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LETTERS TO THE EDITOR

LONGITUDINAL VIBRATIONS OF RODS COUPLED BY SEVERAL SPRING–MASS SYSTEMS

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1. INTRODUCTION

The study of the dynamical behaviour of longitudinally vibrating rods has stimulated the interest of researchers for many years. Recently, a study by Kukla *et al.* considered the problem of the natural longitudinal vibrations of two rods coupled by many translational springs where the Green's function method was employed [1]. Since then, Mermertas and Gürgöze [2] investigated a system made up of two clamped–free axially vibrating rods carrying tip masses, to which a double spring–mass system is attached as a secondary system across the span. As an extension of that publication, the present letter deals with a similar mechanical system, which is not only more complicated but also has more general application than the earlier studies.

The present study is concerned with longitudinal vibrations of a mechanical system consisting of two clamped–free rods carrying tip masses (as the primary system, ps) to which several double spring–mass systems are attached (as secondary systems, ss) across the span.

The major contribution of this study is to derive a general formulation for the exact solution of the system described by using the Green's function method.

2. THEORY

The problem to be considered in the present study is the natural vibration of the system shown in Figure 1; i.e., a longitudinally vibrating system consisting of two clamped–free rods carrying tip masses to which several double spring–mass systems are attached across the span.

However, in order to aid the explanation and to clarify the physics of the system, the Green's function method will first be applied to the $n = 1$ case; the results can then easily be generalized for the $n = n$ case.

2.1. THE CASE OF ONE SS, $n = 1$

The combined system, which has already been studied in reference [2] and which is to be investigated initially, consists of two clamped–free rods carrying tip masses to which a double spring–mass system is attached across the span, (see Figure 2). L_i, m_i, $\eta_i L_i$ and $E_i A_i$ denote length, mass per unit length, location of the spring attachment point and axial rigidity of the ith rod respectively ($i = 1, 2$). The secondary system consists of two springs of stiffness k_1, k_2 and mass M_s. The longitudinal vibration displacements of the first and second rods are denoted as $u_1(x, t)$ and $u_2(x, t)$, respectively, and $z(t)$ represents the displacement of the mass M_s.

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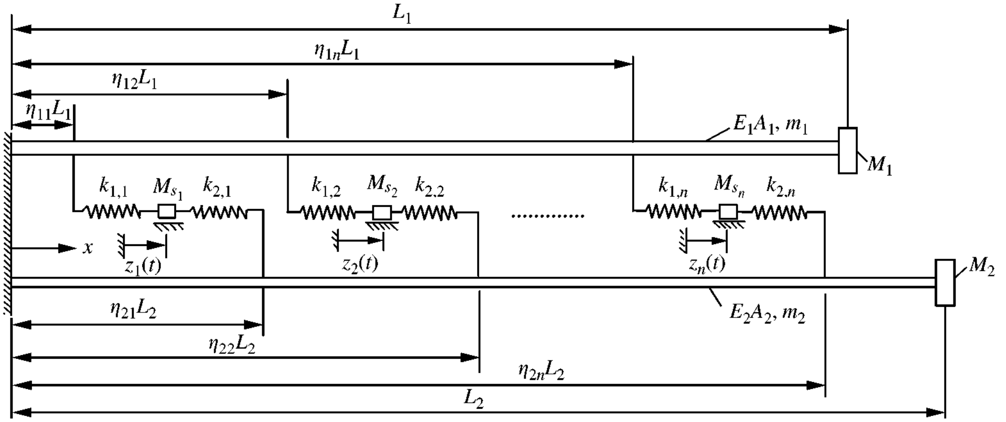

Figure 1. Two clamped–free axially vibrating rods carrying tip masses to which several double spring–mass systems are attached across the span.

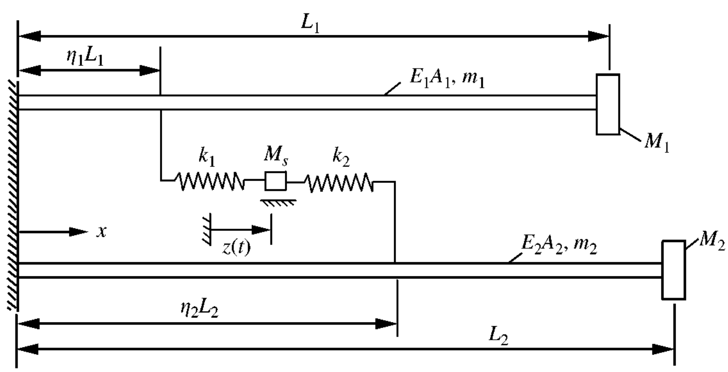

Figure 2. Two clamped–free axially vibrating rods carrying tip masses to which a double spring–mass system is attached across the span.

Assuming that the spring forces, which are generated by the ss, are singular effects for both the rods, the longitudinal vibration equations of the system can be written in the following form:

$$E_iA_iu_i''(x,t) - m_i\ddot{u}_i(x,t) = -k_i[z - u_i(\eta_iL_i,t)]\delta(x-\eta_iL_i), \quad i = 1, 2. \tag{1}$$

The motion of the secondary mass is governed by

$$M_s\ddot{z} = -k_1[z - u_1(\eta_1L_1,t)] + k_2[u_2(\eta_2L_2,t] - z], \tag{2}$$

Here $\delta(\cdot)$ denotes the Dirac delta function and dots and primes denote partial derivates with respect to time t and position co-ordinate x respectively. Using separation of variables according to

$$u_i(x,t) = U_i(x)\cos\omega t, \quad i = 1, 2,$$

$$z(t) = Z\cos\omega t, \tag{3}$$

where $U_i(x)$ and Z are the corresponding amplitude functions and ω is the unknown eigenfrequency of the combined system, and putting them into equations (1, 2),

$$U_1''(x) + \beta^2 U_1(x) = -\frac{k_1}{E_1A_1}\left\{\frac{k_2[U_2(h_2) - U_1(h_1)] + M_s\omega^2 U_1(h_1)}{k_1 + k_2 - M_s\omega^2}\right\}\delta(x - h_1),$$

$$U_2''(x) + \mu^2\beta^2 U_2(x) = -\frac{k_2}{E_2A_2}\left\{\frac{k_1[U_1(h_1) - U_2(h_2)] + M_s\omega^2 U_2(h_2)}{k_1 + k_2 - M_s\omega^2}\right\}\delta(x - h_2) \quad (4)$$

are obtained as the equations of motion, where

$$h_i = \eta_i L_i, \quad \beta^2 = \frac{m_1\omega^2}{E_1A_1}, \quad \mu^2 = \frac{\alpha_m}{\chi}, \quad \alpha_m = \frac{m_2}{m_1}, \quad \chi = \frac{E_2A_2}{E_1A_1}, \quad i = 1, 2. \quad (5)$$

Using the non-dimensional parameters below

$$\xi_i = \frac{x}{L_i}, \quad \bar{U}_i = \frac{U_i}{L_i}, \quad \bar{\beta} = \beta L_1, \quad \alpha_{k_i} = \frac{k_i}{E_iA_i/L_i}, \quad \alpha_k = \frac{k_2}{k_1}, \quad \alpha_M = \frac{M_s}{m_1L_1},$$

$$\alpha_L = \frac{L_2}{L_1}, \quad \bar{\delta} = \mu\alpha_L, \quad (\cdot)' \mathrel{\hat{=}} \frac{\partial}{\partial\xi}, \quad i = 1, 2, \quad (6)$$

the equations of motion can be reformulated as

$$\bar{U}_1''(\xi_1) + \bar{\beta}^2\bar{U}_1(\xi_1) = -\left\{\frac{\alpha_k\alpha_{k_1}[\alpha_L\bar{U}_2(\eta_2) - \bar{U}_1(\eta_1)] + \alpha_M\bar{\beta}^2\bar{U}_1(\eta_1)}{1 + \alpha_k - (\alpha_M/\alpha_{k_1})\bar{\beta}^2}\right\}\delta(\xi_1 - \eta_1),$$

$$\bar{U}_2''(\xi_2) + \bar{\delta}^2\bar{\beta}^2\bar{U}_2(\xi_2) = -\left\{\frac{\alpha_{k_2}[(1/\alpha_L)\bar{U}_1(\eta_1) - \bar{U}_2(\eta_2)] + (\alpha_{k_2}/\alpha_{k_1})\alpha_M\bar{\beta}^2\bar{U}_2(\eta_2)}{1 + \alpha_k - (\alpha_M/\alpha_{k_1})\bar{\beta}^2}\right\} \times \delta(\xi_2 - \eta_2). \quad (7)$$

For the solution of the above differential equations, the Green's function method will be employed. For convenience, the derivation of the corresponding Green's function is given in Appendix B. Therefore, via an analogy with (B3), i.e., using $\xi_1, \eta_1, \bar{\beta}, G_1(\xi_1, \eta_1), 1$ for the first rod and $\xi_2, \eta_2, \bar{\delta}\bar{\beta}, G_2(\xi_2, \eta_2), 1$, for the second rod, instead of $x, \xi, \bar{\beta}, G(x, \xi), L$, respectively, Green's functions, which correspond to the combined system, can be written as follows:

$$G_1,(\xi_1, \eta_1) = \frac{1}{\bar{\beta}}\left\{\sin(\bar{\beta}(\xi_1 - \eta_1))H(\xi_1 - \eta_1) - \frac{\cos(\bar{\beta}(1 - \eta_1)) - \bar{\alpha}_{M_1}\bar{\beta}\sin(\bar{\beta}(1 - \eta_1))}{\cos\bar{\beta} - \bar{\alpha}_{M_1}\bar{\beta}\sin\bar{\beta}}\sin(\bar{\beta}\xi_1)\right\}$$

$$G_2,(\xi_2, \eta_2) = \frac{1}{\bar{\delta}\bar{\beta}}\left\{\sin(\bar{\delta}\bar{\beta}(\xi_2 - \eta_2))H(\xi_2 - \eta_2) - \frac{\cos(\bar{\delta}\bar{\beta}(1 - \eta_2)) - \bar{\alpha}_{M_2}(\alpha_L/\chi\bar{\delta})\bar{\beta}\sin(\bar{\delta}\bar{\beta}(1 - \eta_2))}{\cos\bar{\delta}\bar{\beta} - \bar{\alpha}_{M_2}(\alpha_L/\chi\bar{\delta})\bar{\beta}\sin\bar{\delta}\bar{\beta}}\sin(\bar{\delta}\bar{\beta}\xi_2)\right\}, \quad (8)$$

where

$$\bar{\alpha}_{M_1} = \frac{M_1}{m_1 L_1}, \quad \bar{\alpha}_{M_2} = \frac{M_2}{m_1 L_1}. \tag{9}$$

Here $H(\cdot)$ denotes the Heaviside unit step function. Now, the displacements of the points $\xi_1 = \eta_1$ and $\xi_2 = \eta_2$ can be given in the form

$$\bar{U}_1(\eta_1) = G_1(\eta_1, \eta_1)\left\{ -\frac{\alpha_k \alpha_{k_1}[\alpha_L \bar{U}_2(\eta_2) - \bar{U}_1(\eta_1)] + \alpha_M \bar{\beta}^2 \bar{U}_1(\eta_1)}{1 + \alpha_k - (\alpha_M/\alpha_{k_1})\bar{\beta}^2} \right\},$$

$$\bar{U}_2(\eta_2) = G_2(\eta_2, \eta_2)\left\{ -\frac{\alpha_{k_2}[(1/\alpha_L)\bar{U}_1(\eta_1) - \bar{U}_2(\eta_2)] + (\alpha_{k_2}/\alpha_{k_1})\alpha_M \bar{\beta}^2 \bar{U}_2(\eta_2)}{1 + \alpha_k - (\alpha_M/\alpha_{k_1})\bar{\beta}^2} \right\}. \tag{10}$$

These equations represent a set of two homogeneous equations for the solution of the unknowns $\bar{U}_1(\eta_1)$ and $\bar{U}_2(\eta_2)$. A non-trivial solution exists when the determinant of the coefficient matrix vanishes. This condition in turn leads to the following frequency equation:

$$\begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix} = 0, \tag{11}$$

where

$$M_{11} = \bar{\beta}\left(1 + \alpha_k - \frac{\alpha_M}{\alpha_{k_1}}\bar{\beta}^2\right)(-\cos\bar{\beta} + \alpha_{M_1}\bar{\beta}\sin\bar{\beta}) + \sin(\bar{\beta}\eta_1)[\cos(\bar{\beta}(1-\eta_1))$$

$$- \alpha_{M_1}\bar{\beta}\sin(\bar{\beta}(1-\eta_1))](\alpha_M \bar{\beta}^2 - \alpha_k \alpha_{k_1}),$$

$$M_{12} = \alpha_k \alpha_{k_1} \alpha_L \sin(\bar{\beta}\eta_1)[\cos(\bar{\beta}(1-\eta_1)) - \alpha_{M_1}\bar{\beta}\sin(\bar{\beta}(1-\eta_1))],$$

$$M_{21} = \frac{\alpha_{k_2}}{\alpha_L}\sin(\bar{\delta}\bar{\beta}\eta_2)\left[-\cos(\bar{\delta}\bar{\beta}(1-\eta_2)) + \alpha_{M_2}\frac{\alpha_L}{\chi\bar{\delta}}\bar{\beta}\sin(\bar{\delta}\bar{\beta}(1-\eta_2))\right],$$

$$M_{22} = \bar{\delta}\bar{\beta}\left(1 + \alpha_k - \frac{\alpha_M}{\alpha_{k_1}}\bar{\beta}^2\right)\left(\cos\bar{\delta}\bar{\beta} - \alpha_{M_2}\frac{\alpha_L}{\chi\bar{\delta}}\bar{\beta}\sin\bar{\delta}\bar{\beta}\right)$$

$$+\sin(\bar{\delta}\bar{\beta}\eta_2)\left[-\cos(\bar{\delta}\bar{\beta}(1-\eta_2)) + \alpha_{M_2}\frac{\alpha_L}{\chi\bar{\delta}}\bar{\beta}\sin(\bar{\delta}\bar{\beta}(1-\eta_2))\right]\left(\frac{\alpha_{k_2}}{\alpha_{k_1}}\alpha_M\bar{\beta}^2 - \alpha_{k_2}\right). \tag{12}$$

The solution of equation (11) yields the desired non-dimensional frequency parameters $\bar{\beta}$ of the system.

2.2. GENERALIZATION FOR THE CASE OF SEVERAL *ss*'s, $n = n$

Consider a system of two rods that are carrying tip masses and coupled by n ss's in such a way that n points of the first rod of co-ordinates $\eta_{11}, \eta_{12}, \ldots, \eta_{1n}$ are connected to n points of co-ordinates $\eta_{21}, \eta_{22}, \ldots, \eta_{2n}$ belonging to the second rod, by using springs of stiffnesses $k_{1,1}, k_{2,1}, k_{1,2}, k_{2,2}, \ldots, k_{1,n}, k_{2,n}$ and the masses $M_{s_1}, M_{s_2}, \ldots, M_{s_n}$ which make up the ss's, as shown in Figure 1.

Equations (7) which represent the governing differential equations of the combined system having a single ss, i.e., $n = 1$ is taken, can be reformulated for the case of several ss's, i.e., $n = n$ is taken, as below

$$\bar{U}_1''(\xi_1) + \bar{\beta}^2 \bar{U}_1(\xi_1) = -\sum_{j=1}^{n} \left\{ \frac{\alpha_{k_j}\alpha_{k_{1,j}}[\alpha_L \bar{U}_2(\eta_{2j}) - \bar{U}_1(\eta_{1j})] + \alpha_{M_j}\bar{\beta}^2 \bar{U}_1(\eta_{1j})}{1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\,\bar{\beta}^2} \right\} \times \delta(\xi_1 - \eta_{1j}),$$

$$\bar{U}_2''(\xi_2) + \bar{\delta}^2\bar{\beta}^2 \bar{U}_2(\xi_2) = -\sum_{j=1}^{n} \left\{ \frac{\alpha_{k_{2,j}}[(1/\alpha_L)\bar{U}_1(\eta_{1j}) - \bar{U}_2(\eta_{2j})] + (\alpha_{k_{2,j}}/\alpha_{k_{1,j}})\,\alpha_{M_j}\bar{\beta}^2 \bar{U}_2(\eta_{2j})}{1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\,\bar{\beta}^2} \right\} \times \delta(\xi_2 - \eta_{2j}). \quad (13)$$

Similar, for this case, equations (10) can be rearranged as

$$\bar{U}_1(\xi_1) = \sum_{j=1}^{n} G_1(\xi_1, \eta_{1j}) \left\{ -\frac{\alpha_{k_j}\alpha_{k_{1,j}}[\alpha_L \bar{U}_2(\eta_{2j}) - \bar{U}_1(\eta_{1j})] + \alpha_{M_j}\bar{\beta}^2 \bar{U}_1(\eta_{1j})}{1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\,\bar{\beta}^2} \right\}$$

$$\bar{U}_2(\xi_2) = \sum_{j=1}^{n} G_2(\xi_2, \eta_2) \left\{ -\frac{\alpha_{k_{2,j}}[(1/\alpha_L)\bar{U}_1(\eta_{1j}) - \bar{U}_2(\eta_{2j})] + (\alpha_{k_{2,j}}/\alpha_{k_{1,j}})\,\alpha_{M_j}\bar{\beta}^2 \bar{U}_2(\eta_{2j})}{1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\,\bar{\beta}^2} \right\}. \quad (14)$$

For simplicity, these equations can be written in the following form, after some re-arrangements

$$\bar{U}_1(\xi_1) = -\sum_{j=1}^{n} [(C_{1j}^{(1)} \bar{U}_1(\eta_{1j}) + C_{2j}^{(1)} \bar{U}_2(\eta_{2j}))\, G_1(\xi_1, \eta_{1j})],$$

$$\bar{U}_2(\xi_2) = -\sum_{j=1}^{n} [(C_{1j}^{(2)} \bar{U}_1(\eta_{1j}) + C_{2j}^{(2)} \bar{U}_2(\eta_{2j}))\, G_2(\xi_2, \eta_{2j})], \quad (15)$$

where the following abbreviations are introduced:

$$C_{1j}^{(1)} = \frac{\alpha_{M_j}\bar{\beta}^2 - \alpha_{k_j}\alpha_{k_{1,j}}}{1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\,\bar{\beta}^2}, \quad C_{2j}^{(1)} = \frac{\alpha_{k_j}\alpha_{k_{1,j}}\alpha_L}{1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\,\bar{\beta}^2},$$

$$C_{1j}^{(2)} = \frac{\alpha_{k_{2,j}}/\alpha_L}{1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\,\bar{\beta}^2}, \quad C_{2j}^{(2)} = \frac{(\alpha_{k_{2,j}}/\alpha_{k_{1,j}})\alpha_{M_j}\bar{\beta}^2 - \alpha_{k_{2,j}}}{1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\,\bar{\beta}^2},$$

$$\alpha_{k_j} = \frac{k_{2,j}}{k_{1,j}}, \quad \alpha_{M_j} = \frac{M_{s_j}}{m_1 L_1}, \quad \alpha_{k_{1,j}} = \frac{k_{1,j}}{E_1 A_1/L_1}, \quad \alpha_{k_{2,j}} = \frac{k_{2,j}}{E_2 A_2/L_2},$$

$$\alpha_L = \frac{L_2}{L_1}, \qquad j = 1, 2, \ldots, n. \quad (16)$$

Equations (15) represent displacement fields on the axes ξ_1 and ξ_2. In order to find the displacements at all the attachment points along the axes,

$$\xi_1 \to \eta_{11}, \eta_{12}, \ldots, \eta_{1n}, \qquad \xi_2 \to \eta_{21}, \eta_{22}, \ldots, \eta_{2n} \quad (17)$$

have to be substituted into equations (15). Thus, $2n$ equations are obtained for the $2n$ unknowns $\bar{U}_1(\eta_{1j})$ and $\bar{U}_2(\eta_{2j})$, $j = 1, 2, \dots, n$. Using matrix notation

$$\mathbf{Ax} = \mathbf{0} \tag{18}$$

can be written, where

$$\mathbf{x}^{\mathrm{T}} = \{\bar{U}_1(\eta_{11}), \bar{U}_1(\eta_{12}), \dots, \bar{U}_1(\eta_{1n}) \mid \bar{U}_2(\eta_{21}), \bar{U}_2(\eta_{22}), \dots, \bar{U}_2(\eta_{2n})\},$$

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{C}_1^{(1)} + \mathbf{I}_n & \mathbf{C}_2^{(1)} \\ \hline \mathbf{C}_1^{(2)} & \mathbf{C}_2^{(2)} + \mathbf{I}_n \end{array}\right], \tag{19}$$

$$[\mathbf{C}_1^{(1)}]_{ij} = C_{1j}^{(1)} G_{ij}^{(1)}, \quad [\mathbf{C}_2^{(1)}]_{ij} = C_{2j}^{(1)} G_{ij}^{(1)}, \quad [\mathbf{C}_1^{(2)}]_{ij} = C_{1j}^{(2)} G_{ij}^{(2)},$$

$$[\mathbf{C}_2^{(2)}]_{ij} = C_{2j}^{(2)} G_{ij}^{(2)},$$

$$G_{ij}^{(m)} \triangleq G_m(\eta_{mi}, \eta_{mj}), \qquad i, j = 1, 2, \dots, n, \tag{20}$$

$$m = 1, 2.$$

Here $[\cdot]_{ij}$, denotes an element of the matrix located at the ith row and jth column. To make it more comprehensible, $G_{ij}^{(m)}$ can be given as follows:

$$G_1(\eta_{1i}, \eta_{1j}) = \frac{1}{\bar{\beta}} \left\{ \sin(\bar{\beta}(\eta_{1i} - \eta_{1j})) H(\eta_{1i} - \eta_{1j}) - \frac{\cos(\bar{\beta}(1 - \eta_{1j})) - \alpha_{M_1} \bar{\beta} \sin(\bar{\beta}(1 - \eta_{1j}))}{\cos \bar{\beta} - \alpha_{M_1} \bar{\beta} \sin \bar{\beta}} \sin(\bar{\beta}\eta_{1i}) \right\}$$

$$G_2(\eta_{2i}, \eta_{2j}) = \frac{1}{\bar{\delta}\bar{\beta}} \left\{ \sin(\bar{\delta}\bar{\beta}(\eta_{2i} - \eta_{2j})) H(\eta_{2i} - \eta_{2j}) - \frac{\cos(\bar{\delta}\bar{\beta}(1 - \eta_{2j})) - \alpha_{M_2}(\alpha_L/\chi\bar{\delta}) \bar{\beta} \sin(\bar{\delta}\bar{\beta}(1 - \eta_{2j}))}{\cos \bar{\delta}\bar{\beta} - \alpha_{M_2}(\alpha_L/\chi\bar{\delta}) \bar{\beta} \sin \bar{\delta}\bar{\beta}} \sin(\bar{\delta}\bar{\beta}\eta_{2i}) \right\},$$

$$i, j = 1, 2, \dots, n. \tag{21}$$

A non-trivial solution exists when the determinant of the coefficient matrix $\mathbf{A}$ vanishes. Thus, the following frequency equation can be obtained.

$$\det(\mathbf{A}) = 0. \tag{22}$$

The solution of equation (22) yields the non-dimensional frequency parameters $\bar{\beta}$ of the system.

3. NUMERICAL RESULTS

This section is devoted to the numerical evaluations of the formulae established in the preceding sections. As an example, the $n = 2$ case, i.e., the two-secondary-system case, is

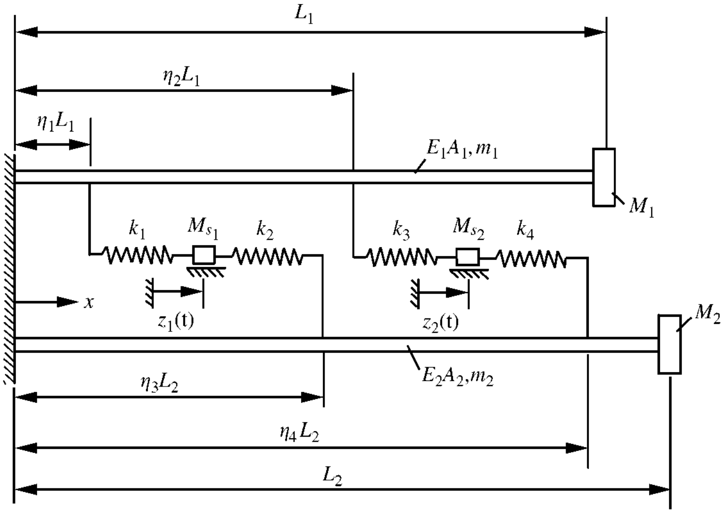

Figure 3. Two clamped–free axially vibrating rods carrying tip masses to which two double spring–mass systems are attached across the span.

considered. Normally, the classical approach of deriving the frequency equation based on the boundary value problem formulation is fairly complicated even for the case of two secondary systems which is presented in Appendix A. Practically, as the number of secondary systems exceeds two, the solution of the problem becomes nearly impossible and extremely tedious. Since the solution of the problem for an n-number of secondary systems does not exist in the literature to the best of our knowledge, the only way to prove the validity of the Green's function method is to compare the formulation with the classical approach with two secondary systems, i.e., $n = 2$ (Figure 3).

The physical parameter values whose definitions are given in Appendix A are chosen as

$$^{2}\alpha_k = 0{\cdot}6, \quad ^{3}\alpha_k = {}^{4}\alpha_k = 2{\cdot}5, \quad \alpha_{k_1} = 1{\cdot}3, \quad \alpha_{k_2} = 1{\cdot}5, \quad ^{1}\alpha_M = {}^{2}\alpha_M = 10,$$

$$\alpha_{M_1} = 2{\cdot}5,$$

$$\alpha_{M_2} = 3{\cdot}5, \quad \alpha_L = \bar{\delta} = 1{\cdot}5, \quad \eta_1 = 0{\cdot}25, \quad \eta_2 = 0{\cdot}75, \quad \eta_3 = 0{\cdot}5, \quad \eta_4 = 1.$$

In Table 1, the first 12 dimensionless eigenfrequency parameters $\bar{\beta}$ of the described system are given. The values in the first column are values from the solution of equation (A7), whereas those in the second column are the roots of equation (22) derived via the Green's function method. It is seen clearly that the values in the columns are identical, which justifies the lengthy and complicated expressions obtained by the application of the Green's function method.

4. CONCLUSION

This study is concerned with longitudinal vibrations of a combined system consisting of two clamped–free rods carrying tip masses to which several double spring–mass systems are attached across the span. Using the Green's function method, the frequency equation of the

TABLE 1

Dimensionless eigenfrequency parameters $\bar{\beta}$ of the system in Figure 3, i.e., $n = 2$, ${}^2\alpha_k = 0{\cdot}6$, ${}^3\alpha_k = {}^4\alpha_k = 2{\cdot}5$, $\alpha_{k_1} = 1{\cdot}3$, $\alpha_{k_2} = 1{\cdot}5$, ${}^1\alpha_M = {}^2\alpha_M = 10$, $\alpha_{M_1} = 2{\cdot}5$, $\alpha_{M_2} = 3{\cdot}5$, $\alpha_L = \bar{\delta} = 1{\cdot}5$, $\eta_1 = 0{\cdot}25$, $\eta_2 = 0{\cdot}75$, $\eta_3 = 0{\cdot}5$, $\eta_4 = 1$ are chosen

From equation (A7)	From equation (22)
0·2871184548	0·2871184548
0·4300885248	0·4300885248
0·9210149661	0·9210149661
1·1808429443	1·1808429443
2·4393725240	2·4393725240
3·7660297675	3·7660297675
4·2259977171	4·2259977171
6·4108351523	6·4108351523
6·9801434970	6·9801434970
8·3955243247	8·3955243247
9·7141410409	9·7141410409
10·5493604996	10·5493604996

system with n ss's is established. Then, in order to prove the validity of the expressions derived, for a special system with $n = 2$, the results are compared with those obtained on the basis of a boundary value problem formulation. The two results are in excellent agreement which clearly indicates the validity of the formulae obtained via the Green's function method.

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APPENDIX A: BOUNDARY VALUE PROBLEM FORMULATION SOLUTION FOR TWO ss's

In order to test the reliability of equation (22), corresponding to the Green's function solution for n ss's, the following example is considered for two ss's; i.e., $n = 2$ is chosen.

The combined system consist of two clamped–free rods carrying tip masses to which two double spring–mass systems are attached across the span, as seen in Figure 3. L_i, m_i, $\eta_1 L_1, \eta_2 L_1, \eta_3 L_2, \eta_4 L_2$ and $E_i A_i$ denote the length, mass per unit length, locations of the spring attachment points and axial rigidity of the ith rod respectively ($i = 1, 2$). Secondary systems consist of two springs of stiffnesses k_1, k_2 and k_3, k_4 and the masses M_{s_j} ($j = 1, 2$). Furthermore, the longitudinal vibration displacements of the first and second rods to the left and right of the spring attachment points are denoted as $u_{11}(x, t)$, $u_{12}(x, t)$, $u_{13}(x, t)$ and $u_{21}(x, t)$, $u_{22}(x, t)$, $u_{23}(x, t)$, respectively, and $z_1(t)$, $z_2(t)$ represent the displacements of the masses M_{s_j}.

The equation of longitudinal vibration of the six rod portions are governed by the following partial differential equations:

$$E_iA_i\frac{\partial^2}{\partial x^2}u_{ij}(x,t)=m_i\frac{\partial^2}{\partial t^2}u_{ij}(x,t)\quad (i=1,2,\ j=1,2,3). \tag{A1}$$

The corresponding boundary and continuity conditions at the spring attachment points are as follows:

$$u_{11}(0,t)=0,\qquad u_{11}(\eta_1L_1,t)=u_{12}(\eta_1L_1,t),$$

$$-E_1A_1u'_{11}(\eta_1L_1,t)+E_1A_1u'_{12}(\eta_1L_1,t)+k_1[z_1(t)-u_{11}(\eta_1L_1,t)]=0,$$

$$u_{12}(\eta_2L_1,t)=u_{13}(\eta_2L_1,t),$$

$$-E_1A_1u'_{12}(\eta_2L_1,t)+E_1A_1u'_{13}(\eta_2L_1,t)+k_3[z_2(t)-u_{12}(\eta_2L_1,t)]=0,$$

$$E_1A_1u'_{13}(L_1,t)+M_1\ddot{u}_{13}(L_1,t)=0,$$

$$M_{s_1}\ddot{z}_1(t)=-k_1[z_1(t)-u_{11}(\eta_1L_1,t)]+k_2[u_{21}(\eta_3L_2,t)-z_1(t)],$$

$$M_{s_2}\ddot{z}_2(t)=-k_3[z_2(t)-u_{12}(\eta_2L_1,t)]+k_4[u_{22}(\eta_4L_2,t)-z_2(t)],$$

$$u_{21}(0,t)=0,\qquad u_{21}(\eta_3L_2,t)=u_{22}(\eta_3L_2,t),$$

$$-E_2A_2u'_{21}(\eta_3L_2,t)+E_2A_2u'_{22}(\eta_3L_2,t)+k_2[z_1(t)-u_{21}(\eta_3L_2,t)]=0,$$

$$u_{22}(\eta_4L_2,t)=u_{23}(\eta_4L_2,t),$$

$$-E_2A_2u'_{22}(\eta_4L_2,t)+E_2A_2u'_{23}(\eta_4L_2,t)+k_4[z_2(t)-u_{22}(\eta_4L_2,t)]=0,$$

$$E_2A_2u'_{23}(L_2,t)+M_2\ddot{u}_{23}(L_2,t)=0. \tag{A2}$$

Here, dots and primes denote partial derivatives with respect to time t and position co-ordinate x respectively. Using the standard method of separation of variables, one assumes

$$u_{ij}(x,t)=U_{ij}(x)\cos\omega t\quad (i=1,2,\ j=1,2,3), \tag{A3}$$

$$z_i(t)=Z_i\cos\omega t\quad (i=1,2), \tag{A4}$$

where $U_{ij}(x)$ and Z_i are the corresponding amplitude functions of the rods and secondary masses, respectively, and ω is the unknown eigenfrequency of the combined system. Substituting these into equations (A1), results in the following ordinary differential equations:

$$U''_{1j}(x)+\beta^2U_{1j}(x)=0,\qquad U''_{2j}(x)+\mu^2\beta^2U_{2j}(x)=0,\quad (j=1,2,3). \tag{A5}$$

The general solutions of the ordinary differential equations (A5) are simply

$$U_{1j}(x) = C_{1j}\sin\beta x + C_{2j}\cos\beta x, \quad U_{2j}(x) = C_3\sin\mu\beta x + C_{4j}\cos\mu\beta x$$

$$(j = 1, 2, 3), \tag{A6}$$

where $C_{1j} - C_{4j}$ are 12 integration constants to be evaluated via conditions (A2). The application of these boundary and matching conditions to solutions (A6) and the amplitudes Z_i yields a set of 14 homogeneous equations for the 14 unknown constants $C_{1j} - C_{4j}$ $(j = 1, 2, 3)$ and Z_i $(i = 1, 2)$. A non-trivial solution of this set of equations is possible only if the characteristic determinant of the coefficient matrix vanishes. Taking into account the fact that C_{21} and C_{41} vanish, the characteristic equation reduces the following form:

$$\det(\mathbf{K}) = 0. \tag{A7}$$

Here, $\mathbf{K}$ is a 12×12 matrix, the elements of which are shown below where all of the unwritten elements are equal to zero.

$$K_{11} = \sin\eta_1\bar{\beta}, \qquad K_{12} = -\sin\eta_1\bar{\beta}, \qquad K_{13} = -\cos\eta_1\bar{\beta},$$

$$K_{21} = \bar{\beta}\cos\eta_1\bar{\beta} + \alpha_{k_1}\sin\eta_1\bar{\beta}, \qquad K_{22} = -\bar{\beta}\cos\eta_1\bar{\beta}, \qquad K_{23} = \bar{\beta}\sin\eta_1\bar{\beta},$$

$$K_{2,11} = -\alpha_{k_1},$$

$$K_{32} = \sin\eta_2\bar{\beta}, \qquad K_{33} = \cos\eta_2\bar{\beta}, \qquad K_{34} = -\sin\eta_2\bar{\beta}, \qquad K_{35} = -\cos\eta_2\bar{\beta},$$

$$K_{42} = \bar{\beta}\cos\eta_2\bar{\beta} + {}^3\alpha_k\alpha_{k_1}\sin\eta_2\bar{\beta}, \qquad K_{43} = \bar{\beta}\sin\eta_2\bar{\beta} + {}^3\alpha_k\alpha_{k_1}\cos\eta_2\bar{\beta},$$

$$K_{44} = -\bar{\beta}\cos\eta_2\bar{\beta}, \qquad K_{45} = \bar{\beta}\sin\eta_2\bar{\beta}, \qquad K_{4,12} = -{}^3\alpha_k\alpha_{k_1},$$

$$K_{54} = \cos\bar{\beta} - \bar{\alpha}_{M_1}\bar{\beta}\sin\bar{\beta}, \qquad K_{55} = -(\sin\bar{\beta} + \bar{\alpha}_{M_1}\bar{\beta}\cos\bar{\beta}),$$

$$K_{61} = -\alpha_{k_1}\sin\eta_1\bar{\beta}, \qquad K_{66} = -{}^2\alpha_k\alpha_{k_1}\sin\psi_3\bar{\beta},$$

$$K_{6,11} = -{}^1\alpha_M\bar{\beta}^2 + \alpha_{k_1}(1 + {}^2\alpha_k),$$

$$K_{72} = -{}^3\alpha_k\alpha_{k_1}\sin\eta_2\bar{\beta}, \qquad K_{73} = -{}^3\alpha_k\alpha_{k_1}\cos\eta_2\bar{\beta}, \qquad K_{77} = -{}^4\alpha_k\alpha_{k_1}\sin\psi_4\bar{\beta},$$

$$K_{78} = -{}^4\alpha_k\alpha_{k_1}\cos\psi_4\bar{\beta}, \qquad K_{7,12} = -{}^2\alpha_M\bar{\beta}^2 + \alpha_{k_1}({}^3\alpha_k + {}^4\alpha_k),$$

$$K_{86} = \sin\psi_3\bar{\beta}, \qquad K_{87} = -\sin\psi_3\bar{\beta}, \qquad K_{88} = -\cos\psi_3\bar{\beta},$$

$$K_{96} = \bar{\delta}\bar{\beta}\cos\psi_3\bar{\beta} + \alpha_{k_2}\sin\psi_3\bar{\beta}, \qquad K_{97} = -\bar{\delta}\bar{\beta}\cos\psi_3\bar{\beta}, \qquad K_{98} = \bar{\delta}\bar{\beta}\sin\psi_3\bar{\beta},$$

$$K_{9,11} = -\alpha_{k_2},$$

$$K_{10,7} = \sin\psi_4\bar{\beta}, \qquad K_{10,8} = \cos\psi_4\bar{\beta}, \qquad K_{10,9} = -\sin\psi_4\bar{\beta},$$

$$K_{10,10} = -\cos\psi_4\bar{\beta},$$

$$K_{11,7} = \bar{\delta}\bar{\beta}\cos\psi_4\bar{\beta} + \frac{{}^4\alpha_k\alpha_{k_2}}{{}^2\alpha_k}\sin\psi_4\bar{\beta}, \qquad K_{11,8} = -\bar{\delta}\bar{\beta}\sin\psi_4\bar{\beta} + \frac{{}^4\alpha_k\alpha_{k_2}}{{}^2\alpha_k}\cos\psi_4\bar{\beta},$$

$$K_{11,9} = -\,\overline{\delta}\overline{\beta}\cos\psi_4\overline{\beta}, \qquad K_{11,10} = \overline{\delta}\overline{\beta}\sin\psi_4\overline{\beta}, \qquad K_{11,12} = -\frac{{}^4\alpha_k\alpha_{k_2}}{{}^2\alpha_k},$$

$$K_{12,9} = \cos\overline{\delta}\overline{\beta} - \frac{\bar{\alpha}_{M_2}}{\mu\chi}\overline{\beta}\sin\overline{\delta}\overline{\beta}, \qquad K_{12,10} = -\,(\sin\overline{\delta}\overline{\beta} + \frac{\bar{\alpha}_{M_2}}{\mu\chi}\overline{\beta}\cos\overline{\delta}\overline{\beta}). \tag{A8}$$

Here, in addition to the abbreviation above, the following definitions are introduced:

$${}^f\alpha_k = \frac{k_f}{k_1} \quad (f = 2, 3, 4), \quad {}^r\alpha_M = \frac{M_{s_r}}{m_1 L_1} \quad (r = 1, 2) \quad \psi_1 = \mu\alpha_L\eta_t \quad (t = 3, 4). \tag{A9}$$

APPENDIX B

As is known [1], the corresponding Green's function for the clamped–free rod carrying a tip mass, is the solution of the differential equation

$$\frac{\mathrm{d}^2 G(x,\xi)}{\mathrm{d}x^2} + \overline{\beta}^2 G(x,\xi) = \delta(x-\xi) \tag{B1}$$

subject to the following boundary conditions:

$$G(0,\xi) = 0,$$

$$G'(L,\xi) - A\overline{\beta}^2 G(L,\xi) = 0. \tag{B2}$$

The solution $G(x, \xi)$ satisfying differential equation (B1) is the Green's function that is looked for. Thus, $G(x, \xi)$ can be found as

$$G(x,\xi) = \frac{1}{\overline{\beta}}\left\{\sin(\overline{\beta}(x-\xi))H(x-\xi) - \frac{\cos(\overline{\beta}(L-\xi)) - A\overline{\beta}\sin(\overline{\beta}(L-\xi))}{\cos\overline{\beta}L - A\overline{\beta}\sin\overline{\beta}L}\sin(\overline{\beta}x)\right\}, \tag{B3}$$

where A denotes α_{M_1} for the first rod and $\alpha_{M_2}(\alpha_L/\chi\overline{\delta})$ for the second rod.