Journal of Sound and Vibration (2000) **235**(2), 175–200
doi:10.1006/jsvi.1999.2926, available online at http://www.idealibrary.com on IDEAL®

CHAOTIC ATTITUDE MOTION OF SATELLITES UNDER SMALL PERTURBATION TORQUES

JINLU KUANG AND SOON HIE TAN

The School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798, *Singapore*

AND

A. Y. T. LEUNG†

*School of Engineering, University of Manchester, Manchester, M*13 9*PL, U.K.*

(*Received* 3 *August* 1999, *and in final form* 24 *November* 1999)

In this paper, the chaotic attitude motion of dissipative satellites under small perturbation torques is investigated by using the Deprit's canonical variables in the standard Hamiltonian form. Melnikov's integral is used to predict the transversal intersections of the stable and unstable manifolds for the satellites in perturbation. The theoretical criterion of chaotic attitude motion of the perturbed satellites will be derived from the Melnikov integral. Two models of satellites are studied. The first model is a quasi-rigid, energy-dissipating satellite subject to the time-periodic, non-Hamiltonian perturbation torques. The second model is a gyrostat satellite under small perturbation torques. It will be shown that, in terms of Deprit's variables, the equations of the attitude motion of the satellites can be easily transformed into the Hamiltonian form which is suitable for the application of Melnikov's method to some cases of complex, small perturbation torques.

1. INTRODUCTION

The research on the dissipative motion of a rigid body under small perturbation torques is often found to have to do with the motion of satellites or natural celestial bodies. The earlier research about the dynamics of the gyrostat could be traced back to the end of the last century [1]. Volterra solved analytically the ordinary differential equations describing the attitude motion of the gyrostat with wheels of constant angular momentum under no external torques and he analysed the stability of the steady solution of its attitude. Rumyantsev [2] reviewed the motion stability of rigid bodies with fluid-filled cavities based on Lyapunov's methods. Pure rigid bodies are special cases of liquid-filled bodies. The Lyapunov–Rumyantsev theorem is widely utilized in the design of artificial satellites and the liquid-filled projectiles. Wittenburg [3] investigated the polhode curves on the inertia ellipsoid of the base body for different rotor speeds. Recently, the chaotic dynamics has attracted many scientists. The main theoretical approach used to study the chaotic dynamics of a non-linear system is Melnikov's integral method (see e.g. Chen and Leung [4]). Melnikov [5] proved that a transverse heteroclinic (homoclinic) orbit occurred in the

† Present address: The Department of Building and Construction, City University of Hong Kong, Hong Kong.

0022-460X/00/320175 + 26 $35.00/0

Poincaré map of the perturbed system by measuring "the distance" between the stable and unstable manifolds associated with the saddle points. Holmes and Marsden [6] employed the Kolmogorov–Arnold–Moser (KAM) theory and a vectorial version of Melnikov's method to address the general question of perturbations of integrable multidimensional Hamiltonian systems with at least three degrees of freedom (d.o.f.). Holmes and Marsden [7] studied the Smale horseshoes and the Arnold diffusion for Hamiltonian systems on Lie groups. They established the existence of Smale horseshoes and the Arnold diffusion for some nearly integrable Hamiltonian systems associated with Lie groups. The presence of horseshoes in the motion of a nearly symmetric heavy top implied that the dynamics was complex, and that the dynamics had periodic orbits of arbitrary high multiple periods embedded in an invariant cantor set, and that the dynamical system admitted no additional analytic integrals. They developed Melnikov's theory when the phase space was a product of the dual of a Lie algebra and a set of action angles. Tong *et al.* [8] introduced Deprit's canonical variables to establish the Hamiltonian structure of the attitude motion of he perturbed asymmetrical gyrostat in the gravitational field. They derived the Hamiltonian equations which were readily suitable for the application of Melnikov's integral generalized by Holmes and Marsden for a two d.o.f. Hamiltonian system with S^1 symmetry. By using a version of Melnikov integral extended by Holmes and Marsden [9], Tong and Tabarrok [10, 11] showed that the attitude motion of the asymmetric gyrostat in the gravitational field was chaotic in the sense of Smale's horseshoes. Tong and Tabarrok [11] revealed the analogy between the motion of self-excited rigid bodies and slowly varying oscillators by using Deprit's canonical variables. They investigated the chaotic attitude motion of self-excited rigid bodies subjected to small perturbation torques by utilizing the version of the Melnikov method developed by Wiggins and Shaw [12] for the case of slowly varying oscillators. They found out the existence of transversal intersections of heteroclinic orbits for certain parameter domains. Gray *et al.* [13] researched into the chaotic dynamics of an attitude transition manoeuvre of a torque-free rigid body in going from minor-axis spin to major-axis spin under the influence of small damping by using Melnikov's method. Their model was a Hamiltonian system perturbed by a non-Hamiltonian perturbation in the form of oscillations of sub-bodies and damping in the satellite. They found that the chaotic motion was due to the formation of Smale's horseshoes. Gray *et al.* [14] studied the analytical criterion for chaotic dynamics in flexible satellites with non-linear controller damping. Gray *et al.* [13, 14] applied a spherical co-ordinate transformation involving very complex calculations via *Mathematica* [15]. Their analytic criterion gave a useful design tool to spacecraft engineers concerned with the avoidance of the potentially problematic chaotic dynamics. Or [16] investigated the chaotic motion of a dual-spin body. He developed the mathematical equations of chaotic motions of the dual-spin body and derived the Melnikov function using the residue theory. Or obtained the conditions for the existence of transverse homoclinic points and discussed the bifurcation to the Smale's horseshoes.

In this paper, the Melnikov integral is employed to investigate the non-linear attitude motion of the satellites under small perturbation torques. The Euler equations of the attitude motion are transformed into the standard Hamiltonian form in terms of Deprit's canonical variables. For the case of a quasi-rigid satellite model subject to small time-periodic, non-Hamiltonian perturbations, the Hamiltonian equations obtained are readily suitable for the application of Melnikov's method developed by Wiggins and Shaw [12]. Reference [13] studied an attitude transition manoeuvre of a quasi-rigid satellite in going from minor-axis spin to a major-axis spin under the influence of small damping and non-Hamiltonian, time-periodic perturbations. The paper includes the studies in references [11, 13] as special cases. The analytical criterion of chaotic motions of

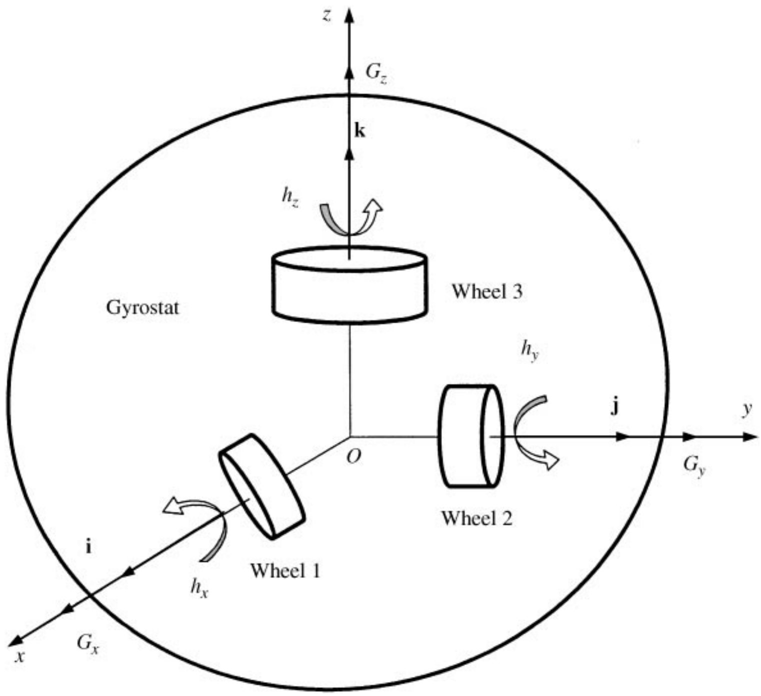

Figure 1. The configuration of a gyrostat rotating about the center of mass O.

the perturbed satellites under small perturbation torques is also obtained by using Melnikov's integral.

2. THE MATHEMATICAL FORMULATION IN TERMS OF DEPRINT'S VARIABLE

Equations of attitude dynamics of satellites can be derived from Euler's moment equations. The system consists of a rigid body with rotating elements inside the satellite. The rotating elements are known as "momentum exchange devices". The satellite is assumed to rotate about the centre of mass under the action of external disturbance torques. The moment of momentum of the system is made up of the moment of momentum of the gyrostat satellite $(I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k})$ and the moment of momentum of the momentum exchange devices $(h_x\mathbf{i} + h_y\ \mathbf{j} + h_z\mathbf{k})$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit direction vectors of the body-axis frame (see Figure 1). In Figure 1, the variables are $G_x = I_x\omega_x + h_x$, $G_y = I_y\omega_y + h_y$, and $G_z = I_z\omega_z + h_z$. The parameters h_x, h_y, and h_z are the components of the vector sum of the angular momentum of all the momentum exchange devices with respect to the body-fixed axis frame. The variables ω_x, ω_y, and ω_z are the angular velocities of the gyrostat satellite in the body-axis frame. The parameters I_x, I_y, and I_z are the principal moments of inertia of the gyrostat satellite (including the wheels) in the body-axis frame. Without loss of generality, throughout this paper, it is assumed that $I_x > I_y > I_z$. With these definitions, the theorem of moment of momentum can be expressed as

$$\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}t} + \boldsymbol{\omega} \times \mathbf{G} = \mathbf{T}_0$$

where

$$\mathbf{G} = G_x\mathbf{i} + G_y\mathbf{j} + G_z\mathbf{k}, \qquad \frac{\mathrm{d}\mathbf{G}}{\mathrm{d}t} = \frac{\mathrm{d}G_x}{\mathrm{d}t}\mathbf{i} + \frac{\mathrm{d}G_y}{\mathrm{d}t}\mathbf{j} + \frac{\mathrm{d}G_z}{\mathrm{d}t}\mathbf{k},$$

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}, \qquad \mathbf{T}_0 = T_x\mathbf{i} + T_y\mathbf{j} + T_z\mathbf{k},$$

where T_x, T_y, and T_z are the components of the vector of disturbance external torques in the body-axis frame, and the symbol $\times$ stands for the cross-product of the vectors. From the theorem of moment of momentum the general equations of the attitude motion become [17]

$$I_x\frac{\mathrm{d}\omega_x}{\mathrm{d}t} - (I_y - I_z)\omega_y\omega_z + \omega_y h_z - \omega_z h_y = T_x, \tag{1}$$

$$I_y\frac{\mathrm{d}\omega_y}{\mathrm{d}t} - (I_z - I_x)\omega_z\omega_x + \omega_z h_x - \omega_x h_z = T_y, \tag{2}$$

$$I_z\frac{\mathrm{d}\omega_z}{\mathrm{d}t} - (I_x - I_y)\omega_x\omega_y + \omega_x h_y - \omega_y h_x = T_z. \tag{3}$$

The Euler angles ψ, θ, and ϕ are defined as in Figure 2, where $Oxyz$ is the body-fixed reference frame that is coinciding with the principal axis of the entire system; $OXYZ$ is the inertial reference frame. The sequence of rotation is $\psi \to \theta \to \phi$. The variables ω_x, ω_y, and ω_z may also be expressed in terms of the Euler angles and the Euler angular velocities as

$$\omega_x = \frac{\mathrm{d}\psi}{\mathrm{d}t}\sin\theta\sin\phi + \frac{\mathrm{d}\theta}{\mathrm{d}t}\cos\phi, \tag{4}$$

$$\omega_y = \frac{\mathrm{d}\psi}{\mathrm{d}t}\sin\theta\cos\phi - \frac{\mathrm{d}\theta}{\mathrm{d}t}\sin\phi, \tag{5}$$

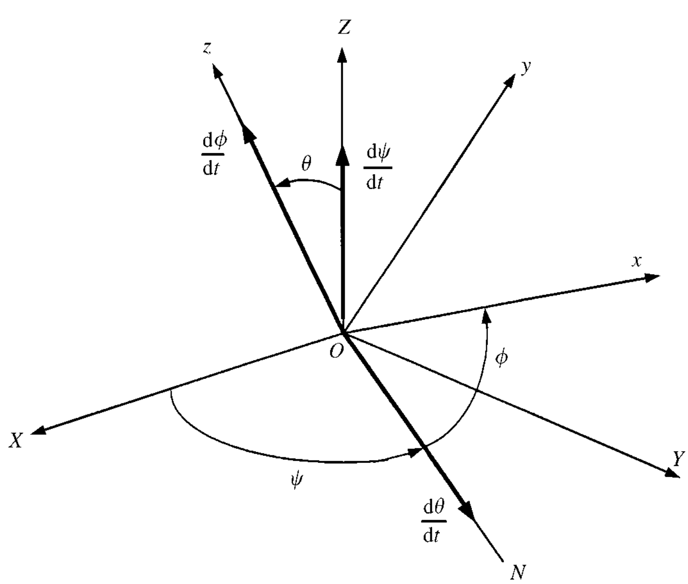

Figure 2. The body-fixed axis frame $Oxyz$.

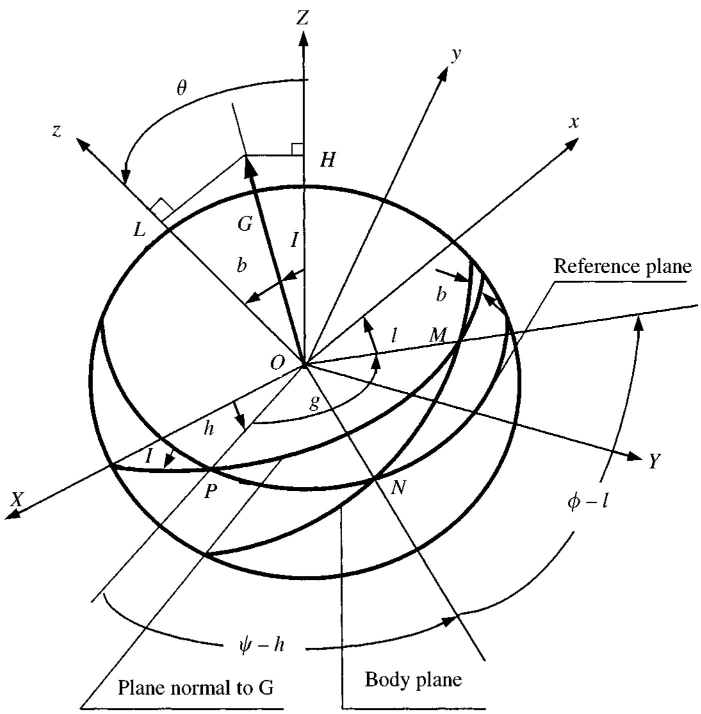

Figure 3. Euler angles (ψ, θ, ϕ) and the Deprit variables (l, g, h, L, G, H) (PNM is a spherical triangle).

$$\omega_z = \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt}. \tag{6}$$

Following Deprit [18], the canonical variables L, G, H, l, g, and h (see Figure 3) are introduced. In Figure 3, the big circle stands for the unit sphere. The reference plane perpendicular to the axis OZ is in the co-ordinate plane OXY. The body plane perpendicular to the axis Oz is in the co-ordinate plane Oxy. G is the magnitude of the moment of the momentum of the entire system. The plane normal to G is defined as the plane perpendicular to the vector of the moment of momentum of the entire system and passing through point O. The projection of the vector of the moment of momentum of the entire system on the axis OZ is H. The projection of the vector of the moment of momentum of the entire system on the axis Oz is L. The angle between the body plane and the plane normal to G is b. The angle between the reference plane and the plane normal to G is I. The region PNM on the surface of the unit sphere is a spherical triangle. The angle h is the angle between the axis OX and the line OP on the reference plane. The angle l is the angle between the axis Ox and the line OM on the body plane. The angle g is the angle between the line OP and the line OM on the plane normal to G. The detailed description of the Deprit's variables can be found in reference [18]. According to Figure 3 and the knowledge of advanced analytical mechanics, the angular velocities ω_x, ω_y, and ω_z can be expressed in terms of the Deprit's canonical variables:

$$I_x\omega_x = G\sin b\sin l - h_x, \tag{7}$$

$$I_y\omega_y = G\sin b\cos l - h_y, \tag{8}$$

$$I_z\omega_z = L - h_z, \tag{9}$$

where the angle b is a function of new momenta G and L given by

$$\cos b = \frac{L}{G}. \tag{10}$$

Differentiating equations (7)–(9) with respect to time t, respectively, the following equations can be derived:

$$I_x\frac{\mathrm{d}\omega_x}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}t}\sin b\sin l + \frac{\mathrm{d}b}{\mathrm{d}t}G\cos b\sin l + \frac{\mathrm{d}l}{\mathrm{d}t}G\sin b\cos l, \tag{11}$$

$$I_y\frac{\mathrm{d}\omega_y}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}t}\sin b\cos l + \frac{\mathrm{d}b}{\mathrm{d}t}G\cos b\cos l - \frac{\mathrm{d}l}{\mathrm{d}t}G\sin b\sin l, \tag{12}$$

$$I_z\frac{\mathrm{d}\omega_z}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}t}\cos b - \frac{\mathrm{d}b}{\mathrm{d}t}G\sin b = \frac{\mathrm{d}L}{\mathrm{d}t}. \tag{13}$$

It is easy to solve for the unknown variables ($\mathrm{d}l/\mathrm{d}t$, $\mathrm{d}G/\mathrm{d}t$, $\mathrm{d}b/\mathrm{d}t$) among the quasi-linear equations (11)–(13) as follows:

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{1}{G\sin b}\left[I_x\frac{\mathrm{d}\omega_x}{\mathrm{d}t}\cos l - I_y\frac{\mathrm{d}\omega_y}{\mathrm{d}t}\sin l\right], \tag{14}$$

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sin b\left[I_x\frac{\mathrm{d}\omega_x}{\mathrm{d}t}\sin l + I_y\frac{\mathrm{d}\omega_y}{\mathrm{d}t}\cos l\right] + \cos b\, I_z\frac{\mathrm{d}\omega_z}{\mathrm{d}t}, \tag{15}$$

$$\frac{\mathrm{d}b}{\mathrm{d}t} = \frac{\cos b}{G}\left[I_x\frac{\mathrm{d}\omega_x}{\mathrm{d}t}\sin l + I_y\frac{\mathrm{d}\omega_y}{\mathrm{d}t}\cos l\right] - \frac{\sin b}{G}I_z\frac{\mathrm{d}\omega_z}{\mathrm{d}t}, \tag{16}$$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}t}\cos b - \frac{\mathrm{d}b}{\mathrm{d}t}G\sin b. \tag{17}$$

Substituting equations (1)–(3) into equations (14)–(17), one obtains

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{\partial H_a}{\partial L} + \frac{T_x\cos l - T_y\sin l}{G\sin b}, \tag{18}$$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{\partial H_a}{\partial l} + T_z, \tag{19}$$

$$\frac{\mathrm{d}G}{\mathrm{d}t} = T_z\cos b + (T_x\sin l + T_y\cos l)\sin b, \tag{20}$$

where

$$H_a = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_y^2)$$

$$= \frac{1}{2}\left(\frac{\sin^2 l}{I_x} + \frac{\cos^2 l}{I_y}\right)(G^2 - L^2) - \left[\left(\frac{h_x \sin l}{I_x} + \frac{h_y \cos l}{I_y}\right)\sqrt{(G^2 - L^2)} + \frac{h_z L}{I_z}\right] \tag{21}$$
$$+ \frac{1}{2}\frac{L^2}{I_z} + \frac{1}{2}\left(\frac{h_x^2}{I_x} + \frac{h_y^2}{I_y} + \frac{h_z^2}{I_z}\right),$$

$$\begin{aligned}\frac{\partial H_a}{\partial L} &= \frac{(-G\cos b\sin^2 l)}{I_x} + \frac{(-G\cos b\cos^2 l)}{I_y} + \frac{(G\cos b)}{I_z}\\ &\quad + \frac{h_x\cos b\sin l}{I_x\sin b} + \frac{h_y\cos b\cos l}{I_y\sin b} - \frac{h_z}{I_z},\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\partial H_a}{\partial l} &= \frac{(G^2\sin^2 b\sin l\cos l)}{I_x} + \frac{(-G^2\sin^2 b\sin l\cos l)}{I_y}\\ &\quad + \frac{(-h_x G\sin b\cos l)}{I_x} + \frac{(h_y G\sin b\sin l)}{I_y},\end{aligned} \tag{23}$$

$$\frac{\partial H_a}{\partial G} = \frac{(G\sin^2 l)}{I_x} + \frac{(G\cos^2 l)}{I_y} - \frac{h_x\sin l}{I_x\sin b} - \frac{h_y\cos l}{I_y\sin b}. \tag{24}$$

The remaining differential equations for the canonical variables g, H, and h are derived from equations (4)–(6) (see Appendix A). In Figure 3, the angles $(\psi - h)$, g, and $(\phi - l)$ are obtained in terms of the angles I, θ, and b from the usual spherical trigonometry applied to the spherical triangle PNM. From the research of Tong and Tabarrok [10,11], the following identities exist:

$$\cos\theta = \cos I\,\cos b - \sin I\sin b\cos g, \tag{25}$$

$$\sin\theta\sin\phi = \sin I\sin g\cos l + (\cos I\sin b + \sin I\cos b\cos g)\sin l, \tag{26}$$

$$\sin\theta\cos\phi = -\sin I\sin g\sin l + (\cos I\sin b + \sin I\cos b\cos g)\cos l, \tag{27}$$

where I is also the angle between the momentum H and G, and $\cos I = H/G$.

3. CHAOTIC MOTION USING MELNIKOV'S METHOD

Two cases will be studied separately: a quasi-rigid satellite model due to small perturbing torques and a dissipative gyrostat under small perturbation torques.

3.1. CASE 1: QUASI-RIGID SATELLITE MODEL DUE TO SMALL PERTURBING TORQUES

In the case, one assumes that there is no momentum of the momentum exchange devices within the satellite, i.e. $h_x = h_y = h_z = 0$, and that the perturbed external torques are of the following form:

$$T_x = \varepsilon M_x = \varepsilon(M_{x1} + M_{x2}), \tag{28}$$

$$T_y = \varepsilon M_y = \varepsilon(M_{y1} + M_{y2}), \tag{29}$$

$$T_z = \varepsilon M_z = \varepsilon(M_{z1} + M_{z2}), \tag{30}$$

where ε is a small parameter, and

$$M_{x1} = -\beta_x\omega_x\omega_y^2 + \gamma_x\omega_y\omega_z[a_x + b_x\cos(\Omega t) + c_x\cos(2\Omega t)], \tag{31}$$

$$M_{y1} = \beta_y\omega_y(I_x^2\omega_x^2 - I_z^2\omega_z^2) + \gamma_y\omega_x\omega_z[a_y + b_y\cos(\Omega t) + c_y\cos(2\Omega t)], \tag{32}$$

$$M_{z1} = \beta_z\omega_z\omega_y^2 + \gamma_z\omega_x\omega_y[a_z + b_z\cos(\Omega t) + c_z\cos(2\Omega t)], \tag{33}$$

$$M_{x2} = -\mu_x\omega_x + d_x\sin(\Omega t), \tag{34}$$

$$M_{y2} = -\mu_y\omega_y + d_y\sin(\Omega t) + M_0, \tag{35}$$

$$M_{z2} = -\mu_z\omega_z + d_z\sin(\Omega t), \tag{36}$$

where β_i, γ_i, a_i, b_i, c_i, d_i $(i = x, y, z)$ and M_0 are constants and μ_i $(i = x, y, z)$ are positive constants and Ω is the external excitation frequency.

The above-discussed problem could be degenerated to the case studied by Tong and Tabarrok [10, 11] using Deprit's canonical variables if $M_{x1} = M_{y1} = M_{z1} = 0$. The above-discussed problem could also be degenerated to the case investigated by Gray *et al.* [13] using the spherical co-ordinate transformation if $M_{x2} = M_{y2} = M_{z2} = 0$ (i.e., for the case of a quasi-rigid satellite model in going from minor-axis spin to a major-axis spin under the influence of small damping and non-Hamiltonian, time-periodic perturbation) (see Figure 4). The complex dynamics using Deprit's canonical variables in conjunction with

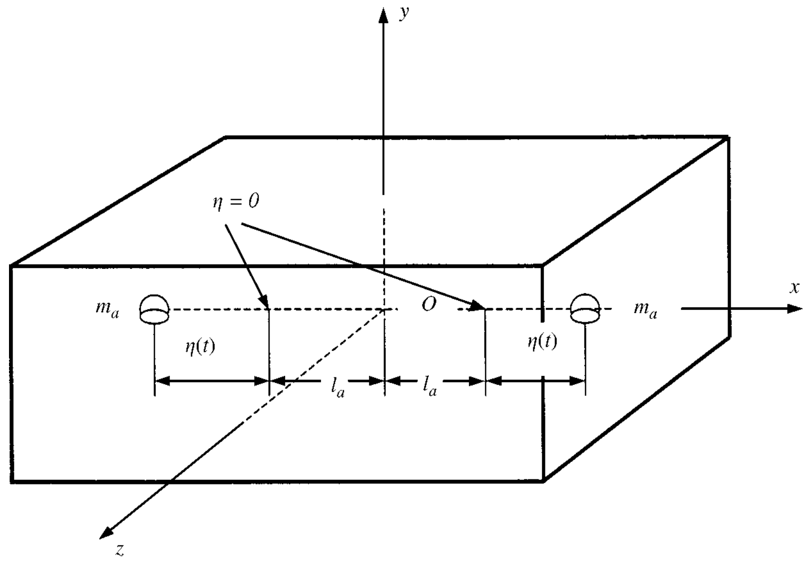

Figure 4. The configuration of model spacecraft with internal moving masses. The two sub-bodies oscillate symmetrically with respect to O along the x-axis. The positions of the moving masses relative to the spacecraft are known periodic functions of time, i.e., $\eta(t) = \eta_0 \sin(\Omega t)$.

Melnikov's method will be examined based upon the development of Wiggins and Shaw [12].

For the torque-free satellite, i.e. $M_x = M_y = M_z = 0$, there are two equilibria for equations (18)–(20). These two equilibria are

$$E_1 : l = 0,\ \pi; \quad L = 0, \tag{37}$$

$$E_2 : l = \frac{\pi}{2},\ \frac{3\pi}{2}; \quad L = 0. \tag{38}$$

According to the researches by Tong and Tabarrok [11], Gray *et al.* [13] and Hughes [17], the solutions for the angular velocities of a torque-free rigid body along the heteroclinic orbits which connect the unstable equilibrium points E_1 are given by

$$\bar{\omega}_x(t) = \omega_{x0} \frac{1}{\cosh(\beta t)}, \tag{39}$$

$$\bar{\omega}_y(t) = \omega_{y0} \frac{\sinh(\beta t)}{\cosh(\beta t)}, \tag{40}$$

$$\bar{\omega}_z(t) = \omega_{z0} \frac{1}{\cosh(\beta t)}, \tag{41}$$

where

$$\omega_{x0} = \pm \frac{G_0}{I_x} \sqrt{\frac{I_x(I_y - I_z)}{I_y(I_x - I_z)}}, \quad \omega_{y0} = \pm \frac{G_0}{I_y}, \quad \omega_{z0} = \pm \frac{G_0}{I_z} \sqrt{\frac{I_z(I_x - I_y)}{I_y(I_x - I_z)}},$$

$$(I_x \bar{\omega}_x(t))^2 + (I_y \bar{\omega}_y(t))^2 + (I_z \bar{\omega}_z(t))^2 = G_0^2 = \text{constant},$$

$$\beta = \sqrt{\left(\frac{G_0^2}{I_y}\right) \frac{(I_x - I_y)(I_y - I_z)}{I_x I_y I_z}}.$$

For the sake of convenience, the dynamical equations (18)–(20) could be rewritten as follows:

$$\frac{\mathrm{d}L}{\mathrm{d}l} = -\frac{\partial H_a}{\partial t}(l(t), L(t)) + \varepsilon f_L(l(t), L(t), t), \tag{42}$$

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{\partial H_a}{\partial L}(l(t), L(t)) + \varepsilon f_l(l(t), L(t), t), \tag{43}$$

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \varepsilon f_G(l(t), L(t), t), \tag{44}$$

where

$$f_L(l(t), L(t), t) = M_z, \tag{45}$$

$$f_l(l(t), L(t), t) = \frac{M_x \cos l - M_y \sin l}{G \sin b} \tag{46}$$

$$f_G(l(t), L(t), t) = M_z \cos b + (M_x \sin l + M_y \cos l) \sin b, \tag{47}$$

with M_y, M_z, and M_x defined in (28)–(30), and

$$\frac{\partial H_a}{\partial L}(l(t),\ L(t)) = \frac{\partial H_a}{\partial L}, \quad \frac{\partial H_a}{\partial l}(l(t),\ L(t)) = \frac{\partial H_a}{\partial l}$$

defined in (22) and (23).

Following Wiggins and Shaw [12] the appropriate Melnikov integral may then be introduced below:

$$\begin{aligned} M(t_0) = &\int_{-\infty}^{+\infty} \frac{\partial H_a}{\partial L}(\bar{l}(t), \bar{L}(t)) f_L(\bar{l}(t),\ \bar{L}(t),\ t + t_0)\,\mathrm{d}t \\ &+ \int_{-\infty}^{+\infty} \frac{\partial H_a}{\partial l}(\bar{l}(t), \bar{L}(t)) f_l(\bar{l}(t),\ \bar{L}(t),\ t + t_0)\,\mathrm{d}t \\ &+ \int_{-\infty}^{+\infty} \frac{\partial H_a}{\partial G}(\bar{l}(t), \bar{L}(t)) f_G(\bar{l}(t),\ \bar{L}(t),\ t + t_0)\,\mathrm{d}t \\ &- \frac{\partial H_a}{\partial G}(0,0) \int_{-\infty}^{+\infty} f_G(\bar{l}(t),\ \bar{L}(t),\ t + t_0)\,\mathrm{d}t, \end{aligned} \tag{48}$$

where

$$\frac{\partial H_a}{\partial G}(0,0) = \frac{\partial H_a}{\partial G}(l(t),\ L(t))\bigg|_{l=0,\,L=0} = \frac{G_0}{I_y}. \tag{49}$$

$(\bar{l}(t),\ \bar{L}(t))$ represents the solution for the heteroclinic orbits in the unperturbed system, where

$$\bar{L}(\mathrm{t}) = L_0 \frac{1}{\cosh(\beta t)}, \tag{50}$$

in which $L_0 = \pm G_0 \sqrt{I_z(I_x - I_y)/I_y(I_x - I_z)}$, and

$$\tan[\bar{l}(t)] = \pm \sqrt{\frac{I_x(I_y - I_z)}{I_y(I_x - I_z)}} \frac{1}{\sinh(\beta t)}. \tag{51}$$

Substituting equations (22)–(24) and (45)–(47) into the Melnikov integral (48), one obtains

$$\begin{aligned} M(t_0) = &\int_{-\infty}^{+\infty} \left[M_x(\bar{l}(t), \bar{L}(t), t + t_0) \left(\frac{1}{I_x} - \frac{1}{I_y} \right) I_x \bar{\omega}_x(t) \right] \mathrm{d}t \\ &+ \int_{-\infty}^{+\infty} \left[M_z(\bar{l}(t), \bar{L}(t), t + t_0) \left(\frac{1}{I_z} - \frac{1}{I_y} \right) \bar{L}(t) \right] \mathrm{d}t \end{aligned}$$

$$= \int_{-\infty}^{+\infty} [A'' \mathrm{sech}^2(\beta t) \tanh^2(\beta t)]\, \mathrm{d}t$$

$$+ \int_{-\infty}^{+\infty} [B'' \cos(\Omega(t+t_0))\, \mathrm{sech}^2(\beta t) \tanh(\beta t)]\, \mathrm{d}t$$

$$+ \int_{-\infty}^{+\infty} [C'' \cos(2\Omega(t+t_0))\, \mathrm{sech}^2(\beta t) \tanh(\beta t)]\, \mathrm{d}t$$

$$+ \int_{-\infty}^{+\infty} [D'' \mathrm{sec}h^2(\beta t) + E'' \sin(\Omega(t+t_0))\, \mathrm{sech}(\beta t)]\, \mathrm{d}t, \quad (52)$$

where

$$A'' = -\left(\frac{1}{I_x} - \frac{1}{I_y}\right) I_x \omega_{x0}^2 \omega_{y0}^2 \beta_x + \left(\frac{1}{I_z} - \frac{1}{I_y}\right) L_0 \omega_{z0} \omega_{y0}^2 \beta_z,$$

$$B'' = \left(\frac{1}{I_x} - \frac{1}{I_y}\right) I_x \omega_{x0} \omega_{y0} \omega_{z0} \gamma_x b_x + \left(\frac{1}{I_z} - \frac{1}{I_y}\right) L_0 \omega_{x0} \omega_{y0} \gamma_z b_z,$$

$$C'' = \left(\frac{1}{I_x} - \frac{1}{I_y}\right) I_x \omega_{x0} \omega_{y0} \omega_{z0} \gamma_x c_x + \left(\frac{1}{I_z} - \frac{1}{I_y}\right) L_0 \omega_{x0} \omega_{y0} \gamma_z c_z,$$

$$D'' = -\left(\frac{1}{I_x} - \frac{1}{I_y}\right) I_x \omega_{x0}^2 \mu_x - \left(\frac{1}{I_z} - \frac{1}{I_y}\right) L_0 \omega_{z0} \mu_z,$$

$$E'' = \left(\frac{1}{I_x} - \frac{1}{I_y}\right) I_x \omega_{x0} d_x + \left(\frac{1}{I_z} - \frac{1}{I_y}\right) L_0 d_z.$$

Therefore, the Melnikov's function could be simplified into the following form via the residue theorem of complex variable theory [19]:

$$M(t_0) = A + B \sin(\Omega t_0) + C \sin(2\Omega t_0), \quad (53)$$

where

$$A = \frac{2(A'' + 3D'')}{3\beta},$$

$$B = -B'' \frac{\Omega^2 \pi}{2\beta^3 \sinh(\Omega\pi/2\beta)} + E'' \frac{\pi}{\beta} \mathrm{sech}\left(\frac{\Omega\pi}{2\beta}\right),$$

$$C = -C'' \frac{2\Omega^2 \pi}{\beta^3 \sinh(\Omega\pi/\beta)}.$$

From formulae (53), the condition of existence of simple zeros for $M(t_0) = 0$ is

$$\left|\frac{A}{F_{max}}\right| \leqslant 1, \quad (54)$$

where

$$F_{max} = \frac{3}{4}(3B + \sqrt{B^2 + 32C^2})\sqrt{\frac{1}{2} + \frac{B}{32C^2}(\sqrt{B^2 + 32C^2} - B)}$$

Inequality (54) is the theoretic condition for the exhibition of chaotic dynamics near the heteroclinic orbits for sufficiently small ε. Figure 5(a)–5(c) show the bifurcation curve (A/F_{max} versus Ω) defined by equation (54) for the case

$$I_x = 1.0\,\mathrm{kg\,m^2},\quad I_y = 0.8\,\mathrm{kg\,m^2},\quad I_z = 0.4\,\mathrm{kg\,m^2},\quad m_a = 0.5\,\mathrm{kg},$$

$$\mu_x = 0.01\,\mathrm{N\,m\,s},\quad \mu_y = 0.01\,\mathrm{N\,m\,s},\quad \mu_z = 0.01\,\mathrm{N\,m\,s},$$

$$d_x = 350\,\mathrm{N\,m},\quad d_y = 350\,\mathrm{N\,m},\quad d_z = 350\,\mathrm{N\,m},$$

$$l_a = 0.5\,\mathrm{m},\quad \eta_0 = 0.001\,\mathrm{m},\quad \beta_c = 0.05.$$

Remark. In the case of quasi-rigid satellite model studied by Gray *et al.* [13], the corresponding parameters can be identified as (see Figure 4)

$$\begin{cases} a_x = a_y = a_z = m_a\eta_0^2, \\ b_x = b_y = b_z = 4m_a\eta_0 l_a, \\ c_x = c_y = c_z = m_a\eta_0^2, \end{cases} \qquad \begin{cases} I_x = I_{x\,rigid}, \\ I_y = I_{y\,rigid} + 2m_a l_a^2, \\ I_z = I_{z\,rigid} + 2m_a l_a^2, \end{cases}$$

$$\begin{cases} \gamma_x = \dfrac{(I_z^2 - I_y^2)}{I_z I_y}\left(\dfrac{I_y}{m_a\, l_a^2}\right), \\[2ex] \gamma_y = \dfrac{I_x}{I_z}\left(\dfrac{I_y}{m_a\, l_a^2}\right), \\[2ex] \gamma_z = -\dfrac{I_x}{I_y}\left(\dfrac{I_y}{m_a l_a^2}\right), \end{cases} \qquad \begin{cases} \beta_x = \beta_c\, I_y^2\, I_x\left(\dfrac{I_y}{m_a\, l_a^2}\right), \\[2ex] \beta_y = \beta_c I_y\left(\dfrac{I_y}{m_a\, l_a^2}\right), \\[2ex] \beta_z = \beta_c\, I_y^2\, I_z\left(\dfrac{I_y}{m_a l_a^2}\right), \end{cases}$$

where m_a is the mass of one of the sub-bodies inside the satellites, η_0 is the amplitude of the oscillation of the sub-bodies, l_a is the characteristic position of the sub-body, and β_c is a positive-valued control gain to be specified. In reference [13], the small parameter is defined as $\varepsilon = m_a\, l_a^2/I_y$. It is noted that the physical parameters I_x, I_y, and I_z (the moment of inertia) consist of the contributions of the rigid part ($I_{x\,rigid}$, $I_{y\,rigid}$, and $I_{z\,rigid}$) and the sub-bodies respectively.

Figure 5(a) shows the bifurcation curve for a satellite with sub-bodies. The initial magnitude of the angular momentum is $G_0 = 18$ N m s. Its maximum frequency that will excite the chaotic dynamics is approximately 7.7 rad/s. Figure 5(b) shows the bifurcation curve for a satellite with sub-bodies. The initial magnitude of the angular momentum is $G_0 = 20$ N m s. Its maximum frequency that will excite the chaotic dynamics is approximately 5.4 rad/s. Figure 5(c) shows the bifurcation curve for a satellite with sub-bodies. The initial magnitude of the angular momentum is $G_0 = 21$ N m s. Its maximum frequency that will excite the chaotic dynamics is approximately 3.7 rad/s. From Figure 5(a)–(c), one can conclude that for the quasi-rigid satellite model, the combined physical parameters (A/F_{max} in equation (54)) vary according to the external excitation frequencies.

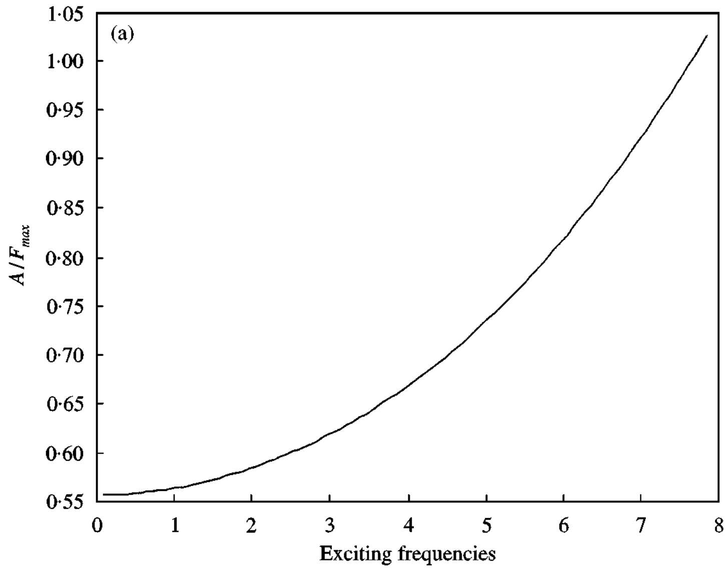

Figure 5(a). The bifurcation curve for a satellite with sub-bodies when $G_0 = 18\,\mathrm{N\,m\,s}$.

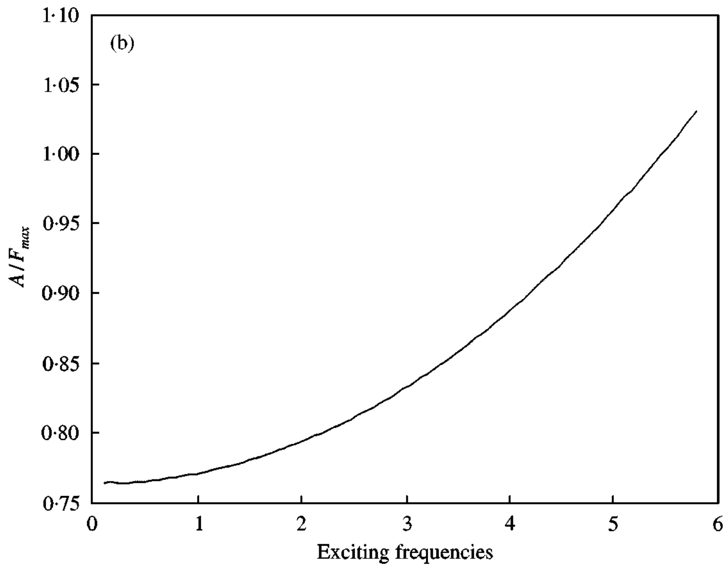

Figure 5(b). The bifurcation curve for a satellite with sub-bodies when $G_0 = 20\,\mathrm{N\,m\,s}$.

The range of the excitation frequencies that can excite the occurring of the chaotic dynamics will decrease as the initial moment of momentum of the entire system increases.

3.2. CASE 2: GYROSTAT UNDER SMALL PERTURBATION TORQUES BEING DISSIPATIVE

For the torque-free rotation of the gyrostat there are two saddle points at $l = 0$ and $l = \pi$ in case $h_x = h_y = 0$, and $h_z \neq 0$ [8]. Analyzing the equations (42)–(44), in case

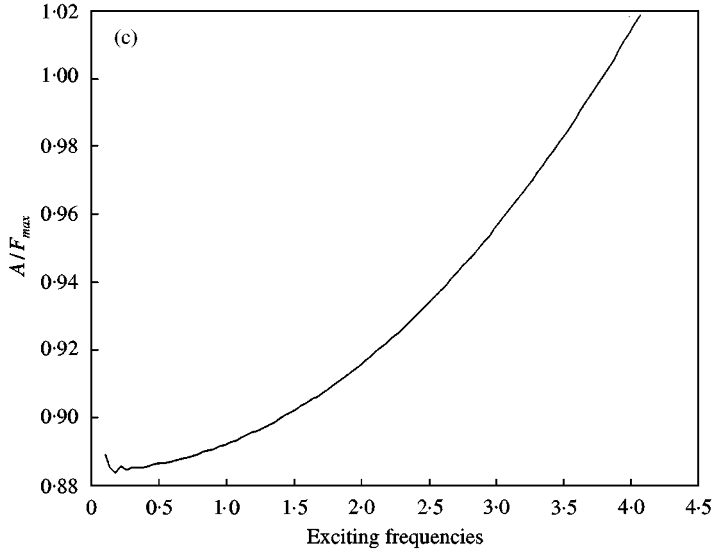

Figure 5(c). The bifurcation curve for a satellite with sub-bodies when $G_0 = 21\,\mathrm{N\,m\,s}$.

$M_x = M_y = M_z = 0$, one obtains the solution $(\bar{\omega}_x(t), \bar{\omega}_y(t), \bar{\omega}_z(t))$ or $(\bar{L}(t), \bar{l}(t))$ along the heteroclinic orbits. Here one considers the attitude motion of a dissipative gyrostat under small perturbation torques. In this case, the Melnikov function can be derived as

$$
\begin{aligned}
M(t_0) &= \int_{-\infty}^{+\infty} \frac{\partial H_a}{\partial L}(\bar{l}(t), \bar{L}(t)) f_L(\bar{l}(t), \bar{L}(t), t + t_0)\,\mathrm{d}t \\
&\quad + \int_{-\infty}^{+\infty} \frac{\partial H_a}{\partial l}(\bar{l}(t), \bar{L}(t)) f_l(\bar{l}(t), \bar{L}(t), t + t_0)\,\mathrm{d}t \\
&\quad + \int_{-\infty}^{+\infty} \frac{\partial H_a}{\partial G}(\bar{l}(t), \bar{L}(t)) f_G(\bar{l}(t), \bar{L}(t), t + t_0)\,\mathrm{d}t \\
&\quad - \frac{\partial H_a}{\partial G}\left(0, \frac{I_y}{I_y - I_z} h_z\right) \int_{-\infty}^{+\infty} f_G(\bar{l}(t), \bar{L}(t), t + t_0)\,\mathrm{d}t \\
&= \int_{-\infty}^{+\infty} M_x(\bar{l}(t), \bar{L}(t), t + t_0)\left[\bar{\omega}_x(t) - \left(\frac{G_0 - h_y}{I_y}\right)\frac{I_x\bar{\omega}_x(t) + h_x}{G_0}\right]\mathrm{d}t \\
&\quad + \int_{-\infty}^{+\infty} M_y(\bar{l}(t), \bar{L}(t), t + t_0)\left[\bar{\omega}_y(t) - \left(\frac{G_0 - h_y}{I_y}\right)\frac{I_y\bar{\omega}_y(t) + h_y}{G_0}\right]\mathrm{d}t \qquad (55) \\
&\quad + \int_{-\infty}^{+\infty} M_z(\bar{l}(t), \bar{L}(t), t + t_0)\left[\bar{\omega}_z(t) - \left(\frac{G_0 - h_y}{I_y}\right)\frac{I_z\bar{\omega}_z(t) + h_z}{G_0}\right]\mathrm{d}t,
\end{aligned}
$$

where $\partial H_a/\partial L$, $\partial H_a/\partial l$, and $\partial H_a/\partial G$ are defined in equations (22)–(24).

According to Or [16] and Masaitis [20], the solutions along the heteroclinic orbits when $h_x = 0$, $h_y = 0$, and $h_z \neq 0$ are given as follows:

$$\bar{\omega}_x(t) = \frac{A_x \operatorname{sech}(\lambda(t - t_b))}{\tanh^2(\lambda(t - t_b)) + k}, \tag{56}$$

$$\bar{\omega}_y(t) = \frac{A_y \tanh(\lambda(t - t_b)) \operatorname{sech}(\lambda(t - t_b))}{\tanh^2(\lambda(t - t_b)) + k}, \tag{57}$$

$$\bar{\omega}_z(t) = \frac{r_1 \tanh^2(\lambda(t - t_b)) + k r_3}{\tanh^2(\lambda(t - t_b)) + k}, \tag{58}$$

where the initial time t_b starts at the saddle point.

$$a_1 = \frac{I_z(I_y - I_z)}{I_x(I_x - I_y)}, \quad b_1 = -\frac{2h_z I_z}{I_x(I_x - I_y)}, \quad c_1 = \frac{G_0^2 - 2TI_y - h_z^2}{I_x(I_x - I_y)},$$

$$a_2 = \frac{I_z(I_x - I_z)}{I_y(I_y - I_x)}, \quad b_2 = -\frac{2h_z I_z}{I_y(I_y - I_x)}, \quad c_2 = \frac{G_0^2 - 2TI_x - h_z^2}{I_y(I_y - I_x)},$$

$$I_x(\bar{\omega}_x(t))^2 + I_y(\bar{\omega}_y(t))^2 + I_z(\bar{\omega}_z(t))^2 = 2T,$$

$$(I_x\bar{\omega}_x(t))^2 + (I_y\bar{\omega}_y(t))^2 + (I_z\bar{\omega}_z(t) + h_z)^2 = G_0^2,$$

$$r_{1,2} = \frac{-b_1 \pm \sqrt{b_1^2 - 4a_1c_1}}{2a_1} \quad (\text{if } b_1^2 - 4a_1c_1 > 0),$$

$$r_{3,4} = \frac{-b_2 \pm \sqrt{b_2^2 - 4a_2c_2}}{2a_2} \quad (\text{if } b_2^2 - 4a_2c_2 > 0),$$

$$k = \frac{r_2 - r_1}{r_3 - r_2}, \quad \lambda = \frac{(I_x - I_y)}{2I_z}\sqrt{a_1a_2(r_3 - r_2)(r_1 - r_2)},$$

$$A_x = k\sqrt{a_1(r_3 - r_1)(r_3 - r_2)}, \quad A_y = \sqrt{ka_2(r_1 - r_3)(r_3 - r_2)}.$$

One can derive the condition of existence for the heteroclinic solutions (56)–(58) to the equations (1)–(3) under no external torques as follows (see Appendix B):

$$G_0^2 = h_z^2 + 2TI_z \pm 2\sqrt{2h_z^2 I_z T}.$$

Further, one assumes that the external disturbance is as follows:

$$T_x = \varepsilon M_x = \varepsilon(-\mu_x\omega_x + d_x \sin(\Omega t)), \tag{59}$$

$$T_y = \varepsilon M_y = \varepsilon(-\mu_y\omega_y + d_y \sin(\Omega t)), \tag{60}$$

$$T_z = \varepsilon M_z = \varepsilon(-\mu_z\omega_z + d_z \sin(\Omega t)), \tag{61}$$

where μ_i $(i = x, y)$ are positive constants and d_i $(i = x, y, z)$ are constants; hereafter one assumes that $\mu_z = d_z = 0$.

Using the transformations (7)–(9), the Melnikov function (55) can be changed into the following form:

$$\begin{aligned} M(t_0) &= \int_{-\infty}^{+\infty} [-\mu_x \bar{\omega}_x(t) + d_x \sin(\Omega(t+t_0))] \left(\frac{1}{I_x} - \frac{1}{I_y}\right) I_x \bar{\omega}_x(t)\,\mathrm{d}t \\ &= J_1 + J_2 \sin(\Omega t_0), \end{aligned} \tag{62}$$

where

$$\begin{aligned} J_1 &= \int_{-\infty}^{+\infty} \frac{\alpha\cosh(2\lambda t) + \beta}{(\sinh^2(\lambda t) + k\cosh^2(\lambda t))^2}\,\mathrm{d}t \\ &= \frac{2\alpha\left(\frac{\pi}{2} - \theta_a\right)\sin(2\theta_a) + (\alpha\cos(2\theta_a) + \beta)}{2\lambda(k+1)^2\cos^2\theta_a\sin^2\theta_a} \\ &\quad + \frac{2\alpha\left(\frac{\pi}{2} - \theta_b\right)\sin(2\theta_b) + (\alpha\cos(2\theta_b) + \beta)}{2\lambda(k+1)^2\cos^2\theta_b\sin^2\theta_b} \\ &\quad + \frac{\left(\frac{\pi}{2} - \theta_a\right)(\alpha\cos(2\theta_a) + \beta)(\cos^2\theta_a - \sin^2\theta_a)}{2\lambda(k+1)^2\cos^3\theta_a\sin^3\theta_a} \\ &\quad + \frac{\left(\frac{\pi}{2} - \theta_b\right)(\alpha\cos(2\theta_b) + \beta)(\cos^2\theta_b - \sin^2\theta_b)}{2\lambda(k+1)^2\cos^3\theta_b\sin^3\theta_b}. \end{aligned} \tag{63}$$

Here,

$$\alpha = -\frac{1}{2}\mu_x I_x A_x^2\left(\frac{1}{I_x} - \frac{1}{I_y}\right), \quad \beta = -\frac{1}{2}\mu_x I_x A_x^2\left(\frac{1}{I_x} - \frac{1}{I_y}\right),$$

where $\mathrm{i}\theta_a + \mathrm{i}\theta_b = \mathrm{i}\pi$ (i is the imaginary unit) and $0 < \theta_a < \pi$ and $0 < \theta_b < \pi$; $\mathrm{i}\theta_a$ and $\mathrm{i}\theta_b$ are the imaginary roots of the following equations:

$$\sinh(\mathrm{i}\theta_a) - \mathrm{i}\sqrt{k}\cosh(\mathrm{i}\theta_a) = 0,$$

$$\sinh(\mathrm{i}\theta_b) + \mathrm{i}\sqrt{k}\cosh(\mathrm{i}\theta_b) = 0$$

and

$$\begin{aligned} J_2 &= \int_{-\infty}^{+\infty} \frac{\alpha'\cosh(\lambda t)\cos(\Omega t)}{\sinh^2(\lambda t) + k\cosh^2(\lambda t)}\,\mathrm{d}t \\ &= \frac{\alpha'\pi}{\lambda(k+1)(1+\cosh(\pi\Omega/\lambda))}\left[\frac{\cosh((\Omega/\lambda)/\theta_a)}{\sin\theta_a} + \frac{\cosh((\Omega/\lambda)/\theta_b)}{\sin\theta_b}\right]. \end{aligned} \tag{64}$$

Here, $\alpha' = (1/I_x - 1/I_y)\, d_x I_x A_x$ (see Appendix C for details).

According to the Melnikov function (62), it can be verified that if

$$\left|\frac{J_1}{J_2}\right| \leqslant 1, \tag{65}$$

then there is a t_0 such that

$$M(t_0) = 0 \quad \text{and} \quad \frac{\partial M}{\partial t}(t)\bigg|_{t=t_0} \neq 0. \tag{66}$$

Thus, there exist transversal intersections of stable and unstable manifolds for the perturbed gyrostats. Figure 6(a)–6(c) show the bifurcation curves (J_1/J_2 (defined by equations (65)) versus Ω) for the following parameters:

$$I_x = 1.0\ \mathrm{kg\,m^2}, \quad I_y = 0.8\ \mathrm{kg\,m^2} \quad I_z = 0.4\ \mathrm{kg\,m^2}, \quad G_0 = 18\ \mathrm{N\,m\,s},$$

$$\mu_x = 0.08\ \mathrm{N\,m\,s}, \quad \mu_z = 0.0, \quad d_x = -100\ \mathrm{N\,m}, \quad d_z = 0.0, \quad T = 80\ \mathrm{N\,m}.$$

Figure 6(a) shows the bifurcation curve for a gyrostat under the small external disturbance torques when the magnitude of the angular momentum of the wheel $h_z = 9$ N m s. The maximum frequency that will excite the chaotic dynamics is approximately 16.7 rad/s. Figure 6(b) shows the bifurcation curve for a gyrostat under the small external disturbance torques when the magnitude of the angular momentum of the wheel $h_z = 10$ N m s. The maximum frequency that will excite the chaotic dynamics is approximately 13.7 rad/s. Figure 6(c) shows the bifurcation curve for a gyrostat under the small external disturbance torques when the magnitude of the angular momentum of the wheel $h_z = 11$ N m s. The maximum frequency that will excite the chaotic dynamics is approximately 9.5 rad/s. The

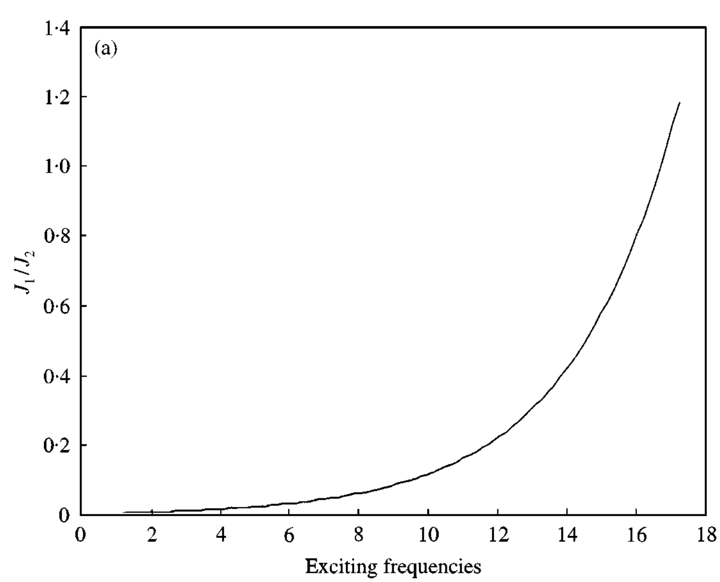

Figure 6(a). The bifurcation curve for a perturbed gyrostat when $h_z = 9$ N m s.

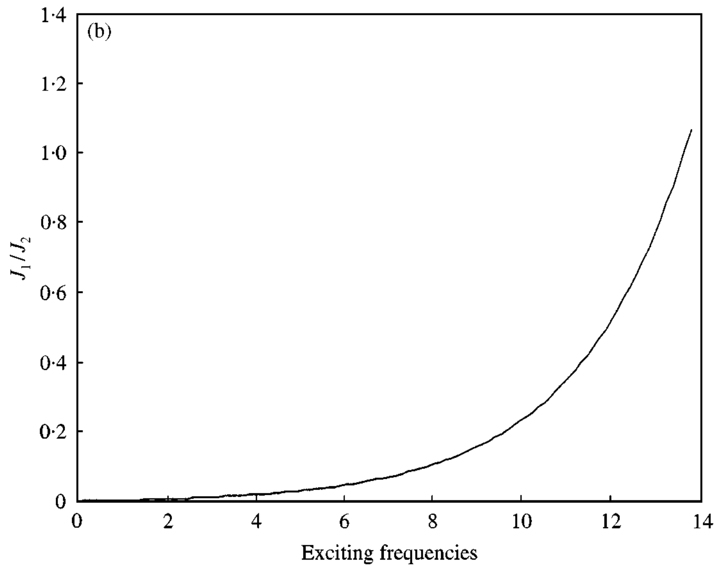

Figure 6(b). The bifurcation curve for a perturbed gyrostat when $h_z = 10\,\mathrm{N\,m\,s}$.

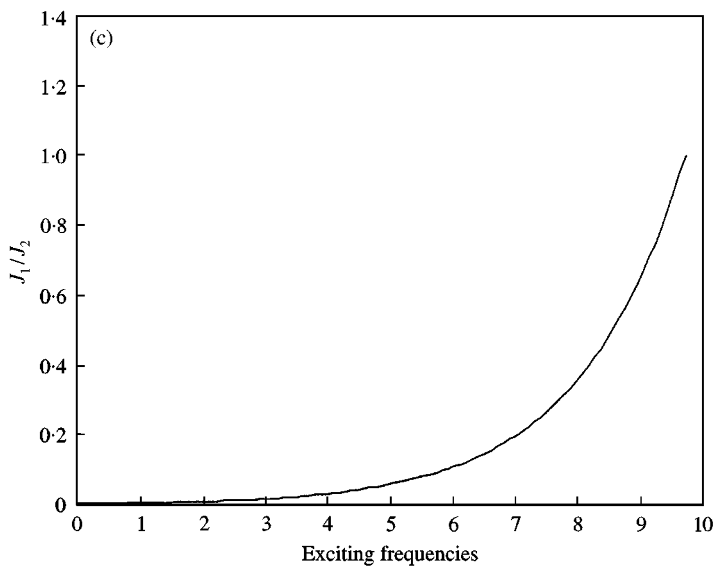

Figure 6(c). The bifurcation curve for a perturbed gyrostat when $h_z = 11\,\mathrm{N\,m\,s}$.

range of the excitation frequencies that can excite the occurring of the chaotic dynamics will decrease as the moment of the momentum of the wheel increases.

One chooses the external exciting frequency as $\Omega = 10$ rad/s in order to account for the steady state behavior of the attitude motion equations (1)–(3) by using a fourth order Runge–Kutta scheme. Figure 7a and 7b is the two-dimensional phase portraits showing the chaotic motion about the homoclinic solutions. The rest of the corresponding parameters

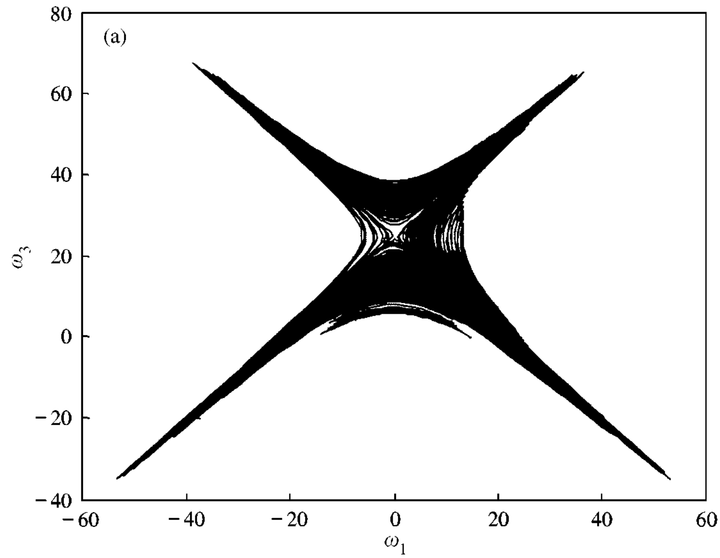

Figure 7(a). A projection (ω_1 versus ω_3) of three-dimensional phase portrait.

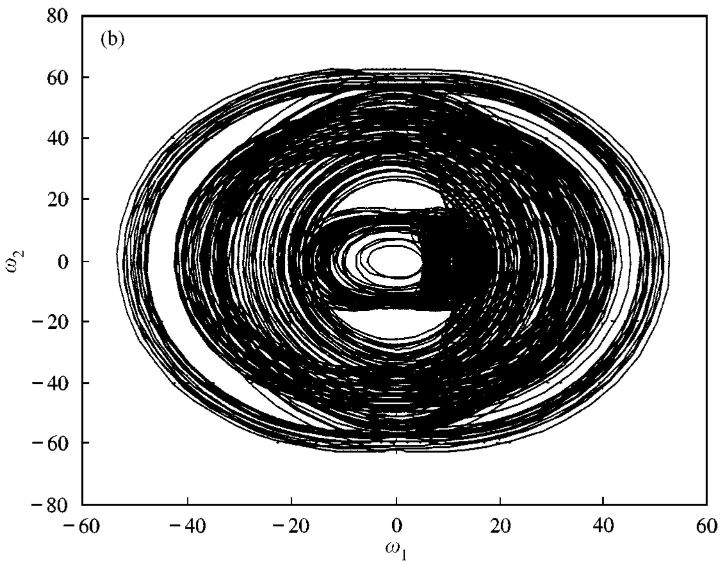

Figure 7(b). A projection (ω_1 versus ω_2) of three-dimensional phase portrait.

are given as follows:

$$\mu_y = 0.08\,\mathrm{N\,m\,s}, \quad d_y = -100\,\mathrm{N\,m}, \quad \varepsilon = 0.05, \quad h_z = 10\,\mathrm{N\,m\,s}.$$

From the phase curves (Figure 7a and 7b) and the printed-out data, one knows that the perturbed gyrostats exhibit non-periodic solutions that possess many characteristics of the randomness.

4. CONCLUSIONS

In this paper, the six-dimensional ordinary differential equations governing the attitude motion of the gyrostats in terms of Deprit canonical variables are derived in the form of Hamiltonian equations directly from the ordinary differential equations of the Euler's angular momentum and from Euler angles' attitude kinematic equations, by using the Deprit's transformations (see Appendix A). With the help of *Mathematica* software, the conditions of physical parameters for the existence of heteroclinic solutions to the attitude motion of gyrostats are formulated (see Appendix B). By using the Melnikov integral, one obtains the theoretical criteria of the chaotic attitude motion of two model configurations, which are the quasi-rigid, energy-dissipating satellite model subject to non-Hamiltonian, time-periodic and small perturbation torques, and a gyrostat model under small perturbation torques. For the quasi-rigid satellite the bifurcation curves are computed, which describe the combined physical parameters (A/F_{max} in equation (54)) varying according to the external excitation frequencies. The range of the excitation frequencies that can excite the occurring of the chaotic dynamics will decrease as the initial moment of momentum of the entire system increases. For the gyrostat model, the bifurcation curves are also computed, which depict the combined physical parameters (J_1/J_2 in equation (65)) varying according to the external excitation frequencies. The range of the excitation frequencies that can excite the occurring of the chaotic dynamics will decrease as the moment of momentum of the wheel increases. From the fourth-order Runge–Kutta integration method, the complex attitude solutions of the perturbed gyrostats are solved to show that the perturbed gyrostat possesses many random characteristics of a non-periodic solution which are theoretically proved to be chaotic by using the Melnikov technique.

APPENDIX A

The differential equations about canonical variables g, H, and h can be derived from equations (4)–(6). From equations (4)–(6) one obtains

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega_z - \frac{(\omega_x \sin\theta \sin\phi + \omega_y \sin\theta \cos\phi)\cos\theta}{\sin^2\theta}, \tag{A.1}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{(\omega_x \sin\theta \cos\phi - \omega_y \sin\theta \sin\phi)}{\sin\theta}, \tag{A.2}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{(\omega_x \sin\theta \sin\phi + \omega_y \sin\theta \cos\phi)}{\sin^2\theta}, \tag{A.3}$$

Using the following identities from the spherical triangle PNM (see Figure 3), one has

$$\cos\theta = \cos I \cos b - \sin I \sin b \cos g, \tag{A.4}$$

$$\sin\theta \cos(\phi - l) = \cos I \sin b + \sin I \cos b \cos g, \tag{A.5}$$

$$\sin\theta \sin(\phi - l) = \sin I \sin g, \tag{A.6}$$

$$\sin\theta \cos(\psi - h) = \sin I \cos b + \cos I \sin b \cos g, \tag{A.7}$$

$$\sin\theta \sin(\psi - h) = \sin b \sin g. \tag{A.8}$$

From the transformation (7)–(9), one obtains

$$\omega_x = \frac{G \sin b \sin l - h_x}{I_x}, \tag{A.9}$$

$$\omega_y = \frac{G \sin b \cos l - h_y}{I_y}, \tag{A.10}$$

$$\omega_z = \frac{G \cos b - h_z}{I_z}. \tag{A.11}$$

One sets

$$S_1(\theta, \phi, l) = \sin\theta \cos(\phi - l), \tag{A.12}$$

$$F_1(I, b, g) = \cos I \sin b + \sin I \cos b \cos g, \tag{A.13}$$

$$S_2(\theta, \phi, l) = \sin\theta \sin(\phi - l), \tag{A.14}$$

$$F_2(I, b, g) = \sin I \sin g, \tag{A.15}$$

$$S_3(\theta, \psi, h) = \sin\theta \cos(\psi - h), \tag{A.16}$$

$$F_3(I, b, g) = \sin I \cos b + \cos I \sin b \cos g. \tag{A.17}$$

According to the identities (A.4)–(A.7), one knows that

$$S_1(\theta, \phi, l) = F_1(I, b, g), \tag{A.18}$$

$$S_2(\theta, \phi, l) = F_2(I, b, g), \tag{A.19}$$

$$S_3(\theta, \psi, h) = F_3(I, b, g), \tag{A.20}$$

Taking the time derivative of equations (A.18)–(A.20), one obtains

$$\frac{\partial S_1}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\partial S_1}{\partial \phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\partial S_1}{\partial l}\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{\partial F_1}{\partial I}\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{\partial F_1}{\partial g}\frac{\mathrm{d}g}{\mathrm{d}t} + \frac{\partial F_1}{\partial b}\frac{\mathrm{d}b}{\mathrm{d}t}, \tag{A.21}$$

$$\frac{\partial S_2}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\partial S_2}{\partial \phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\partial S_2}{\partial l}\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{\partial F_2}{\partial I}\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{\partial F_2}{\partial g}\frac{\mathrm{d}g}{\mathrm{d}t} + \frac{\partial F_2}{\partial b}\frac{\mathrm{d}b}{\mathrm{d}t}, \tag{A.22}$$

$$\frac{\partial S_3}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\partial S_3}{\partial \psi}\frac{\mathrm{d}\psi}{\mathrm{d}t} + \frac{\partial S_3}{\partial h}\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial F_3}{\partial I}\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{\partial F_3}{\partial g}\frac{\mathrm{d}g}{\mathrm{d}t} + \frac{\partial F_3}{\partial b}\frac{\mathrm{d}b}{\mathrm{d}t}, \tag{A.23}$$

where

$$\begin{cases} \dfrac{\partial S_1}{\partial \theta} = \dfrac{\cos\theta}{\sin\theta}(\cos I \sin b + \sin I \cos b \cos g), \\ \dfrac{\partial S_1}{\partial \phi} = -\sin I \sin g, \\ \dfrac{\partial S_1}{\partial l} = \sin I \sin g, \end{cases} \quad \begin{cases} \dfrac{\partial S_2}{\partial \theta} = \dfrac{\cos\theta}{\sin\theta}\sin I \sin g, \\ \dfrac{\partial S_2}{\partial \phi} = \cos I \sin b + \sin I \cos b \cos g, \\ \dfrac{\partial S_2}{\partial l} = -(\cos I \sin b + \sin I \cos b \cos g), \end{cases}$$

$$\begin{cases} \dfrac{\partial S_3}{\partial \theta} = \dfrac{\cos\theta}{\sin\theta}(\sin I \cos b + \cos I \sin b \cos g), \\ \dfrac{\partial S_3}{\partial \psi} = -\sin b \sin g, \\ \dfrac{\partial S_3}{\partial h} = \sin b \sin g, \end{cases} \quad \begin{cases} \dfrac{\partial F_1}{\partial I} = -\sin I \sin b + \cos I \cos b \cos g, \\ \dfrac{\partial F_1}{\partial g} = -\sin I \cos b \sin g, \\ \dfrac{\partial F_1}{\partial b} = \cos I \cos b - \sin I \sin b \cos g, \end{cases}$$

$$\begin{cases} \dfrac{\partial F_2}{\partial I} = \cos I \sin g, \\ \dfrac{\partial F_2}{\partial g} = \sin I \cos g, \\ \dfrac{\partial F_2}{\partial b} = 0, \end{cases} \quad \begin{cases} \dfrac{\partial F_3}{\partial I} = \cos I \cos b - \sin I \sin b \cos g, \\ \dfrac{\partial F_3}{\partial g} = -\cos I \sin b \sin g, \\ \dfrac{\partial F_3}{\partial b} = -\sin I \sin b + \cos I \cos b \cos g. \end{cases}$$

Differentiating the following equations:

$$G\cos b = L, \quad G\cos I = H$$

with respect to the time t, one obtains

$$\frac{dG}{dt}\cos b - G\frac{db}{dt}\sin b = \frac{dL}{dt}, \tag{A.24}$$

$$\frac{dG}{dt}\cos I - G\frac{dI}{dt}\sin I = \frac{dH}{dt}, \tag{A.25}$$

where dl/dt, dL/dt, and dG/dt are defined in equations (18), (19), and (20) respectively. The quantities $d\theta/dt$, $d\phi/dt$, and $d\psi/dt$ are defined in equations (A.1), (A.2) and (A.3) respectively. Using the identities (A.5)–(A.8) and their respective partial derivatives, one can find out the quasi-linear equations (A.21)–(A.25) with respect to the unknowns db/dt, dI/dt, dH/dt, dh/dt, and dg/dt.

One may obtain

$$\begin{aligned} \frac{dH}{dt} = {} & T_x[(\sin I \sin g)\cos l + (\cos I \sin b + \sin I \cos b \cos g)\sin l] \\ & + T_y[-(\sin I \sin g)\sin l + (\cos I \sin b + \sin I \cos b \cos g)\cos l] \\ & + T_z[\cos I \cos b - \sin I \sin b \cos g], \end{aligned} \tag{A.26}$$

$$\begin{aligned}\frac{\mathrm{d}g}{\mathrm{d}t} = {} & \frac{(G\sin b\sin l - h_x)\sin l}{I_x\sin b} + \frac{(G\sin b\cos l - h_y)\cos l}{I_y\sin b} \\ & + T_x\,\frac{-(\sin I\cos b + \cos I\sin b\cos g)\cos l + (\cos I\sin b\cos b\sin g)\sin l}{G\sin I\sin b} \\ & + T_y\,\frac{(\sin I\cos b + \cos I\sin b\cos g)\sin l + (\cos I\sin b\cos b\sin g)\cos l}{G\sin I\sin b} \\ & + T_z\,\frac{(-\cos I\sin b\sin g)}{G\sin I},\end{aligned} \tag{A.27}$$

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} = {} & T_x\,\frac{\cos g\cos l - \cos b\sin g\sin l}{G\sin I} \\ & - T_y\,\frac{\cos g\sin l + \cos b\sin g\cos l}{G\sin I} \\ & + T_z\,\frac{\sin b\sin g}{G\sin I}.\end{aligned} \tag{A.28}$$

Equations (18)–(20) and (A.26)–(A.28) are the six-dimensional ordinary differential equations governing the attitude motion of the satellite using Deprit variables. These six-dimensional ordinary differential equations are of Hamiltonian form. The Hamiltonian function is written in equation (21).

APPENDIX B

When $h_x = h_y = 0$, $h_z \neq 0$, there are two constants of motion for the ordinary differential equations (1)–(3) as follows:

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T = \text{constant}, \tag{B.1}$$

$$(I_x\omega_x)^2 + (I_y\omega_y)^2 + (I_z\omega_z + h_z)^2 = G_0^2 = \text{constant}. \tag{B.2}$$

From equations (B.1) and (B.2), one can obtain

$$\omega_x^2 = a_1\omega_z^2 + b_1\omega_z + c_1 = a_1(\omega_z - r_1)(\omega_z - r_2), \tag{B.3}$$

$$\omega_y^2 = a_2\omega_z^2 + b_2\omega_z + c_2 = a_2(\omega_z - r_3)(\omega_z - r_4), \tag{B.4}$$

where a_i, b_i, c_i, and r_i $(i = 1, 2)$ are defined in Case 2 (see equations (56)–(58)). Substituting equations (B.3) and (B.4) into equation (3), one obtains

$$\frac{\mathrm{d}\omega_z}{\mathrm{d}t} = \pm\frac{(I_x - I_y)}{I_z}\sqrt{a_1a_2(\omega_z - r_1)(\omega_z - r_2)(\omega_z - r_3)(\omega_z - r_4)}. \tag{B.5}$$

The conditions of existence for the heteroclinic solutions (56)–(58) to equations (1)–(3) are

$$r_1 > r_2 = r_4 > r_3. \tag{B.6}$$

From the equation $r_2 = r_4$, one gets

$$\frac{-b_1 - \sqrt{b_1^2 - 4a_1c_1}}{2a_1} = \frac{-b_2 - \sqrt{b_2^2 - 4a_2c_2}}{2a_2}. \tag{B.7}$$

Substituting a_i, b_i, and c_i $(i = 1, 2)$ into this equation and using the software *Mathematica* [15], one finds

$$\frac{16(I_x - I_z)^2(I_y - I_z)^2 I_z^6 [G_0^4 - 2(h_z^2 + 2TI_z)G_0^2 + (h_z^2 - 2TI_z)^2]}{I_x^4(I_x - I_y)^6 I_y^4} = 0$$

Because of the assumption $I_x > I_y > I_z$, the condition of existence for the heteroclinic solutions (56)–(58) to equations (1)–(3) can be further derived as follows:

$$G_0^2 = h_z^2 + 2TI_z \pm 2\sqrt{2h_z^2 I_z T}. \tag{B.8}$$

APPENDIX C

Using the idea suggested by Or [16], in order to apply the residue theory to the Melnikov integrals, one constructs a closed circuit C enclosing the two poles $z = \mathrm{i}\theta_a$ and $z = \mathrm{i}\theta_b$. One chooses the circuit C as a rectangular strip in the upper-half complex plane. It runs counter-clockwise from $z = -R$ to $z = R$, to $z = R + \mathrm{i}\pi$, then $z = -R + \mathrm{i}\pi$, and then back to $z = -R$. By applying $R \to \infty$ and applying the residue theorem one obtains

$$J_1 = \int_{-\infty}^{+\infty} \frac{(\alpha\cosh(2\lambda t) + \beta)}{(\sinh^2(\lambda t) + k\cosh^2(\lambda t))^2}\,\mathrm{d}t = \left(\frac{1}{\pi \mathrm{i}}\right)\oint_C \frac{f(z)}{h(z)}\,\mathrm{d}z, \tag{C.1}$$

where

$$f(z) = \left(\frac{\pi \mathrm{i}}{2} - z\right)(\alpha\cosh(2z) + \beta), \quad h(z) = \lambda(\sinh^2 z + k\cosh^2 z)^2.$$

Expanding $f(z)$ and $h(z)$ into Taylor series in the neighbourhood of z_0, one obtains

$$f(z) = f_0(z_0) + f_1(z_0)(z - z_0) + \tfrac{1}{2} f_2(z_0)(z - z_0)^2 + \tfrac{1}{6} f_3(z_0)(z - z_0)^3 + \cdots,$$

$$h(z) = h_0(z_0) + h_1(z_0)(z - z_0) + \tfrac{1}{2} h_2(z_0)(z - z_0)^2 + \tfrac{1}{6} h_3(z_0)(z - z_0)^3 + \cdots,$$

where $h_0 = h(z_0) = 0$, $h_1(z_0) = (\mathrm{d}h/\mathrm{d}z)(z_0) = 0$,

$$h_2(z_0) = \frac{\mathrm{d}^2 h}{\mathrm{d}z^2}(z_0) = 8\lambda(k + 1)^2\sinh^2(z_0)\cosh^2(z_0),$$

$$h_3(z_0) = \frac{\mathrm{d}^3 h}{\mathrm{d}z^3}(z_0) = 24\lambda(k + 1)^2(\sinh^2(z_0) + \cosh^2(z_0))\sinh(z_0)\cosh(z_0),$$

$$f_0(z_0) = \left(\frac{\pi i}{2} - z_0\right)(\alpha\cosh(2z_0) + \beta),$$

$$f_1(z_0) = \frac{\mathrm{d}f}{\mathrm{d}z}(z_0) = \left(\frac{\pi i}{2} - z_0\right)(2\alpha\cosh(2z_0)) - (\alpha\cosh(2z_0) + \beta).$$

Therefore,

$$\begin{aligned} J_1 &= \frac{2\pi \mathrm{i}}{\pi \mathrm{i}}\left[\lim_{z\to a}\frac{\mathrm{d}}{\mathrm{d}z}\left((z-a)^2\frac{f(z)}{h(z)}\right) + \lim_{z\to b}\frac{\mathrm{d}}{\mathrm{d}z}\left((z-b)^2\frac{f(z)}{h(z)}\right)\right] \\ &= 4\left(\frac{f_1(a)}{h_2(a)} - \frac{f_0(a)h_3(a)}{3[h_2(a)]^2}\right) + 4\left(\frac{f_1(b)}{h_2(b)} - \frac{f_0(b)h_3(b)}{3[h_2(b)]^2}\right), \end{aligned} \tag{C.2}$$

where $a = \mathrm{i}\theta_a$, $b = \mathrm{i}\theta_b$.

From equation (C.2), it is easy to derive the analytical form of the integral J_1 in equation (63). Similarly

$$\begin{aligned} J_2 &= \int_{-\infty}^{+\infty}\frac{\alpha\cosh(\lambda t)\cos(\Omega t)}{(\sinh^2(\lambda t) + k\cosh^2(\lambda t))}\mathrm{d}t \\ &= \int_{-\infty}^{+\infty}\frac{\alpha'\cosh z\cos(\Omega(z/\lambda))}{\lambda(\sinh^2 z + k\cosh^2 z)}\mathrm{d}z \\ &= \frac{\alpha'}{(1+\cosh(\Omega(\pi/\lambda)))}\oint_C\frac{\cosh z\cos(\Omega(z/\lambda))}{\lambda(\sinh^2 z + k\cosh^2 z)}\mathrm{d}z \\ &= \frac{\alpha' 2\pi \mathrm{i}}{(1+\cosh(\Omega(\pi/\lambda)))}\lim_{z\to a}\left((z-a)\frac{\cosh z\cos(\Omega(z/\lambda))}{\lambda(\sinh^2 z + k\cosh^2 z)}\right) \\ &\quad + \frac{\alpha' 2\pi \mathrm{i}}{(1+\cosh(\Omega(\pi/\lambda)))}\lim_{z\to b}\left((z-b)\frac{\cosh z\cos(\Omega(\mathrm{z}/\lambda))}{\lambda(\sinh^2 z + k\cosh^2 z)}\right) \\ &= \frac{\alpha'\pi \mathrm{i}}{\lambda(k+1)(1+\cosh(\Omega(\pi/\lambda)))}\left[\frac{\cos(\Omega(a/\lambda))}{\sinh a} + \frac{\cos(\Omega(b/\lambda))}{\sinh b}\right]. \end{aligned} \tag{C.3}$$

From equation (C.3), integral J_2 in equation (64) can be derived easily.

REFERENCES

1. V. VOLTERRA 1898 *Acta Mathematica* **22**, 1898, 201–357. Sur la Theóries des Variations des Latitudes.
2. V. V. RUMYANTSEV 1964 *Advances in Applied Mechanics*, **8**, 183–232. On the Lyapunov's methods in the study of stability of motions of rigid bodies with fluid-filled cavities.
3. J. WITTENBURG 1975 *Accademia Nazionale dei Lincei*, *Quaderno N*. **217**, 1–187. Beitrage Zur Dynämik von Gynostaten.
4 Y. S. CHEN and A. Y. T. LEUNG 1998 *Bifurcation and Chaos in Engineering*. London: Springer-Verlag.
5. V. K. MELNIKOV 1963 *Transactions Moscow Mathematical Society* **12**, 1–57. On the stability of the centre for time-periodic perturbations.
6. P. J. HOLMES and J. E. MARSDEN 1982 *Journal of Mathematics and Physics* **23**, 669–675. Melnikov's method and Arnold diffusion for perturbations of integrable Hamiltonian systems.

7. P. J. Holmes and J. E. Marsden 1983 *Indiana University Mathematical Journal* **32**, 273–309. Horseshoe and Arnold diffusion for Hamiltonian system on Lie groups.
8. X. Tong, B. Tabarrok and F. P. J. Rimrott 1995 *International Journal of Non-linear Mechanics* **30**, 191–203. Chaotic motion of an asymmetric gyrostat in the gravitational field.
9. P. J. Holmes and J. E. Marsden 1982 *Communications in Mathematical Physics* **82**, 523–544. Horseshoes in perturbations of Hamiltonian systems with two degrees of freedom.
10. X. Tong and B. Tabarrok 1996 *Archives of Applied Mechanics* **66**, 215–230. Melnikov's method for rigid bodies subject to small perturbation torques.
11. X. Tong and B. Tabarrok 1997 *Journal of Guidance, Control, and Dynamics* **20**, 1997, 123–128. Bifurcation of self-excited rigid bodies subjected to small perturbation torques.
12. S. Wiggins and S. W. Shaw 1998 *Transactions ASME/Journal of Applied Mechanics* **55**, 959–968. Chaos and three-dimensional horseshoe in slowly varying oscillators.
13. G. L. Gray, I. Dobson and D. C. Kammer 1996, *Transactions ASME/Journal of Applied Mechanics* **63**, 501–508. Chaos in a spacecraft attitude manoeuvre due to time-periodic perturbations.
14. G. L. Gray, A. P. Mazzoleni and D. R. Campbell 1998 *Journal of Guidance, Control, and Dynamics* **21**, 558–565. Analytical criterion for chaotic dynamics in flexible satellites with nonlinear controller damping.
15. S. Wolfram, 1991 *Mathematica: A System for Doing Mathematics by Computers*. Red Wood City, CA: Addison-Wesley, second edition.
16. A. C. Or 1998 *Transactions ASME/Journal of Applied Mechanics* **65**, 150–156 Chaotic motions of a dual-spin body.
17. P. C. Hughes 1986 *Spacecraft Attitude Dynamics*. New York: Wiley.
18. A. Deprit 1967 *American Journal of Physics* **35**, 424–427. A free rotation of a rigid body studied in the phase plane.
19. I. S. Gradshteyn and I. M. Ryzhik 1983 *Table of Integrals, Series, and Products: Corrected and Enlarged Edition*. New York: Academic Press.
20. C. Masaitis 1961 *Archives for Rational Mechanics and Analysis* **8**, 23–35. One the motion of two linked bodies.