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FREE VIBRATION OF ELASTICALLY RESTRAINED FLEXURAL-SHEAR PLATES WITH VARYING CROSS-SECTION

Q. S. Li

Department of Building and Construction, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong

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This paper presents an analytical approach to determining natural frequencies and mode shapes of non-uniform flexural-shear plates with line translational spring and rotational spring supports and line masses under action of axial forces. The governing differential equation for vibration of a non-uniform flexural-shear plate under axial forces is established first. It is shown that it is possible to separate a flexural-shear plate as two beams for free vibration analysis, one is a flexural beam, and the other is a shear beam. The natural frequency of the plate is equal to the square root of the square sum of the two natural frequencies of the two beams, and the mode shape of the plate is the product of the corresponding two mode shapes of the two beams. In this paper, power functions and exponential functions are adopted for describing the distributions of mass and stiffness along the height of the plate as well as the axial forces acting on the plate. The exact solutions for free vibrations of non-uniform flexural-shear plates for several cases that are important in engineering practices are derived. A numerical example shows that the calculated results are in good agreement with the experimental data and it is convenient to apply the proposed method to free vibration analysis of elastically restrained flexural-shear plates with varying cross-section.

1. INTRODUCTION

Experimental results on structural dynamic behavior obtained by Wang [1], Li [2], He *et al.* [3], Li *et al.* [4–6], Jeary [7] and others have shown that the flexural deformation is usually dominant in the total deformation of tall buildings with shear-wall structures in their horizontal vibrations. Li *et al.* [4] suggested that for certain cases these shear-wall buildings can be simplified as cantilever flexural beams or elastically restrained flexural beams for free vibration analysis. An approach to determining free vibration of flexural beams with variably distributed mass and stiffness was proposed by Li *et al.* [5, 6]. However, if a shear-wall building has a narrow rectangular plane configuration (narrow building), e.g., $B/L < \frac{1}{4}$, where B and L are width and length of the rectangular plane respectively, the stiffness of each floor of the building may not be treated as infinitely rigid [8]. Hence, such a narrow building may not be simplified as a cantilever flexural beam for free vibration analysis. It was reported by Li *et al.* [8] that the whole deformation characteristics of a narrow building with shear walls are similar to those of a flexural-shear plate, i.e., the shear deformation in the longitudinal direction (the x-axis in Figure 1) is dominant, and the flexural deformation in the y direction is dominant. This is due to the fact that the flexural deformation of shear walls is dominant in the lateral deformation of such a narrow building. On the other hand, the main connections of shear-wall structures in the longitudinal direction (the x direction in Figure 1) are floors, and shear deformation of each floor, in-plane of that floor, is dominant. It is necessary to point out that the displacement caused by shear deformation and flexural deformation are all in the z direction, i.e., the displacement is a function of x, y and t. This analytical model of a flexural-shear plate is adopted in this paper for free vibration analysis of narrow buildings with shear-wall structures. In general, a tall building with shear-wall structures has variably distributed mass and stiffness along its height; thus, such a building is treated as a flexural-shear plate with variably distributed mass and stiffness for vibration analysis.

Exact solutions for free vibration of flexural plates or shear plates with variably distributed mass and stiffness have been obtained only for certain plate shapes and boundary conditions. For example, Chopra [9] developed an analytical approach for the free vibration of a simply supported flexural plate with one change in thickness. Guo *et al.* [10] recently found the analytical solutions for the free vibration of a stepped, simply supported flexural plate with uniform thickness and abrupt thickness changes. Wang [1] derived the closed-form solutions for the free vibration of cantilever shear plates with uniformly distributed mass and stiffness. However, it is obvious that the distributions of mass and stiffness of most narrow buildings are actually not uniform, especially, along the building height. The concept of shear orthotropic plates was developed and used by Beiner and Librescu [11]. They have presented an analysis of weight minimization for rectangular flat panels with fixed flutter speed. To simplify the problem, a structural model that considers transverse shear deformation only and neglects the bending stiffness of the plate was adopted in their study. This has the effect of reducing the linear partial differential equation for this problem from the fourth to the second order. Li *et al.* [8] found the closed-form solutions for free vibration of non-uniform shear plates. It should be pointed out that vibration analysis of flexural plates or shear plates and effect of shear deformation on flexural plate vibration have been extensively studied in the past. The concept and analytical model of the flexural-shear plates which are different from those of flexural plates or shear plates were recently proposed by Li *et al.* [12]. An exact approach for determining natural frequencies and mode shapes of flexural-shear plates with uniformly distributed mass and stiffness was presented by them.

Apart from the several analytical methods for analyzing limited classes of plates, many approximate and numerical methods have been developed. These include the Ritz method,

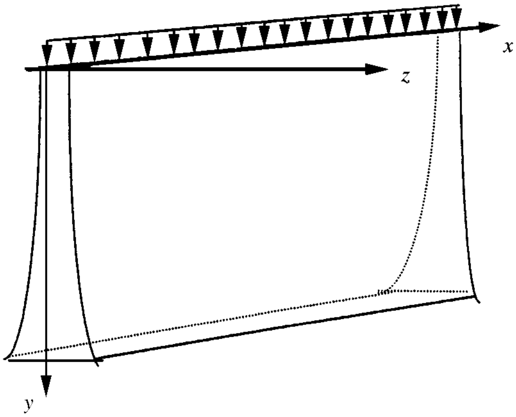

Figure 1. A flexural-shear plate.

the finite strip method (FSM) and the finite element method (FEM). In general, the Ritz method, can provide accurate solutions; however, it depends on the choice of global admissible functions. Liew and his co-workers [13–18] have developed efficient Ritz algorithms for the free vibration of various non-uniform plates having arbitrary boundary conditions. The first known study on the problem of free vibration of symmetric cross-ply laminated plates with elastically restrained edges was conducted by Liew *et al.* [18] using the Rayleigh–Ritz method. Recently, Cheung and Zhou [19] studied the free vibrations of tapered rectangular plates with an arbitrary number of intermediate line supports in one or two directions using the Ritz method. The FSM presented by Cheung [20] has been developed and applied vibration analysis of various plates over the years. Cheung and Kong [21] investigated the free vibration of line-supported rectangular plates by applying the FSM method. Guo *et al.* [10] studied the free vibration analysis of a stepped flexural plate by applying FEM and FSE. Compared with FEM, the main advantage of FSE is its efficiency, in particular for plates with regular geometry.

In this paper, an attempt is made to present an exact approach to determining free vibrations of non-uniform flexural-shear plates with line translational spring and rotational spring supports and line masses under the action of axial forces. In order to derive closed-form solutions for the title problem, the functions for describing the distributions of mass, stiffness and axial forces are selected as suitable expressions, such as power functions and exponential functions. All exact solutions derived are expressed in terms of Bessel functions and trigonometric functions. It is shown through a numerical example that the selected expressions are suitable for describing the distributions of mass, stiffness and axial forces for typical multi-story narrow buildings with shear-wall structures. The numerical example also demonstrates that the calculated results are in good agreement with the experimental data and it is convenient to apply the proposed method to free vibration analysis of elastically restrained flexural-shear plates with varying cross-section.

A flexural-shear plate representing a narrow building with shear walls, in general, has free–free edges in the longitudinal direction and clamped–free or spring–free edges in the vertical direction. In order to extend practical applications of the methods proposed in this paper, free vibrations of flexural-shear plates with various boundary conditions, including classical and non-classical ones, are investigated.

The main purpose of this work is to present exact solutions and an efficient computational method for the free vibration analysis of elastically restrained flexural-shear plates with varying cross-section. In the absence of the exact solutions, this problem may be solved using approximated methods (e.g., the Ritz method) or numerical methods (e.g., the finite element method and the finite strip method). However, the present exact solutions could provide adequate insight into the physics of the problem and can be easily implemented. The availability of the exact solutions will help in examining the accuracy of the approximate or numerical solutions. Therefore, it is always desirable to obtain the exact solutions to such problems.

2. THEORY

As discussed above, a flexural-shear plate is a special orthotropic plate which deforms in one direction (y) by bending only, and in the other direction (x) by shear only.

In order to establish the governing differential equation for vibration of a non-uniform flexural shear plate under the action of line axial forces (Figure 1), an infinitesimal plate element is cut from the plate. The size of the element is $\mathrm{d}x \times \mathrm{d}y$. The dynamic loading acting on the element is $q(x, y, t)\mathrm{d}x\, \mathrm{d}y$. The inertial force is ($\bar{m}_{xy}\, \partial^2 W/\partial t^2\, \mathrm{d}x\, \mathrm{d}y$) and the damping force is $(-C_{xy}\,\partial W/\partial t)$, where $\bar{m}_{xy}$, $W(x, y, t)$ and C_{xy} are the mass intensity (mass per unit area), dynamic displacement in the z direction and viscous damping coefficient at point (x, y) respectively. The element shown in Figure 2 is rotated over an angle of 90°. According to d'Alembert principle, all forces acting on the element including the inertial force should satisfy the equilibrium conditions. From $\sum F_z = 0$, we have

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - C_{xy}\frac{\partial W}{\partial t} - \bar{m}_{xy}\frac{\partial^2 W}{\partial t^2} = -q(x, y, t), \tag{1}$$

where Q_x and Q_y are the transverse shear forces in the x and y directions respectively, given by

$$Q_y = -\frac{\partial}{\partial y}\left(K_y\frac{\partial^2 W}{\partial y^2}\right) - N_y\frac{\partial W}{\partial y}, \tag{2}$$

$$Q_x = K_x\frac{\partial W}{\partial x}, \tag{3}$$

where K_x and K_y are the stiffnesses in the x and y directions respectively, and N_y is the axial force in the y-axis.

By substituting equations (2) and (3) into equation (1), we obtain

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial W}{\partial x}\right) - \frac{\partial^2}{\partial y^2}\left(K_y\frac{\partial^2 W}{\partial y^2}\right) - \frac{\partial}{\partial y}\left(N_y\frac{\partial W}{\partial y}\right) - C_{xy}\frac{\partial W}{\partial t} - \bar{m}_{xy}\frac{\partial^2 W}{\partial t^2} = -q(x, y, t). \tag{4}$$

This is the governing differential equation for vibration of a flexural-shear plate considering the effect of axial force in the y direction. Setting $q(x, y, t) = 0$ one obtains the governing differential equation for free vibration of the flexural-shear plate as follows:

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial W}{\partial x}\right) - \frac{\partial^2}{\partial y^2}\left(K_y\frac{\partial^2 W}{\partial y^2}\right) - \frac{\partial}{\partial y}\left(N_y\frac{\partial W}{\partial y}\right) - C_{xy}\frac{\partial W}{\partial t} - \bar{m}_{xy}\frac{\partial^2 W}{\partial t^2} = 0 \tag{5}$$

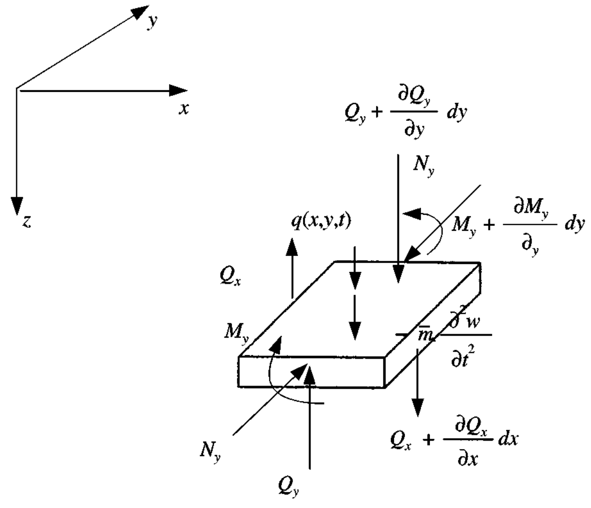

Figure 2. An element of the plate.

In order to solve equation (5) it is assumed that

$$W(x, y, t) = Z(x, y)\exp(\lambda t). \tag{6}$$

Substituting equation (6) into equation (5) leads to

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial Z}{\partial x}\right) - \frac{\partial^2}{\partial y^2}\left(K_y \frac{\partial^2 Z}{\partial y^2}\right) - \frac{\partial}{\partial y}\left(N_y \frac{\partial Z}{\partial y}\right) + \bar{m}_{xy}\omega^2 Z = 0. \tag{7}$$

We assume that

$$C_{xy} = 2C_0 \bar{m}_{xy} \tag{8}$$

and set

$$\omega^2 = -\lambda(2C_0 + \lambda). \tag{9}$$

Obviously, if we set $C_0 = 0$, then equation (7) becomes the governing equation for undamped free vibration of a flexural-shear plate, i.e., the governing differential equation of the damped mode function has the same form with the undamped mode function. This implies that the damped mode shape is the same as the corresponding undamped mode shape under the condition given in equation (8).

Solving equation (9) for λ gives

$$\lambda = -C_0 \pm \mathrm{i}\omega\sqrt{1-\xi^2}, \quad \xi = \frac{C_0}{\omega}, \tag{10}$$

where ξ is the critical damping ratio.

As is well known, the real part of λ is the damping coefficient, and the imaginary part is the damped circular natural frequency denoted as ω_d,

$$\omega_d = k_d\omega, \quad k_d = \sqrt{1-\xi^2}. \tag{11}$$

In general, ξ is in the range from 0·01 to 0·02, 0·02 to 0·04 and 0·03 to 0·06 for steel structures, reinforced concrete structures and brick masonry structures respectively. Even if $\xi = 0{\cdot}06$, $k_d = 0{\cdot}9982$, this means that the damped natural frequency is almost equal to the undamped one.

It is necessary to point out that the damped mode function is the same as the corresponding undamped mode function and the damped natural frequency is equal to the corresponding undamped natural frequency multiplied by the coefficient k_d. All the relationships presented above are obtained subjected to the condition that the viscous damping coefficient is proportional to the mass intensity.

In order to determine the undamped natural frequencies and mode shapes, the method of separation of variables is adopted herein:

$$Z(x, y) = X(x)Y(y). \tag{12}$$

It is assumed that K_y, K_z, $\bar{m}_{xy}$, N_y are functions of y as

$$K_y = K_1 f(y), \quad K_x = K_2\varphi(y), \quad \bar{m}_{xy} = \bar{m}\varphi(y), \quad N_y = N\psi(y), \tag{13}$$

i.e., it is assumed that K_x is directly proportional to $\bar{m}_{xy}$. Since the values of K_x and $\bar{m}_{xy}$ are mainly dependent on the dimensions and materials of building floors, this assumption is reasonable for many narrow buildings. Substituting equations (12) and (13) into equation (7) one obtains

$$\frac{K_2 \dfrac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2}}{X(x)} + \bar{m}\omega^2 = \frac{\dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\left[K_1 f(y) \dfrac{\mathrm{d}^2 Y(y)}{\mathrm{d}y^2}\right] + \dfrac{\mathrm{d}}{\mathrm{d}y}\left[N\psi(y)\dfrac{\mathrm{d}Y(y)}{\mathrm{d}y}\right]}{Y(y)\varphi(y)}. \tag{14}$$

Since the left-hand side of this equation is a function of x, and it is not related to y, the right-hand side is a function of y, and it is not related to x; thus, the entire equation is satisfied for arbitrary values of x and y only if both sides are equal to a constant. If it is assumed that the constant is $\bar{m}\theta^2$, then the following two independent ordinary differential equations are obtained from equation (4):

$$K_2 \frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} + \bar{m}\Omega^2 X(x) = 0, \tag{15}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left[K_1 f(y) \frac{\mathrm{d}^2 Y(y)}{\mathrm{d}y^2}\right] + \frac{\mathrm{d}}{\mathrm{d}y}\left[N\psi(y)\frac{\mathrm{d}Y(y)}{\mathrm{d}y}\right] - \bar{m}\varphi(y)\theta^2 Y(y) = 0, \tag{16}$$

where

$$\Omega^2 = \omega^2 - \theta^2, \quad \omega = \sqrt{\theta^2 + \Omega^2}. \tag{17}$$

It is obvious that equations (15) and (16) are two governing equations of vibration mode shapes of two beams. One is a shear beam in the x direction, K_2, $\bar{m}$, Ω are the shear stiffness, mass intensity and circular natural frequency of this shear beam respectively; the boundary conditions of the shear beam are the same as those of the flexural-shear plate in the x direction. The other one is a flexural beam, $K_1 f(y)$, $\bar{m}\varphi(y)$, $N\psi(y)$ and θ are the flexural stiffness, mass intensity, axial force and circular natural frequency of this flexural beam respectively, the boundary conditions of the flexural beam are the same as those of the flexural-shear plate in the y direction. The natural frequency of the plate is equal to the square sum of the two natural frequencies of the two beams. This suggests that free vibration analysis for a flexural-shear plate can be carried out by analyzing free vibration of two independent beams, one is a shear beam and the other is a flexural beam, with the same boundary conditions as those of the flexural-shear plate.

The general solution of equation (15) is easy to find as

$$X(x) = D_1 \sin\frac{\Omega}{\alpha_2}x + D_2 \cos\frac{\Omega}{\alpha_z}x \tag{18}$$

where

$$\alpha_2 = \sqrt{\frac{K_2}{\bar{m}}}. \tag{19}$$

The frequency equation and mode shape in the x direction can be determined by the use of equation (18) and the boundary conditions in the x direction of the flexural-shear plate are as follows:

1. *A flexural-shear plate with free-free (F–F) edges in the x direction.* In general, the boundary conditions in the longitudinal direction of a narrow building belong to this case, which can be written as

$$\frac{\mathrm{d}X(x)}{\mathrm{d}x} = 0 \quad \text{at } x = 0 \text{ and } x = L. \tag{20}$$

Using equations (20) and (18) one obtains

$$\Omega_k = \frac{\alpha_2 (k-1)\pi}{L}, \quad k = 1, 2, \ldots \tag{21}$$

$$X_k(x) = \sin \frac{k\pi x}{L}. \tag{22}$$

2. *A flexural-shear plate with clamped–clamped (C–C) edges or simply supported edges in the x direction.* The boundary conditions for this case are

$$X(x) = 0 \quad \text{at } x = 0 \text{ and } L. \tag{23}$$

Using equations (23) and (18) leads to

$$\Omega_k = \frac{\alpha_2 k\pi}{L}, \quad k = 1, 2, \ldots \tag{24a}$$

$$X_k(x) = \sin \frac{k\pi x}{L}. \tag{24b}$$

3. A *flexural-shear plate with clamped–spring (C–S) edges and a line mass is attached to the spring edge.* If the edge at $x = 0$ is clamped, then the boundary conditions, $X(0) = 0$, is substituted into equation (18), leading to

$D_2 = 0.$

Since the edge at $x = L$ is a spring-supported one with line mass, the boundary condition at this edge is

$$X'(L) = -(a_{uL} - b_{mL}\Omega^2) X(L), \tag{25}$$

where

$$X'(L) = \left.\frac{\mathrm{d}X(x)}{\mathrm{d}x}\right|_{x=L}, \quad a_{uL} = \frac{K_{uL}}{K_2}, \quad b_{mL} = \frac{m_L}{K_2}. \tag{26}$$

K_{uL} and m_L are the spring stiffness and line mass intensity (mass per unit length) attached to the plate at edge $x = L$, respectively.

Using equations (26) and (18) one obtains the frequency equation as

$$\tan\frac{\Omega}{\alpha_2}L = -\frac{\Omega}{\alpha_2(a_{uL} - b_{mL}\Omega^2)}. \quad (27)$$

The kth mode shape in the x direction can be written as

$$X_k(x) = \sin\frac{\Omega_k x}{\alpha_2} \quad (28)$$

in which Ω_k is the kth circular natural frequency of the shear beam.

4. *A flexural-shear plate with spring–spring (S–S) edges and line masses in the x direction.* If the opposite edges in the x direction of a flexural-shear plate are spring-supported ones with line masses, then the boundary conditions can be written as

$$X'(0) = (a_{u0} - b_{m0}\Omega^2)X(0), \quad a_{u0} = \frac{K_{u0}}{K_2}, \quad b_{m0} = \frac{m_0}{K_2}, \quad (29)$$

$$X'(L) = -(a_{uL} - b_{mL}\Omega^2)X(L), \quad (30)$$

where K_{u0} and m_0 are the spring stiffness and line mass intensity attached to the edge at $x = 0$. Using equations (29), (30) and (18) one obtains the frequency equation of the shear beam as follows:

$$\frac{\Omega}{\alpha_2}\tan\frac{\Omega L}{\alpha_2} + (a_{uL} - b_{mL}\Omega^2) + (a_{u0} - b_{m0}\Omega^2)\left[1 + \frac{\alpha_2(a_{uL} - b_{mL}\Omega^2)}{\Omega}\tan\frac{\Omega L}{\alpha_2}\right] = 0. \quad (31)$$

Solving this equation one obtains a set of Ω_k ($k = 1, 2, \ldots$). Substituting Ω_k into equation (21) one obtains the kth mode shape of the shear beam in the x direction.

It is necessary to point out that equations (21), (24) and (27) can be directly obtained from equation (31) by letting $a_{u0} = 0$, $a_{ul} = 0$ and $a_{u0} \to \infty$, $a_{ul} \to \infty$ as well as $a_{u0} \to \infty$ respectively.

The general solution of equation (16) is dependent on the expressions of K_y, N_y and $\bar{m}_{xy}$. Obviously, it is only possible to get the general solution of equation (16) for several special cases which will be investigated as follows.

Case 1: The functions for describing the distributions of the flexural stiffness, axial force and mass intensity are power functions

$$K_y = K_1(1 + \beta y)^{n+2}, \quad N_y = N(1 + \beta y)^{n+1}, \quad \bar{m}_{xy} = \bar{m}(1 + \beta y)^n, \quad (32)$$

Substituting equation (32) into equation (16) and assuming that

$$Y(y) = t^n J_n(t), \quad (33a)$$

$$t = \lambda\sqrt{1 + \beta y}, \quad (33b)$$

one obtains

$$K_1\left(\frac{\lambda\beta}{2}\right)^4 + N\left(\frac{\lambda\beta}{2}\right)^2 - \bar{m}\theta^2 = 0. \tag{34}$$

Solving equation (34) for λ gives

$$\lambda_1 = \frac{2}{\beta}\sqrt{Z_1}, \quad \lambda_2 = \frac{2}{\beta}\sqrt{Z_2}, \qquad \lambda_3 = -\lambda_1, \quad \lambda_4 = -\lambda_2, \tag{35}$$

where

$$Z_1 = N_e + \sqrt{N_e^2 + \theta_e^2}, \qquad Z_2 = N_e - \sqrt{N_e^2 + \theta_e^2}, \qquad N_e = \frac{N}{2K_1}, \quad \theta_e = \frac{\bar{m}\theta^2}{K_1}. \tag{36}$$

It can be seen from equation (36) that $Z_1 > 0$ and $Z_2 < 0$; thus, λ_1 and λ_3 are real roots, and λ_2 and λ_4 are pure imaginary roots.

The general solution of equation (16) for Case 1 can be written as

$$Y(y) = C_1 t_1^{-n} \mathrm{J}_n(t_1) + C_2 t_1^{-n} \mathrm{J}_{-n}(t_1) + C_3 t_2^{-n} \mathrm{I}_n(t_n) + C_4 t_2^{-n} \mathrm{I}_{-n}(t_2), \quad n = \text{a non-integer} \tag{37}$$

or

$$Y(y) = C_1 t_1^{-n} \mathrm{J}_n(t_1) + C_2 t_1^{-n} \mathrm{Y}_n(t_1) + C_3 t_2^{-n} \mathrm{I}_n(t_2) + C_4 t_2^{-n} \mathrm{K}_n(t_2), \quad n = \text{an integer}, \tag{38}$$

where $\mathrm{J}_n(\cdot)$, $\mathrm{Y}_n(\cdot)$, $\mathrm{I}_n(\cdot)$ and $\mathrm{K}_n(\cdot)$ are Bessel functions of the first, second, third and fourth kinds, respectively; t_1 and t_2 can be determined by substituting λ_1 and λ_2 into equation (33b).

Case 2: The functions for describing the distributions of the flexural stiffness, axial force and mass intensity are expressed as

$$K_y = K_1(1 + \beta y)^{n+4}, \quad N_y = N(1 + \beta y)^{n+2}, \quad \bar{m}_{xy} = \bar{m}(1 + \beta y)^n. \tag{39}$$

Substituting equation (39) into equation (16) leads to an Euler's equation, the general solution of which can be written as

$$Y(y) = C_1 \exp(r_1\eta) + C_2 \exp(r_2\eta) + C_3 \exp(r_3\eta) + C_4 \exp(r_4\eta) \tag{40}$$

where

$$\eta = \ln(1 + \beta y),$$

$$r_{1,2} = -\frac{n+1}{2} \pm \sqrt{\frac{(n+1)^2}{4} + \theta_f - N_f},$$

$$r_{3,4} = \frac{n+1}{2} \pm \sqrt{\frac{(n+1)^2}{4} - \theta_f - N_f},$$

$$N_f = \frac{1}{2}(N_d - n - 2), \tag{41}$$

$$\theta_f = \sqrt{\theta_d^2 + N_f^2},$$

$$N_d = \frac{N}{\beta^2 K_1}, \quad \theta_d = \frac{\bar{m}\theta^2}{\beta^4 K_1}.$$

Since $\theta_f \geqslant N_f$, r_1 and r_2 are real roots. If r_3 and r_4 are complex values, then

$$Y(y) = C_1 \exp(r_1\eta) + C_2 \exp(r_2\eta) + \exp\left(-\frac{n+1}{2}\eta\right)(D_3 \cos \varphi y + D_4 \sin \varphi y), \tag{42}$$

where

$$\varphi^2 = \theta_f + N_f - \frac{(n+1)^2}{4}. \tag{43}$$

Case 3: The distributions of flexural stiffness, axial force and mass intensity are given by

$$K_y = K_1 \exp(-by), \quad N_y = N \exp(-by), \quad \bar{m}_{xy} = \bar{m} \exp(-by). \tag{44}$$

Substituting equation (44) into equation (16) one obtains a differential equation with constant coefficients its general solution is

$$Y(y) = \exp\left(\frac{by}{2}\right)\left[C_1 \exp\left(\sqrt{\frac{b^2}{4} - Z_2}\, y\right) + C_2 \exp\left(-\sqrt{\frac{b^2}{4} - Z_2}\, y\right)\right.$$
$$\left. + C_3 \cos\left(\sqrt{Z_1 - \frac{b^2}{4}}\, y\right) + C_4 \sin\left(\sqrt{Z_1 - \frac{b^2}{4}}\, y\right)\right] \tag{45}$$

where Z_1 and Z_2 are given by equation (36).

Case 4: The distribution of flexural stiffness axial force and mass intensity are uniform, i.e.,

$$K_y = K_1, \quad N_y = N, \quad \bar{m}_{xy} = \bar{m}. \tag{46}$$

Equation (16) for this case becomes a differential equation with constant coefficients; its general solution is easily found as

$$Y(y) = C_1 \sin \alpha_1 y + C_2 \cos \alpha_1 y + C_3 \sinh \alpha_2 y + C_4 \cosh \alpha_2 y \tag{47}$$

where

$$\alpha_{1,2} = f\sqrt{\sqrt{1+\left(\frac{\varepsilon}{f}\right)^4} \pm 1}, \qquad f = \frac{N}{2K_1}, \quad \varepsilon^4 = \frac{\bar{m}\theta^2}{K_1}, \tag{48}$$

The general solution for all the cases discussed above can be expressed in a unified form as follows:

$$Y(y) = C_1 S_1(y) + C_2 S_2(y) + C_3 S_3(y) + C_4 S_4(y), \tag{49}$$

where $S_i(y)$ ($i = 1$–4) are four independent solutions of equation (16), which can be found from equations (37), (38), (40), (45) and (47) for one of the four cases discussed previously.

In order to conveniently establish the frequency equation for the title problem, by using $S_i(y)$ ($i = 1$–4), we construct the following four linearly independent fundamental solutions $\bar{S}_i(y)$ ($i = 1$–4) as

$$\begin{bmatrix} \bar{S}_1(y) \\ \bar{S}_2(y) \\ \bar{S}_3(y) \\ \bar{S}_4(y) \end{bmatrix} = \begin{bmatrix} S_1(0) & S_1'(0) & S_1''(0) & S_1'''(0) \\ S_2(0) & S_2'(0) & S_2''(0) & S_2'''(0) \\ S_3(0) & S_3'(0) & S_3''(0) & S_3'''(0) \\ S_4(0) & S_4'(0) & S_4''(0) & S_4'''(0) \end{bmatrix}^{-1} \begin{bmatrix} S_1(y) \\ S_2(y) \\ S_3(y) \\ S_4(y) \end{bmatrix}. \tag{50}$$

Obviously, $\bar{S}_1(y)$ ($i = 1$–4) satisfy the following normalization condition at the origin of co-ordinate system:

$$\begin{bmatrix} \bar{S}_1(0) & \bar{S}_1'(0) & \bar{S}_1''(0) & \bar{S}_1'''(0) \\ \bar{S}_2(0) & \bar{S}_2'(0) & \bar{S}_2''(0) & \bar{S}_2'''(0) \\ \bar{S}_3(0) & \bar{S}_3'(0) & \bar{S}_3''(0) & \bar{S}_3'''(0) \\ \bar{S}_4(0) & \bar{S}_4'(0) & \bar{S}_4''(0) & \bar{S}_4'''(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{51}$$

The advantage of using the fundamental solutions, $\bar{S}_i(y)$, is that the mode shape functions in the y direction can be easily expressed by initial parameters as follows:

$$Y(y) = Y(0)\bar{S}_1(y) + \varphi(0)\bar{S}_2(y) - \frac{M_y(0)}{K_y(0)}\bar{S}_3(y) - \frac{1}{K_y(0)}$$

$$[Q_y(0) + N_y(0)\varphi(0) - \mu(0)M_y(0)]\bar{S}_4(y), \tag{52}$$

where

$$\mu(y) = \frac{K_y'(y)}{K_y(y)}.$$

$Y(0)$, $\varphi(0)$, $M_y(0)$ and $Q_y(0)$ are initial displacement, slope, bending moment and shear force in the y direction at $y = 0$ respectively. Because two of the four initial parameters are known for any type of support conditions, it is easy to establish the frequency equation of the flexural-shear plate in the y direction by use of the fundamental functions. Free vibration of a flexural-shear plate with classical and non-classical boundary conditions is discussed as follows.

1. *A flexural-shear plate with F–F edges in the y direction.* The boundary conditions for this case are given by

$$M_y(0) = 0, \quad Q_y(0) = 0, \quad M_y(H) = 0, \quad Q_y(H) = 0. \tag{53}$$

It can be seen from the above equation that two of the four parameters at the edges $y = 0$ and $y = H$ are known. By using the boundary conditions at $y = 0$, we obtain

$$Y(y) = Y(0)\bar{S}_1(y) + \varphi(0)[\bar{S}_2(y) - N_K(0)\bar{S}_4(y)] \tag{54}$$

where

$$N_K(y) = \frac{N_y(y)}{K_y(y)}. \tag{55}$$

2. *A flexural-shear plate with C–C edges in the y direction.* The boundary conditions can be written as

$$Y(0) = 0, \quad \varphi(0) = 0, \quad Y(H) = 0, \quad \varphi(H) = 0. \tag{56}$$

By using the boundary conditions at $y = 0$ and equation (52) we obtain

$$Y(y) = [\bar{S}_3(y) + \mu(0)\bar{S}_4(y)]M_y(0) + \bar{S}_4(y)Q_y(0). \tag{57}$$

Then, using the boundary conditions at $y = H$ gives the frequency equation as

$$\bar{S}'_4(H)[\bar{S}_3(H) + \mu(0)\bar{S}_4(H)] - \bar{S}_4(H)[\bar{S}'_3(H) + \mu(0)\bar{S}'_4(H)] = 0. \tag{58}$$

3. *A flexural-shear plate with hinged–hinged (H–H) edges in the y direction.* The boundary conditions are given by

$$Y(0) = 0, \quad M_y(0) = 0, \quad Y(H) = 0, \quad M_y(H) = 0. \tag{59}$$

The boundary conditions at $y = 0$ are substituted into equation (52), leading to

$$Y(y) = [\bar{S}_2(y) - N_K(0)\bar{S}_4(y)]\varphi(0) - \frac{1}{K_y(0)}\bar{S}_4(y)Q_y(0) = 0. \tag{60}$$

The frequency equation can be established by the use of equation (60) and the boundary conditions at $y = H$ as follows:

$$\bar{S}''_4(H)[\bar{S}_2(H) - N_K(0)\bar{S}_4(H)] - \bar{S}_4(H)[\bar{S}''_2(H) - N_K(0)\bar{S}''_4(H)] = 0. \tag{61}$$

4. *A flexural-shear plate with C–F edges in the y direction.* If the edge at $y = 0$ is free, $Y(y)$ is given by equation (54). The frequency equation can be established by using the boundary conditions at $y = H$,

$$Y(H) = 0, \quad \varphi(H) = 0 \tag{62}$$

as follows:

$$\bar{S}_1(H)[\bar{S}'_2(H) - N_K(0)\bar{S}'_4(H)] - \bar{S}'_1(H)[\bar{S}'_2(H) - N_K(0)\bar{S}_4(H)] = 0. \tag{63}$$

5. *A flexural-shear plate with S–F edges in the y direction.* If the edge at $y = 0$ is free, then $Y(y)$ is given by equation (54). If translational spring and rotational spring supports with a line mass are attached to the edge at $y = H$, then the boundary conditions can be written as

$$\begin{aligned} M_y(H) &= K_{\varphi H}\varphi(H), \\ Q_y(H) &= -(K_{yH} - m_{yH}\theta^2)Y(H) \end{aligned} \tag{64}$$

or

$$\begin{aligned} K_y(H)Y''(H) + K_{\varphi H}Y'(H) &= 0, \\ Y'''(H) + \mu(H)Y''(H) + N_K(H)Y'(H) - (a_{yH} - b_{yH}\theta^2)Y(H) &= 0, \end{aligned} \tag{65}$$

where

$$a_{yH} = \frac{K_{yH}}{K_y(H)}, \quad b_{yH} = \frac{m_{yH}}{K_y(H)}. \tag{66}$$

$K_{\varphi H}$, K_{yH}, m_{yH} are the rotational spring stiffness, translational spring stiffness and line mass intensity attached to the edge at $y = H$ respectively.

The frequency equation can be determined from equations (54) and (65) as follows:

$$\begin{aligned} &[K_{\varphi H}\bar{S}'_1(H) + K_y(H)\bar{S}''_1(H)]A_\varphi - \{K_{\varphi H}\bar{S}'_2(H) + K_y(H)\bar{S}''_2(H) - N_K(0) \\ &\qquad [K_{\varphi H}\bar{S}'_4(H) + K_y(H)\bar{S}''_4(H)]\}\, A_u = 0, \end{aligned} \tag{67}$$

where

$$\begin{aligned} A_\varphi &= \bar{S}'''_2(H) - N_K(0)\bar{S}'''_4(H) + \mu(H)[\bar{S}''_2(H) - N_K(0)\bar{S}''_4(H)] + N_K(H)[\bar{S}'_2(H) - N_K(0)\bar{S}'_4(N)] \\ &\quad - (a_{yH} - b_{yH}\theta^2)[\bar{S}_2(H) - N_K(0)\bar{S}_4(H)], \\ A_u &= \bar{S}'''_1(H) + \mu(H)\bar{S}''_1(H) + N_K(H)\bar{S}'_1(H) - (a_{yH} - b_{yH}\theta^2)\bar{S}_1(H). \end{aligned} \tag{68}$$

6. *A flexural-shear plate with S–S edges in the y direction.* If translational spring and rotational spring supports with a line mass are attached to the edge at $y = 0$ and the

support conditions at the edge $y = H$ are the same as those at the edge $y = 0$, then the boundary conditions can be written as

$$
\begin{aligned}
M_y(0) &= -K_{\varphi 0}\varphi(0), \\
Q_y(0) &= (K_{y0} - m_{y0}\theta^2)Y(0), \\
M_y(H) &= K_{\varphi H}\varphi(H), \\
Q_y(H) &= -(K_{yH} - m_{yH}\theta^2)Y(H)
\end{aligned}
\tag{69}
$$

or

$$
\begin{gathered}
K_y(0)Y''(0) - K_{\varphi 0}Y'(0) = 0, \\
Y'''(0) + \mu(0)Y''(0) + N_K(0)Y'(0) + (a_{y0} - b_{y0}\theta^2)Y(0) = 0, \\
K_y(H)Y''(H) + K_{\varphi H}Y'(H) = 0, \\
Y'''(H) + \mu(H)Y''(H) + N_K(H)Y'(H) + (a_{yH} - b_{yH}\theta^2)Y(H) = 0,
\end{gathered}
\tag{70}
$$

where $K_{\varphi 0}$, K_{y0}, m_{y0}, are the rotational spring stiffness, translational spring stiffness and line mass intensity attached to the edge at $y = 0$. $K_{\varphi H}$, K_{yH}, m_{yH} are the rotational spring stiffness, translational spring stiffness and line mass intensity attached to the edge at $y = H$.

The boundary conditions at $y = 0$ are substituted into equation (52), leading to

$$
Y(y) = Y(0)\bar{S}_{14}(y) + \varphi(0)\bar{S}_{34}(y),
\tag{71}
$$

where

$$
\begin{aligned}
\bar{S}_{14}(y) &= \bar{S}_1(y) - (a_{y0} - b_{y0}\theta^2)\bar{S}_4(y), \\
\bar{S}_{34}(y) &= C_{\varphi 0}\bar{S}_3(y) - (N_K(0) - \mu_0 C_{\varphi 0})\bar{S}_4(y), \\
C_{\varphi 0} &= \frac{K_{\varphi 0}}{K_y(0)}.
\end{aligned}
\tag{72}
$$

The frequency equation can be determined from equation (71) and the boundary conditions at $y = H$ as follows:

$$
[K_y(0)\bar{S}''_{14}(0) - K_{\varphi 0}\bar{S}'_{14}(0)]B_\varphi - [K_y(0)\bar{S}''_{34}(0) - K_{\varphi 0}\bar{S}'_{34}(0)]B_u = 0,
\tag{73}
$$

where

$$
\begin{aligned}
B_\varphi &= \bar{S}'''_{34}(H) + \mu(H)\bar{S}''_{34}(H) + N_K(H)\bar{S}'_{34}(H) - (a_{yH} - b_{yH}\theta^2)\bar{S}_{34}(H), \\
B_u &= \bar{S}'''_{14}(H) + \mu(H)\bar{S}''_{14}(H) + N_K(H)\bar{S}'_{14}(H) - (a_{yH} - b_{yH}\theta^2)\bar{S}_{14}(H).
\end{aligned}
\tag{74}
$$

7. *A flexural-shear plate with line masses, line translational spring and rotational spring supports at the* $(q-1)$ *intermediate lines* (*Figure* 3). It is assumed that the flexural stiffness, mass intensity and axial force are described by continuous functions denoted by $K_y(y)$, $\bar{m}_{xy}(y)$, $N_y(y)$. The stiffness of the jth line translational spring and that of rotational spring

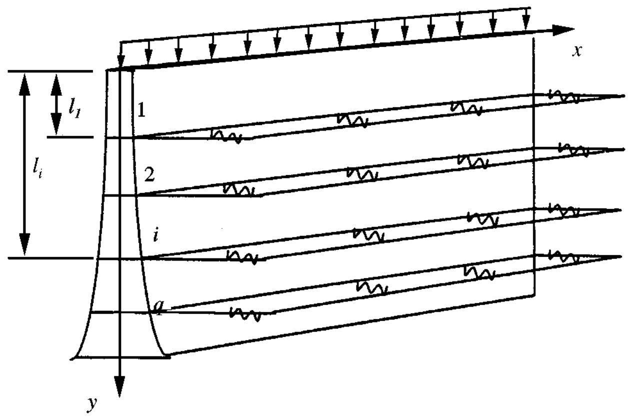

Figure 3. A flexural-shear plate with line masses, line spring at the $(q-1)$ intermediate lines. (Note: the line masses and rotational springs are not shown in Figure 3.)

and the jth line mass intensity are denoted by K_j, $K_{\varphi j}$ and m_j respectively, the fundamental solutions for this case are $\bar{S}_1(y)$, $\bar{S}_2(y)$, $\bar{S}_3(y)$ and $\bar{S}_4(y)$.

The mode shape, $Y_1(y)$, of the first segment, $y \in [0, l_1]$, is dependent on the boundary conditions at $y=0$; the mode shape, $Y_i(y)$, of the ith segment can be written as

$$Y_i(y) = Y_1(y) + \sum_{j=1}^{i-1} \frac{K_{\varphi j}}{K_y(l_j)} \varphi_j(l_j)\bar{S}_3(y-l_j)\mathrm{H}(y-l_j)$$
$$- \sum_{j=1}^{i-1} \frac{1}{K_y(l_i)} [(K_j - m_j\theta^2)Y_j(l_j)$$
$$+ \mu(l_j)K_{\varphi j}\varphi_j(l_j)]\bar{S}_4(y-l_j)\mathrm{H}(y-l_j), \quad j=2,3,\ldots,q, \qquad (75)$$

where $\mathrm{H}(\cdot)$ is Heaviside function.

The frequency equation can be determined by the use of $Y_q(y)$ and the boundary conditions at $y=l_q$.

3. NUMERICAL EXAMPLE

Figure 4 shows a sketch of a 20-story building with narrow rectangular plane; the main structures of the building are shear walls. Based on the field measurement of vibration of this building [4], it can be treated as a flexural-shear plate for free vibration analysis. The building foundation is treated as translational springs and rotational springs attached to the building base. The procedure for determining the natural frequencies and mode shapes of this narrow building is as follows.

1. *Determination of the mass intensity of the flexural-shear plate.* The mass intensity (mass per unit area) of the flexural-shear plate, which represents the building considered, varies in echelon along the building height (Figure 5).

The distributions of mass in different stories are found as

$$m_1 = 6{\cdot}146 \times 10^5 \text{ kg},$$

$$m_2 = m_3 = m_4 = m_5 = 4{\cdot}612 \times 10^5 \text{ kg},$$

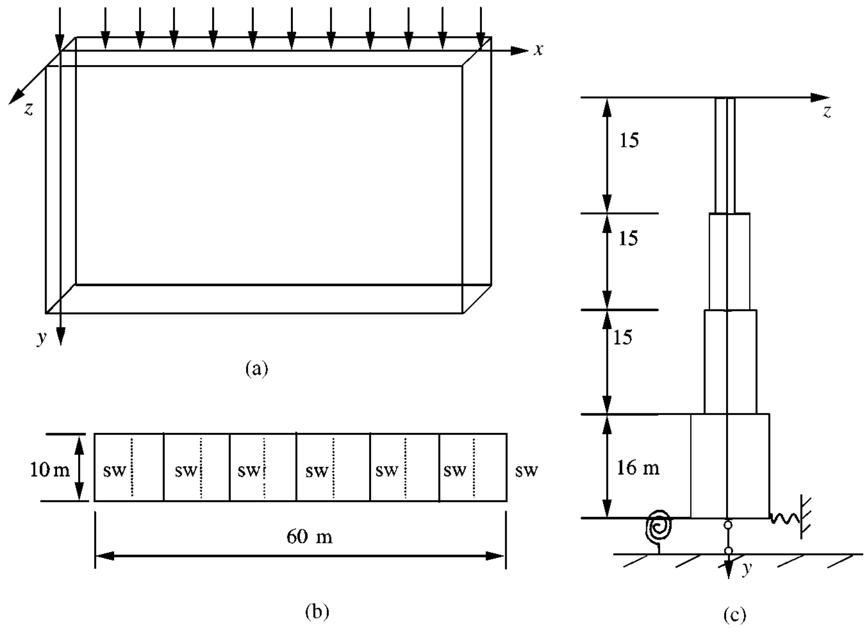

Figure 4. A narrow building: (a) perspective drawing, (b) plane, (c) a transverse shear–wall.

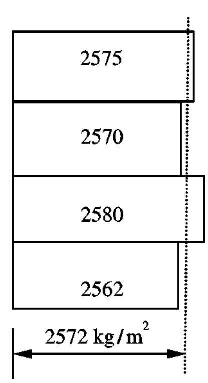

Figure 5. The distribution of mass intensity.

$$m_6 = m_7 = m_8 = m_9 = m_{10} = 4{\cdot}644 \times 10^5\,\text{kg},$$

$$m_{11} = m_{12} = m_{13} = m_{14} = m_{15} = 4{\cdot}626 \times 10^5\,\text{kg},$$

$$m_{16} = m_{17} = m_{18} = m_{19} = m_{20} = 4{\cdot}630 \times 10^5\,\text{kg}.$$

The height of the first story is 4 m, and the height of the other floors is 3 m; so, the equivalent mass for 3 m height of the first story is

$$6{\cdot}146 \times 10^5 \times \tfrac{3}{4} = 4{\cdot}610 \times 10^5\,\text{kg}.$$

It can be seen from the above results that the mass distribution from the first story to the fifth story is almost uniform, the mass intensity of the plate in this story range is

$$\bar{m}_1 = \frac{4{\cdot}612 \times 10^5}{3 \times 5 \times 12} = 2{\cdot}562 \times 10^3\ \text{kg/m}^2.$$

From the sixth story to the 10 story,

$$\bar{m}_2 = \frac{4{\cdot}644 \times 10^5}{3 \times 5 \times 12} = 2{\cdot}580 \times 10^3\ \text{kg/m}^2.$$

From the 11th story to the 15th story,

$$\bar{m}_3 = \frac{4{\cdot}626 \times 10^5}{3 \times 5 \times 12} = 2{\cdot}570 \times 10^3\ \text{kg/m}^2.$$

From the 16th story to the 20th story,

$$\bar{m}_4 = \frac{4{\cdot}635 \times 10^5}{3 \times 5 \times 12} = 2{\cdot}575 \times 10^3\ \text{kg/m}^2.$$

The values of the mass intensity of the four-step plate divided above are shown in Figure 5. It can be seen from the above results that the variation of the mass intensity of the plate is relatively small; thus, it is reasonable to assume $\bar{m}$ as a constant, i.e., $\bar{m} = 2{\cdot}572 \times 10^3$ kg/m^2.

2. *Evaluation of the flexural stiffness* $K_y(y)$. The distribution of shear walls along the longitudinal direction of the building is uniform and the cross-sectional dimensions of the shear wall vary in echelon along the building height. The total flexural stiffness of the shear walls from the first story to the fifth story is found as

$$EI_1 = 25{\cdot}6194 \times 10^{12}\ \text{N m}^2.$$

The flexural stiffness of the plate in this story range is the value of EI_1 divided by the length of the building:

$$K_{1y} = \frac{25{\cdot}6194 \times 10^{12}}{5 \times 12} = 4{\cdot}2699 \times 10^{11}\ \text{N m}.$$

The flexural stiffness of the plate from the sixth story to the 10th story is found as

$$K_{2y} = 3{\cdot}7126 \times 10^{11}\ \text{N m}.$$

From the 11th story to the 15th story,

$$K_{3y} = 3{\cdot}3806 \times 10^{11}\ \text{N m}.$$

From the 16th story to the 20th story,

$$K_{4y} = 2{\cdot}9105 \times 10^{11}\ \text{N m}$$

In order to use the method proposed in this paper to analyze free vibration of the narrow building, the four-step distribution of the flexural stiffness is treated as a continuous one given by

$$K_y = 2{\cdot}9105(1+\beta y)^2 \times 10^{11}, \quad \beta = 3{\cdot}4627 \times 10^{-3}. \tag{76}$$

A comparison between the distribution of the flexural stiffness estimated by equation (76) and the real one is given in Figure 6.

3. *Evaluation of the shear stiffness* K_x. The shear stiffness of all the floors is

$$GF = 3{\cdot}8718 \times 10^{10} \text{ N}.$$

The shear stiffness of the plate is thus a constant that is equal to the value of GF divided by the story height,

$$K_x = K_2 = \frac{3{\cdot}8718 \times 10^{10}}{3} = 1{\cdot}2906 \times 10^{10} \text{ N/m}$$

4. *Evaluation of stiffness for the elastic foundation.* It is assumed that the elastic foundation is treated as translational springs and rotational springs attached to the building base, as shown in Figure 4(c), which are found as

$$K_{yH} = 4{\cdot}9105 \times 10^{10} \text{ N/m}^2,$$

$$K_{\varphi H} = 9{\cdot}7038 \times 10^{10} \text{ N}.$$

5. *Evaluation of axial forces.* Since the distribution of mass intensity of the plate is uniform, the distribution of axial force is described by a linear function as

$$N_y = N_0(1+\beta y). \tag{77}$$

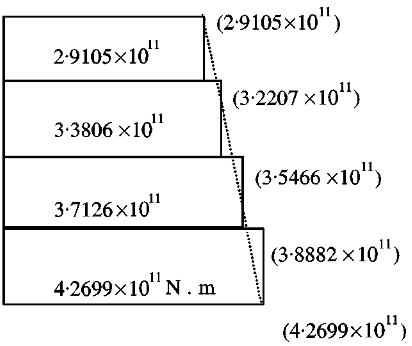

Figure 6. The distribution of flexural stiffness.

In order to use the method proposed in this paper, β must be equal to the same value as that given in equation (76). N_0 is determined from

$$N_0 = \int_0^H (\bar{m}g - \beta N_0)\left(1 - \frac{y}{H}\right)^2 \mathrm{d}y = 4{\cdot}8857 \times 10^5\ \mathrm{N}. \tag{78}$$

6. *Determination of the natural frequencies in the x direction.* Since the plate, which represents the 20-storey building considered, has F–F edges in the x direction Ω_K is given by equation (21), i.e.,

$$\Omega_K = \frac{2240.0636(k-1)\pi}{60}.$$

Letting $k = 1, 2, 3$, one obtains $\Omega_1 = 0$, $\Omega_2 = 117.2897$, $\Omega_3 = 234.5794$.

7. *Determination of the natural frequencies in the y direction.* The boundary conditions of the plate in the y direction can be written as

$$\begin{aligned} M_y(0) &= 0, \\ Q_y(0) &= 0, \\ M_y(H) &= K_{\varphi H}\varphi(H), \\ Q_y(H) &= -K_{yH}Y(H). \end{aligned} \tag{79}$$

The frequency equation is the same as equation (67) for this case, but $b_{yH} = 0$. Solving the frequency equation one obtains

$$\theta_1 = 8{\cdot}4531, \quad \theta_2 = 58{\cdot}4112, \quad \theta_3 = 169{\cdot}0273.$$

If the effect of rotation of the foundation on the natural frequencies is not considered, then

$$\theta_1 = 8{\cdot}5176, \quad \theta_2 = 85{\cdot}9241, \quad \theta_3 = 169{\cdot}9872.$$

If the effects of rotation of the foundation and the foundation elasticity on the natural frequencies are not considered, then

$$\theta_1 = 8{\cdot}5986, \quad \theta_2 = 58{\cdot}9972, \quad \theta_3 = 170{\cdot}2991.$$

If the effects of the foundation elasticity and the axial forces are not considered, then

$$\theta_1 = 8{\cdot}6203, \quad \theta_2 = 59{\cdot}0437, \quad \theta_3 = 170{\cdot}3218.$$

It can be seen from the above results that the foundation of this building can be treated as clamped end support, and the effect of the axial forces can be ignored.

8. *Determination of the natural frequencies of the plate.* After θ_j and Ω_k are found, the circular natural frequency, ω_{jk}, of the plate is given by

$$\omega_{jk} = \sqrt{\theta_j^2 + \Omega_k^2},$$

TABLE 1

The circular natural frequencies of the narrow building

ω_{11}	ω_{21}	ω_{12}	ω_{22}	ω_{31}	ω_{32}	ω_{13}	ω_{23}	ω_{33}
8·6203	59·0437	117·6061	131·3127	170·3218	206·8004	234·7377	241·8962	289·8916
[8·6209]	[59·0441]	[117·6073]	[131·3152]	[170·3821]	[206·8973]	[234·8792]	[241·9104]	[289·9984]
(8·63)	(59·21)	(117·69)	(131·73)	(170·95)				

Note: The data in square brackets are the values calculated by using the uniform four-step model, and the resutls in parentheses are the measured data.

where ω_{jk} corresponds to the jth mode shape in the y direction and the kth mode shape in the x direction.

The calculated values of ω_{jk} are listed in Table 1.

The values of ω_k obtained by using the uniform four-step model shown in Figures 5 and 6 based on the calculation method proposed by Li *et al.* [12] and the measured values of ω_{jk} [4] are presented in Table 1 for comparison purposes.

9. *Determination of mode shapes.* $X_k(x)$ $(k = 1, 2, 3, \dots)$ are given in equation (25). $Y_j(y)$ can be determined from equation (54) and the boundary conditions. Since the effects of the axial forces and the elasticity of the foundation can be ignored, the plate has C–F edges in the y direction, i.e., we have

$$\begin{aligned} M_y = 0, \quad Q_y(0) = 0, \\ Y(H) = 0, \quad \varphi(H) = 0. \end{aligned} \tag{80}$$

Using equations (54), (80) and the calculated value of θ_j one obtains $Y_j(y)$ as listed in Table 2. The values of $Y_j(y)$ determined by using the uniform four-step model and from the field measurement [4] are also listed in Table 2 for comparison purposes.

It can be seen from Tables 1 and 2 that the calculated results are almost the same as those obtained based on the uniform four-step model. This illustrates that a non-uniform flexural-shear plate can be treated as a uniform multi-step flexural-shear plate for free vibration analysis and *vice versa*. It is also shown that all the calculated results are in good agreement with the measured data, suggesting that the proposed methods in this paper are applicable to engineering application.

4. CONCLUSIONS

An analytical procedure for determining natural frequencies and mode shapes of non-uniform flexural-shear plates with line translational spring and rotational spring supports and line masses under the action of axial forces has been proposed in this paper. It is shown that a flexural-shear plate is a special orthotropic plate that can be simplified as two beams for free vibration analysis. One is a shear beam, the other is a flexural beam. The natural frequency of the flexural-shear plate is equal to the square root of the square sum of the two natural frequencies of the two beams; the mode shape of the plate is the product of the corresponding two mode shapes of the two beams. Thus, the analytical procedure for free vibration of a non-uniform flexural-shear plate can be greately simplified. By selecting suitable functions, such as power functions and exponential functions, for describing the

TABLE 2

The mode shapes in y direction

Story no.	0	2	4	6	8	10	12	14	16	18	20
$Y_1(y)$	0	0·018	0·056	0·139	0·230	0·336	0·459	0·580	0·723	0·861	1·00
	[0]	[0·018]	[0·056]	[0·138]	[0·230]	[0·337]	[0·459]	[0·581]	[0·724]	[0·862]	[1·0]
	(0)	(0·020)	(0·057)	(0·139)	(0·230)	(0·338)	(0·460)	(0·582)	(0·724)	(0·863)	(1·0)
$Y_2(y)$	0	0·092	0·295	0·514	0·684	0·701	0·583	0·317	− 0·164	− 0·349	− 1·0
	[0]	[0·092]	[0·295]	[0·515]	[0·685]	[0·701]	[0·583]	[0·318]	[− 0·165]	[− 0·350]	[− 1·0]
	(0)	(0·093)	(0·296)	(0·514)	(0·686)	(0·701)	(0·584)	(0·319)	(− 0·165)	(− 0·350)	(− 1·0)
$Y_3(y)$	0	0·227	0·606	0·757	0·526	0·181	− 0·473	− 0·660	− 0·399	0·302	1·0
	[0]	[0·227]	[0·606]	[0·756]	[0·527]	[0·180]	[− 0·473]	[− 0·661]	[− 0·398]	[0·303]	[1·0]
	(0)	(0·229)	(0·607)	(0·758)	(0·528)	(0·180)	(− 0·433)	(0·662)	(− 0·399)	(0·304)	(1·0)

Note: The data in square brackets are the values calculated by using the uniform four-step model, and the results in parentheses are the measured data.

distributions of mass, flexural stiffness and axial forces, the exact solutions for the title problem for four cases that are important in engineering practices are derived. The numerical example shows that the selected expressions are suitable for describing the distributions of mass, stiffness and axial forces for typical multi-story narrow buildings with shear-wall structures. In fact, the four linearly independent fundamental solutions developed in this paper and satisfying the normalization condition can be easily constructed. The advantage of using the fundamental solutions is that the mode shape functions can be expressed by initial parameters and the frequency equation for the title problem can be conveniently established. In order to extend practical applications of the methods proposed in this paper, free vibrations of flexural-shear plates with various boundary conditions, including classical and non-classical ones, are investigated. The numerical example demonstrates that the effects of elastic foundation and axial forces on structural dynamic characteristics of common tall buildings (about 20 stories) are not significant, and it is possible to regard a multi-step flexural-shear plate as a one-step plate with continuously varying cross-section for free vibration analysis. It is also shown through the numerical example that the calculated results are in good agreement with the experimental data and the proposed procedure is an efficient method.

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