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THE PERIODICITY OF CHAOTIC IMPACT OSCILLATORS IN HAUSDORFF PHASE SPACES

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It is well known that non-periodic behavior is one of the most puzzling characteristics of chaotic oscillators. So far chaotic dynamical systems have been investigated in Euclidean spaces. In this paper, the concept of non-autonomous dynamical systems and that of Hausdorff phase spaces are proposed. The behavior of chaotic impact oscillators is investigated in Hausdorff phase spaces. It is discovered that, although the non-autonomous dynamical systems described by chaotic impact oscillators are non-periodic in Euclidean phase spaces, they are periodic in Hausdorff phase spaces. This shows that Euclidean spaces in which we stayed for hundreds of years may no longer be suitable for the investigation into chaotic phenomena. In addition, the periodicity of chaotic dynamical systems in Hausdorff metric spaces induces a new class of strange invariant sets in Euclidean spaces. Such strange invariant sets may be an ideal symbol of chaotic dynamical systems.

1. INTRODUCTION

Many engineering systems experience intermittent motion of contact and separation due to existing clearances or gaps. When the difference between the stiffness of the two spring is very large, the system is called an impact oscillator. Analysis of impact oscillators has attracted considerable interest in the past and continues to do so [1–14]. Many researches show that a simple impact oscillator may exhibit chaotic motion. Holmes [14] considered an impact system (a mass bouncing on a vibration table) and found not only harmonic and subharmonic motion, but also chaotic ones. In digital simulations Thompson and Ghaffari [13] observed the phenomenon of period-doubling route to chaos of the impact oscillator in the marine structural dynamics. Shaw and Holmes [11] studied the stability, bifurcations and chaos of the system by examining the eigenvalues of the Jacobian matrix of the Poincaré map. Kim and Noah [6] developed a modified harmonic balance/Fourier transform procedure to analyze the stability, bifurcations and chaos of the impact system. Shaw [9] investigated the chaotic motions and global bifurcations of a harmonically excited system having two-sided amplitude constraints. Bishop *et al.* [2] studied the chaotic behavior of a beam impact system. Foale and Bishop [5] considered the non-standard bifurcations of an impact oscillator. Han and Luo [4] investigated the chaotic motion of a horizontal impact pair. Pun *et al.* [3] reported the chaotic windows between the regions of periodic responses of a multi-degree impact oscillator.

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One of the most important properties of chaotic systems is that the responses of the deterministic systems to periodic excitation are non-periodic. This is in conflict with the traditional belief that there should exist periodic elements in the responses of a deterministic dynamical system to periodic external excitation. Therefore, at the beginning of the researches of chaos, chaotic motion was often thought to be a transitional state. Thanks to the great efforts made by many admirable researchers it is generally acknowledged that chaotic motion is a new phenomenon. Nowadays, chaos is regarded as one of the most puzzling phenomena in this century [15–37]. However, it should be pointed out that so far chaotic systems have been investigated in Euclidean spaces. It is well known that an object may appear to have different features from different points of view. We think that all the unusual properties of chaotic systems imply nothing, but that Euclidean metric spaces may no longer fit the study of chaotic phenomena. In this paper, the behavior of chaotic impact oscillators is investigated in Hausdorff metric spaces. It is discovered that chaotic dynamical systems may be periodic in Hausdorff metric spaces.

Long-term analysis of chaotic time series exhibits a few geometrical and dynamical invariants, such as Lyapunov exponents, fractal dimensions, phase plane invariants and so on. Many efforts have been made to extract these invariants [38–46]. Here, we discover that a class of strange invariant sets will be induced by the periodic behavior of chaotic systems in Hausdorff metric spaces. The new invariant sets may be a potential, reliable diagnostic tool for chaos of dynamical systems.

2. NON-AUTONOMOUS DYNAMICAL SYSTEMS AND HAUSDORFF PHASE SPACES

To investigate the periodicity of chaotic impact oscillators, it is necessary to give a basic frame of non-autonomous dynamical systems. Let $(\mathbf{M}, \mu)$ be a metric space, $\phi(\tau, t_0, \cdot)$, $\tau, t_0, \in \mathbf{R}$, denotes a double-parameter family of maps of the metric space $(\mathbf{M}, \mu)$ onto itself.

Definition 1. $\phi(\tau, t_0, \cdot)$ is called a non-autonomous dynamical system (or a non-autonomous flow) in the metric space $(\mathbf{M}, \mu)$, if it satisfies that

$$\begin{aligned} &\phi(0, t_0, p) = p, && p \in \mathbf{M},\ t_0 \in \mathbf{R}, \\ &\phi(s + r, t_0, p) = \phi[s, t_0 + r, \phi(r, t_0, p)], && p \in \mathbf{M}, s, r, t_0 \in \mathbf{R}. \end{aligned} \tag{1}$$

$(\mathbf{M}, \mu)$ is called a phase space of $\phi(\tau, t_0, \cdot)$.

When ϕ is independent of t_0, equation (1) reduces to

$$\begin{aligned} &\phi(0, p) = p, && p \in \mathbf{M}, \\ &\phi(s + r, p) = \phi[s, \phi(r, p)], && p \in \mathbf{M}, s, r, \in \mathbf{R}. \end{aligned} \tag{2}$$

$\phi(\tau, \cdot)$ is the dynamical systems (or flows) studied by the modern theory of dynamical systems [31, 47]. In the following discussions $\phi(\tau, \cdot)$ is called *an autonomous dynamical system* (or *an autonomous flow*) in the metric space $(\mathbf{M}, \mu)$. Both of $\phi(\tau, t_0, \cdot)$ and $\phi(\tau, \cdot)$ are called *dynamical systems* (or *flows*).

From physical point of view, a phase space is a logic one in which the object is observed in a specific manner. For instance, since dynamics was framed by Issac Newton, oscillators have been investigated in displacement–velocity spaces with the Euclidean metric. However, we have to realize that an object can be observed from different points of view, i.e., the same object can be investigated in different metric spaces. A flow in $(\mathbf{M}_1, \mu_1)$ may also be one in $(\mathbf{M}_2, \mu_2)$. A flow in order in $(\mathbf{M}_1, \mu_1)$ may be chaotic in $(\mathbf{M}_2, \mu_2)$. From philosophical point of view, one of the ultimate tasks of science is to find $(\mathbf{M}_1, \mu_1)$, but not $(\mathbf{M}_2, \mu_2)$, i.e., what we try to find is the order in chaos, but not chaos itself. Therefore, the existence of chaotic flows shows that Euclidean metric spaces may no longer be suitable for the observation of the behavior of chaotic oscillators. The order of chaotic flows may be hidden in a special phase space.

$(\mathbf{R}^n, d)$ denotes a n-dimensional Euclidean metric space, where d is the Euclidean metric. Let $\mathbf{H}^n$ be the collection of all non-empty closed subsets of $\mathbf{R}^n$. From the point of view of the observers in $\mathbf{R}^n$, a point of $\mathbf{H}^n$ may be a set containing numerous points of $\mathbf{R}^n$. The distance between $p(\in \mathbf{R}^n)$ and $A(\in \mathbf{H}^n)$ is defined as

$$\varrho(p, A) = \inf\{d(p, r), r \in A\}. \tag{3}$$

The Hausdorff distance [48] between two points A and B of $\mathbf{H}^n$ is defined as

$$\rho(A, B) = \sup\{\sup[\varrho(p, A), p \in B], \sup[\varrho(q, B), q \in A]\}. \tag{4}$$

$(\mathbf{H}^n, \rho)$ is a complete metric space [48]. This metric space was often used by F. Hausdorff (1868–1942). Therefore, we call it a Hausdorff metric space. To investigate the behavior of chaotic dynamical systems in Hausdorff metric spaces, we have to prove that current dynamical systems in Euclidean spaces are also ones in Hausdorff metric spaces.

Theorem 1. *If ϕ is a flow in the Euclidean metric space $(\mathbf{R}^n, d)$, then, it is also a flow in the corresponding Hausdorff metric space $(\mathbf{H}^n, \rho)$.*

Proof. The definition of $\mathbf{H}^n$ shows that, if $A \in \mathbf{H}^n$, then $A \subset \mathbf{R}^n$. Therefore, equation (1) leads to

$$\begin{aligned}\phi(0, t_0, A) &= \{\phi(0, t_0, p): p \in A\} \\ &= \{p: p \in A\} = A, \quad t_0 \in \mathbf{R}, A \in \mathbf{H}^n,\end{aligned} \tag{5}$$

$$\begin{aligned}\phi(r + s, t_0, A) &= \{\phi(r + s, t_0, p): p \in A\} \\ &= \{\phi[s, t_0 + r, \phi(r, t_0, p)]: p \in A\} \\ &= \phi[s, t_0 + r, \phi(r, t_0, A)], \quad s, r, t_0 \in \mathbf{R}, A \in \mathbf{H}^n.\end{aligned} \tag{6}$$

Equations (1), (5) and (6) show that non-autonomous flows in Euclidean metric spaces are also ones in Hausdorff metric spaces. This conclusion is also valid for an autonomous flow. □

Theorem 1 implies that one can observe the behavior of ϕ in $(\mathbf{R}^n, d)$, also in $(\mathbf{H}^n, \rho)$. $(\mathbf{R}^n, d)$ and $(\mathbf{H}^n, \rho)$ are, respectively, called *Euclidean and Hausdorff phase spaces* (*EPS* and *HPS*) of the flow ϕ.

3. NON-AUTONOMOUS DYNAMICAL SYSTEMS DESCRIBED BY IMPACT OSCILLATORS

Most of the dynamical problems in various fields, such as physics, mechanics, civil engineering and so on, are described by following non-autonomous differential equations in $(\mathbf{R}^n, d)$,

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{F}(\mathbf{X}, t), \tag{7}$$

where $\mathbf{X} = [x_1, x_2, \dots, x_n]^{\mathrm{T}} \in \mathbf{R}^n$ is the state vector, $\mathbf{F} = [f_1, f_2, \dots, f_n]^{\mathrm{T}}$.

Theorem 2. *If* p *and* $\psi(\tau, t_0, p)$ *are, respectively, the states of system* (7) *at time* t_0 *and* $t_0 + \tau$, *then* ψ *satisfies*

$$\psi(0, t_0, p) = p, \quad p \in \mathbf{R}^n, t_0 \in \mathbf{R}, \tag{8}$$

$$\psi(s + r, t_0, p) = \psi[s, t_0 + r, \psi(r, t_0, p)], \quad p \in \mathbf{R}^n, s, r, t_0 \in \mathbf{R}. \tag{9}$$

Proof. Equation (8) follows immediately from the definition of $\psi(\tau, t_0, p)$. Let $t = t_0 + \tau$. Equation (7) can be rewritten as

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\tau} = \mathbf{F}(\mathbf{X}, t_0 + \tau), \quad t_0, \tau \in \mathbf{R}, \mathbf{X} \in \mathbf{R}^n. \tag{10}$$

Therefore,

$$\begin{aligned} \mathbf{X}[\psi(r + s, t_0, p)] &= \mathbf{X}(p) + \int_0^{r+s} \mathbf{F}(\mathbf{X}, t_0 + \tau)\,\mathrm{d}\tau \\ &= \mathbf{X}(p) + \int_0^{r} \mathbf{F}(\mathbf{X}, t_0 + \tau)\,\mathrm{d}\tau + \int_r^{r+s} \mathbf{F}(\mathbf{X}, t_0 + \tau)\,\mathrm{d}\tau \\ &= \mathbf{X}[\psi(r, t_0, p)] + \int_0^{s} \mathbf{F}[\mathbf{X}, (t_0 + r) + \bar{\tau}]\,\mathrm{d}\bar{\tau}, \\ &= \mathbf{X}[\psi(s, t_0 + r, \psi(r, t_0, p))], \end{aligned} \tag{11}$$

where $\tau = \bar{\tau} + r$, $\mathbf{X}(\cdot)$ denotes the vector of the co-ordinates of a point. Equation (11) leads to equation (9).

Theorem 2 shows that a non-autonomous differential equation in $(\mathbf{R}^n, d)$ describe a non-autonomous flow in $(\mathbf{R}^n, d)$. It should be clear that property (2) does not hold for system (7), unless it is autonomous. Therefore, equation (7) does not describe an autonomous flow in $(\mathbf{R}^n, d)$. However, one can always make a non-autonomous equation autonomous by redefining time as a new dependent variable. This is done as follows. Equation (7) can be rewritten as

$$\frac{\mathrm{d}\bar{\mathbf{X}}}{\mathrm{d}t} = \bar{\mathbf{F}}(\bar{\mathbf{X}}), \tag{12}$$

where $\bar{\mathbf{X}} = [x_1, x_2, \dots, x_n, t]^{\mathrm{T}} \in \mathbf{R}^{n+1}$ and $\bar{\mathbf{F}} = [f_1, f_2, \dots, f_n, 1]^{\mathrm{T}}$. This implies that a non-autonomous differential equations in $(\mathbf{R}^n, d)$ describes an autonomous flow in $(\mathbf{R}^{n+1}, d)$. From mathematical point of view, this transformation may be clever. However, from the point of view of applications, it may cover some important properties of original systems.

Consider a typical impact oscillator shown in Figure 1. A mass m is attached to a spring of stiffness k_1 and a linear dashpot with damping factor c. When the dynamical additional

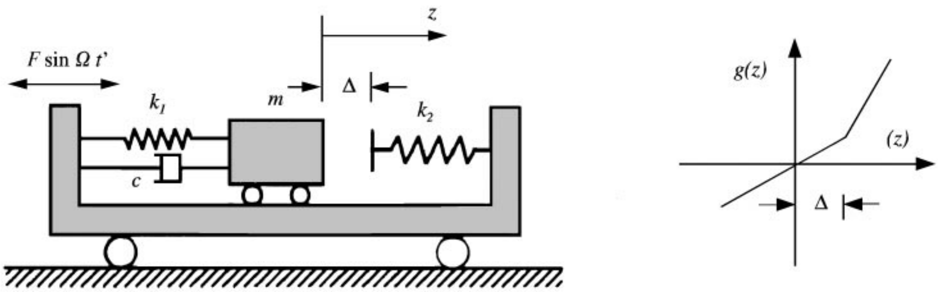

Figure 1. An impact oscillator (a) physical model, (b) piecewise-linear restoring force.

displacement, z, exceeds a certain value, Δ, a second linear spring, k_2, contacts m. When the system is externally excited by a harmonic base movement, the non-dimensional equation of motion may be written as

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\zeta\frac{\mathrm{d}x}{\mathrm{d}t} + g^*(x) = \theta^2\sin(\theta t), \tag{13}$$

where

$$g^*(x) = \begin{cases} x, & x \leqslant \delta, \\ \eta^2 x + (1-\eta^2)\delta, & x > \delta, \end{cases} \tag{14}$$

in which $x = z/F$ is a non-dimensional displacement, $\omega = \sqrt{k_1/m}$ and $\omega_+ = \sqrt{(k_1+k_2)/m}$ are the frequencies, $\zeta = c/(2m\omega)$ is the damping ratio, η^2 is the stiffness ratio $(k_1+k_2)/k_1 = \omega_+^2/\omega^2$, θ has the value of Ω/ω, $\delta = \Delta/F$ and $\omega t' = t$. Let $y = \mathrm{d}x/\mathrm{d}t$. Equation (13) can be written as

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{F}(\mathbf{X}, t), \tag{15}$$

where $\mathbf{X} = [x, y]^{\mathrm{T}} \in \mathbf{R}^2$, $\mathbf{F} = [f_1, f_2]^{\mathrm{T}}$ and

$$f_1 = y, \qquad f_2 = -2\zeta y - g^*(x) + \theta^2\sin(\theta t). \tag{16}$$

Equation (15) and Theorem 2 show that the impact oscillator shown in Figure 1 describes non-autonomous flow in $(\mathbf{R}^2, d)$. We call it *an impact flow*. The analysis of the Poincaré maps, phase trajectories, Lyapunov exponents and so on of equation (13) shows that the oscillator is chaotic when (a) $\theta = 0{\cdot}7$, $\zeta = 0{\cdot}05$, $\eta = 10$ and $\delta = 0{\cdot}05$; (b) $\theta = 2{\cdot}5$, $\zeta = 0{\cdot}01$, $\eta = 80$ and $\delta = 1{\cdot}0$. Let $\psi_a(\tau, t_0, \cdot)$ and $\psi_b(\tau, t_0, \cdot)$ be, respectively the flows determined by (a) and (b).

4. THE PERIODICITY OF THE CHAOTIC IMPACT DYNAMICAL SYSTEMS IN HPS

A flow ϕ is said to be asymptotically periodic (or simply called to be periodic) with T at p in the phase space $(\mathbf{M}, \mu)$, $p \in \mathrm{M}$, if, for any given $\varepsilon > 0$, there exists N satisfying

$$\mu[\phi(\tau + n_1 T, t_0, p), \phi(\tau + n_2 T, t_0, p)] < \varepsilon, \quad n_1, n_2 > N, \tau, t_0 \in \mathbf{R}, \tag{17}$$

where T is a constant.

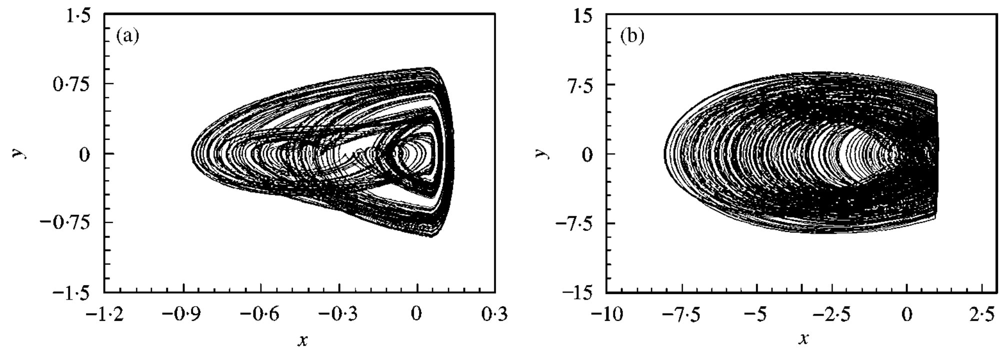

Figure 2. The trajectories of the impact oscillators in the EPS (a) ψ_a, (b) ψ_b.

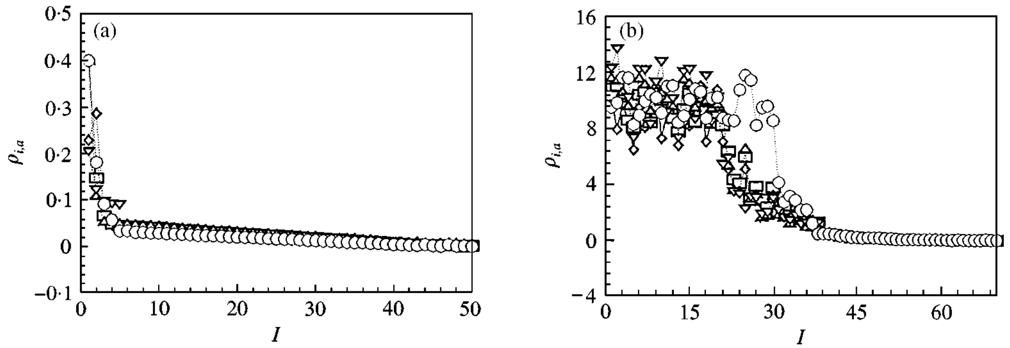

Figure 3. The investigation of the periodicity of the chaotic impact flow in Hausdorff phase spaces. (a) $\rho_{i,a}$: -◇-, $\tau = 0{\cdot}4T_e$, □, $\tau = 0{\cdot}8T_e$, -△-, $\tau = 1{\cdot}2T_e$, -▽-, $\tau = 1{\cdot}6T_e$, -○-, $\tau = 0$. (*b*) $\rho_{i,b}$: -◇-, $\tau = 0{\cdot}2T_e$, □, $\tau = 0{\cdot}4T_e$, -△-, $\tau = 0{\cdot}6T_e$, -▽-, $\tau = 0{\cdot}8T_e$, -○-, $\tau = 0$.

The fact that ψ_a and ψ_b are chaotic means that they are non-periodic in EPS. Figure 2 shows their trajectories in the EPS $(\mathbf{R}^2, d)$. Theorem 1 shows that the impact oscillator also describes a non-autonomous flow in HPS. To investigate the behavior of the impact flow in HPS, we consider

$$\rho_{i,a} = \sup\{\rho[\psi_a(i2T_e + \tau, 0, A), \psi_a((i+j)2T_e + \tau, 0, A)], \quad j = 1, 2, \dots\}, \tag{18}$$

$$\rho_{i,b} = \sup\{\rho[\psi_b(iT_e + \tau, 0, A), \psi_b((i+j)T_e + \tau, 0, A)], \quad j = 1, 2, \dots\}, \tag{19}$$

where $T_e = 2\pi/\theta$ is the period of the external excitation and

$$A = \{(x, y){:}\ -0{\cdot}01 \leqslant x \leqslant 0,\ -0{\cdot}01 \leqslant y \leqslant 0\}, \quad A \in \mathbf{H}^2. \tag{20}$$

Figure 3(a) shows $\rho_{i,a}$, $i = 0, 1, 2, \dots, \tau = 0, 0{\cdot}4T_e, 0{\cdot}8T_e, 1{\cdot}2T_e, 1{\cdot}6T_e$. Figure 3(b) shows $\rho_{i,b}$, $i = 0, 1, 2, \dots, \tau = 0, 0{\cdot}2T_e, 0{\cdot}4T_e, 0{\cdot}6T_e, 0{\cdot}8T_e$. Although the periodicity in HPS cannot be completely expressed in EPS, the description in EPS is more readable. It is necessary to introduce the Euclidean description of some of the characteristics of the periodicity of chaotic dynamical systems in HPS. Let $x_{c,a}$ and $y_{c,a}$ denote the co-ordinates of the centroid of $\psi_a(\tau, 0, A)$ respectively, $x_{c,b}$ and $y_{c,b}$ are, respectively, the co-ordinates of the centroid of $\psi_b(\tau, 0, A)$. Figures 4 and 5 show the time history of $x_{c,a}$, $y_{c,a}$, $x_{c,b}$ and $y_{c,b}$. The quasi-radius

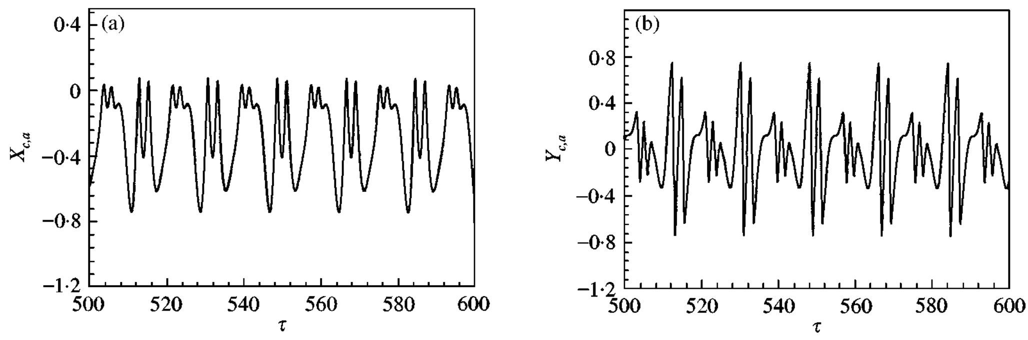

Figure 4. The time history of the co-ordinates of the centroid of $\psi_a(\tau, 0, A)$, (a) $x_{c,a}$, (b) $y_{c,a}$.

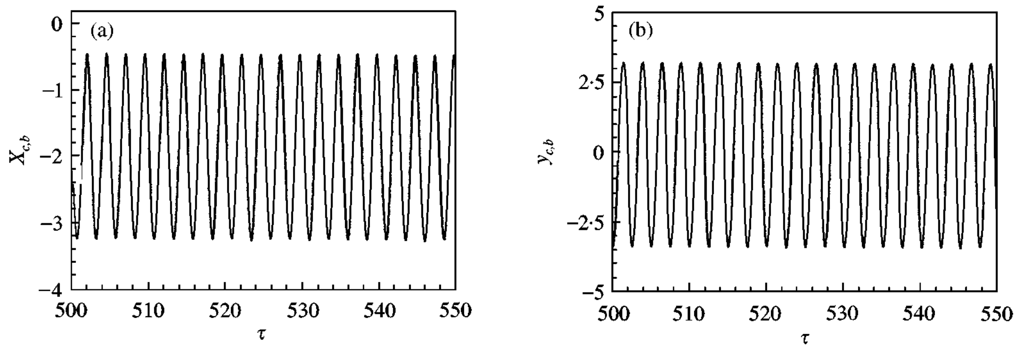

Figure 5. The time history of the co-ordinates of the centroid of $\psi_b(\tau, 0, A)$, (a) $x_{c,b}$, (b) $y_{c,b}$.

r_a of the set $\psi_a(\tau, 0, A)$ can be defined as

$$r_a = \sup\{d(p, c), p \in \psi(\tau, 0, A)\}, \tag{21}$$

where c is the centroid of $\psi(\tau, 0, A)$. Figure 6 shows the time history of r_a and r_b. Figures 3–6 show that ψ_a and ψ_b are periodic in $(\mathbf{H}^2, \rho)$. The periods of ψ_a and ψ_b are, respectively, $2T_e$ and T_e.

In our experiments it is discovered that the periodically excited, chaotic dynamical systems discovered currently are periodic in HPS.

5. THE STRANGE INVARIANT SETS AND POINCARÉ MAPS

If a flow ϕ is periodic with T at p in the metric space $(\mathbf{M}, \mu)$, $\{\phi(\tau + iT, t_0, p), i = 1, 2, \dots\}$ is a Cauchy sequence in $(\mathbf{M}, \mu)$ for any given $\tau \in \mathbf{R}$. Therefore, both of $\{\psi_a(\tau + i2T_e, 0, A),$ $i = 1, 2, \dots\}$ and $\{\psi_b(\tau + iT_e, 0, A),\ i = 1, 2, \dots\}$ are Cauchy sequences in the Hausdorff metric space $(\mathbf{H}^2, \rho)$. Because a Hausdorff metric is complete, for any $\varepsilon > 0$ there exists $\mathbf{S}_a(\tau) \in \mathbf{H}^2$ and N satisfying

$$\rho[\mathbf{S}_a(\tau), \psi_a(\tau + i2T_e, 0, A)] < \varepsilon, \quad i \geqslant N, \quad \tau \in \mathbf{R}, A \in \mathbf{H}^2. \tag{22}$$

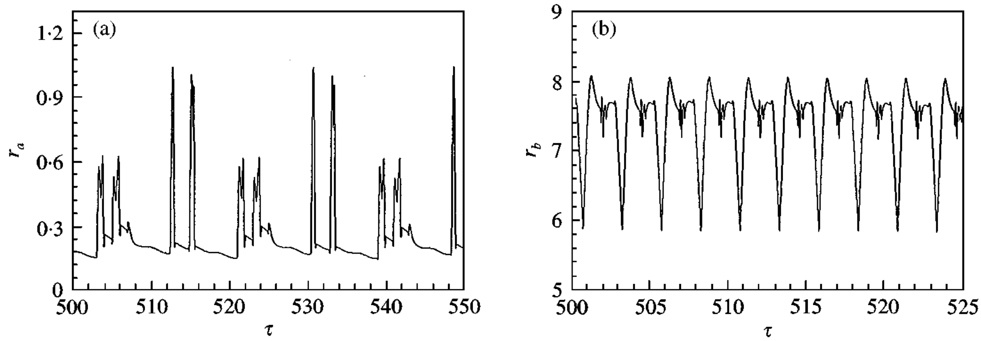

Figure 6. The time history of the quasi-radius of $\psi_a(\tau, 0, A)$ and $\psi_b(\tau, 0, A)$, (a) r_a, (b) r_b.

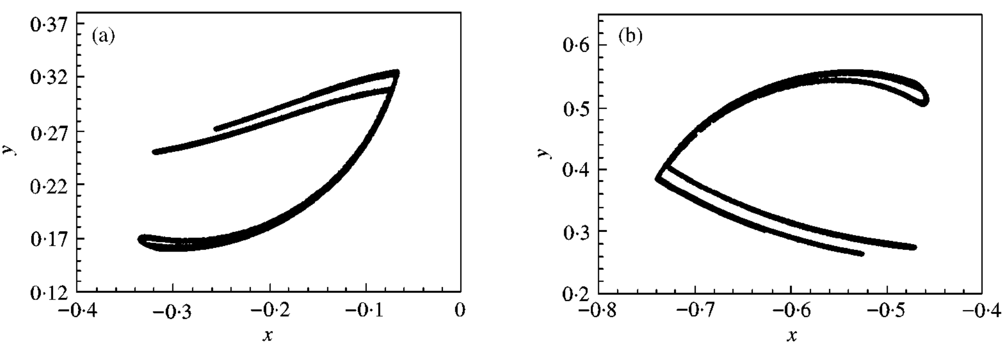

Figure 7. The strange invariant sets: (a) $\mathbf{S}_a(0)$, (b) $\mathbf{S}_a(T_e)$.

Similarly, we have

$$\mathbf{S}_b(\tau) = \lim_{i \to \infty} \psi_b(\tau + iT_e, 0, A), \quad \tau \in \mathbf{R}, \quad A, \mathbf{S}_b\ (\tau) \in \mathbf{H}^2. \tag{23}$$

Because $\mathbf{S}(\tau) \in \mathbf{H}^2$, $\mathbf{S}(\tau) \subset \mathbf{R}^2$. $\mathbf{S}_a(0)$, $\mathbf{S}_a(T_e)$, $\mathbf{S}_b(0)$ and $\mathbf{S}_b(T_e/2)$ are shown in Figures 7 and 8. All this shows that the periodic behavior of a chaotic flow in Hausdorff metric spaces will induce a invariant set in Euclidean metric spaces. We call it *a strange invariant set*. It is evident that the concept of strange invariant set is different from that of Poincaré maps. However, it is very interesting that the numerical experiments show that there may be some strange relationship between them.

A Poincaré map $\mathbf{P}(\tau)$ of a periodically excited, non-autonomous flow ϕ in $(\mathbf{R}^n, d)$ is defined as

$$\mathbf{P}(\tau) = \bigcap_{m=0}^{\infty} \{\phi(\tau + jT_e, t_0, p), p \in \mathbf{R}^n, j \geqslant m\}, \quad \tau \in \mathbf{R}, \tag{24}$$

where T_e is the period of external excitation.

Theorem 3. *The Poincaré map* $\mathbf{P}(\tau)$ *is periodic with the external excitation period* T_e, *i.e.*, $\mathbf{P}(\tau + iT_e) = \mathbf{P}(\tau)$.

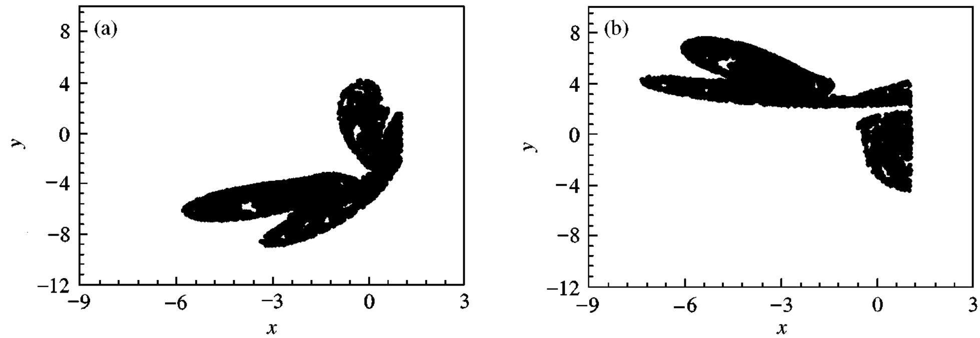

Figure 8. The strange invariant sets: (a) $\mathbf{S}_b(0)$, (b) $\mathbf{S}_b(T_e/2)$.

Proof: Equation (24) leads to

$$\mathbf{P}(\tau + iT_e) = \bigcap_{m=0}^{\infty} [\phi[\tau + (i+j)T_e, t_0, p], p \in \mathbf{R}^n, j \geqslant m\}$$

$$= \bigcap_{m=0}^{\infty} \{\phi[\tau + kT_e, t_0, p], p \in \mathbf{R}^n, k \geqslant m + i\}$$

$$= \bigcap_{l=i}^{\infty} \{\phi[\tau + kT_e, t_0, p], p \in \mathbf{R}^n, k \geqslant l\}. \tag{25}$$

Because $\{\phi[\tau + kT_e, t_0, p], p \in \mathbf{R}^n, k \geqslant l\}$, $l = 0, 1, 2, \ldots$, is a decreasing sequence of sets,

$$\bigcap_{l=i}^{\infty} \{\phi[\tau + kT_e, t_0, p], p \in \mathbf{R}^n, k \geqslant l\} = \bigcap_{l=0}^{\infty} \{\phi[\tau + kT_e, t_0, p], p \in \mathbf{R}^n, k \geqslant l\}. \tag{26}$$

Equations (24)–(26) give

$$\mathbf{P}(\tau + iT_e) = \mathbf{P}(\tau). \tag{27}$$

Therefore, this theorem is true. □

Theorem 3 shows that the Poincaré map of a flow excited periodically is periodic with the external excitation period. This conclusion can be easily verified by numerical experiments.

Unlike the periodicity of Poincaré maps, the periodicity of chaotic flows in HPS cannot be proved mathematically. From the theoretical point of view, there is no relationship between the periodicity to a chaotic flow in HPS and that of its Poincaré maps. Therefore, it is not surprising that ψ_a is periodic with $2T_e$, while its Poincaré maps are periodic with T_e, and that the strange invariant set **S** is different from the Poincaré map (see Figures 7 and 9). However, the results of experiments show that sometimes there is very close relationship between them. The Poincaré maps $\mathbf{P}_b(0{\cdot}157)$ and $\mathbf{P}_b(0{\cdot}157 + T_e/2)$ are shown in Figure 10. The results shown in Figures 8 and 10 are surprising. In the experiments on other chaotic attractors we also discover such phenomenon. The striking similarity is indeed a mystery.

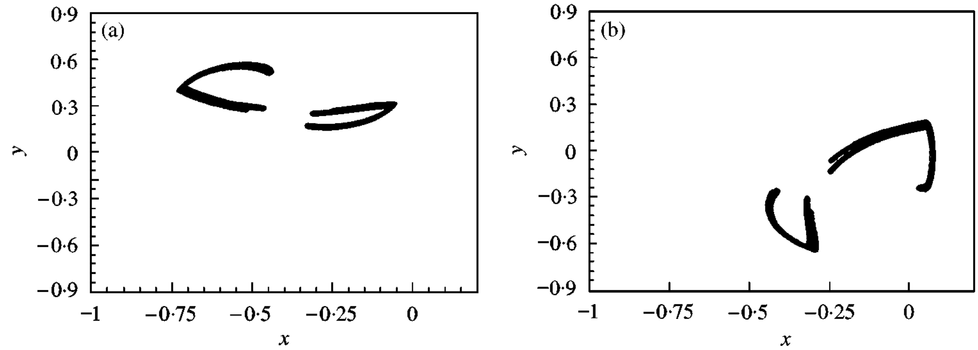

Figure 9. Poincaré maps: (a) $\mathbf{P}_a(0)$, (b) $\mathbf{P}_a(T_e/2)$.

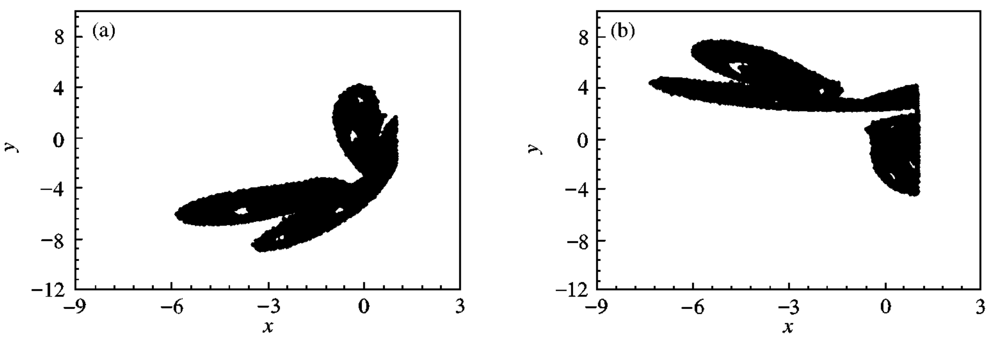

Figure 10. Poincaré maps: (a) $\mathbf{P}_b(0{\cdot}157)$, (b) $\mathbf{P}_a(0{\cdot}157 + T_e/2)$.

6. CONCLUSIONS AND DISCUSSIONS

The concept of non-autonomous flows and that of Hausdorff phase spaces was proposed. An impact oscillator in a n-dimensional Euclidean space describes not only a non-autonomous flow in a n-dimensional Euclidean metric space, but also that in a Hausdorff metric spaces. Although chaotic impact oscillators are non-periodic in Euclidean phase spaces, they may be periodic in Hausdorff phase spaces. It was rigorously proved that the Poincaré map of a chaotic system is periodic with the period of the external excitation. However, the periodicity of chaotic systems in Hausdorff phase spaces cannot be proved in theory. The results of numerical experiments showed that the period of a chaotic system in a Hausdorff phase space may not be that of the external excitation. The periodicity of a chaotic system in a Hausdorff metric space will induce strange invariant sets in the Euclidean space. Poincaré maps were regarded as the symbol of the chaotic dynamical systems until strange non-chaotic dynamical systems were discovered by Romeiras and Ott [49], El Naschie and Kapitaniak [50]. Yet we have not found the non-chaotic system generating the strange invariant sets. They may be a good symbol of the chaos.

Although from theoretical point of view, there is no relation between the Poincaré maps and the strange invariant sets, the numerical experiments showed that sometimes there is striking similarity between them. Yet such mystery similarity has not been explained in theory.

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