)}{\mathrm{J}'_n(\lambda_1)} - \dfrac{\mathrm{I}_n(\lambda_1)}{\mathrm{I}'_n(\lambda_1)}}{\dfrac{\mathrm{Y}_n(\lambda_1)}{\mathrm{Y}'_n(\lambda_1)} - \dfrac{\mathrm{J}_n(\lambda_1)}{\mathrm{J}'_n(\lambda_1)}}\right\},$$

$$F5 = \left(\frac{\mathrm{K}'_n(\lambda_1)}{\mathrm{Y}'_n(\lambda_1)}\right)\left\{\frac{\dfrac{\mathrm{J}_n(\lambda_1)}{\mathrm{J}'_n(\lambda_1)} - \dfrac{\mathrm{K}_n(\lambda_1)}{\mathrm{K}'_n(\lambda_1)}}{\dfrac{\mathrm{Y}_n(\lambda_1)}{\mathrm{Y}'_n(\lambda_1)} - \dfrac{\mathrm{J}_n(\lambda_1)}{\mathrm{J}'_n(\lambda_1)}}\right\}.$$