Journal of Sound and Vibration (2000) **235**(2), 201–217
doi:10.1006/jsvi.1999.2925, available online at http://www.idealibrary.com on IDEAL®

ON INTERACTIONS OF OSCILLATION MODES FOR A WEAKLY NON-LINEAR UNDAMPED ELASTIC BEAM WITH AN EXTERNAL FORCE

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(*Received* 20 *October* 1998, *and in final form* 14 *December* 1999)

In this paper an initial–boundary value problem for the vertical displacement of a weakly non-linear elastic beam with an harmonic excitation in the horizontal direction at the ends of the beam is studied. The initial–boundary value problem can be regarded as a simple model describing oscillations of flexible structures like suspension bridges or iced overhead transmission lines. Using a two-time-scales perturbation method an approximation of the solution of the initial–boundary value problem is constructed. Interactions between different oscillation modes of the beam are studied. It is shown that for certain external excitations, depending on the phase of an oscillation mode, the amplitude of specific oscillation modes changes.

1. INTRODUCTION

Flexible structures, like tall buildings, suspension bridges or iced overhead transmission lines with bending stiffness, are subjected to oscillations due to different causes. Simple models which describe these oscillations can involve non-linear second and fourth order partial differential equations (PDEs), as can be seen, for example, in references [1] or [2]. In many cases, perturbation methods can be used to construct approximations for solutions of this type of second or fourth order equations. Initial–boundary value problems for second order PDEs have been considered for a long time, for instance, in references [3–9]. These problems have been studied in references [2, 10–15], using a two-time-scales perturbation method or a Galerkin-averaging method to construct the approximations. For fourth order PDEs the analysis is more complex. In a number of papers [1, 16–18], approximations for solutions of initial–boundary value problems for fourth order weakly non-linear PDEs are constructed using perturbation methods. In most cases, the solutions are approximated by a single-mode representation, without justification as to whether truncation to one mode is valid. In this paper, approximations are constructed using a two-time-scales perturbation method. The interaction between the different oscillation modes is studied and a justification is given in which cases mode truncation is valid. For fourth order strongly non-linear PDEs numerical finite element methods can be used, as is done for example in reference [19].

In this paper, we will consider the following initial-boundary value problem, which describes, up to $\mathcal{O}(\varepsilon)$, the vertical displacement of an elastic beam with a linear spring force and a constant gravity force acting on it, and with an external force $F(t)$ acting on the ends of the beam in horizontal direction:

$$w_{tt} + w_{xxxx} + p^2 w = \varepsilon \left(F(t) + \frac{2}{\pi} \int_0^\pi w_x^2 \, dx \right) w_{xx}, \quad 0 < x < \pi, \, t > 0, \tag{1}$$

$$w(0, t) = w(\pi, t) = 0, \quad t \geqslant 0, \tag{2}$$

$$w_{xx}(0, t) = w_{xx}(\pi, t) = 0, \quad t \geqslant 0, \tag{3}$$

$$w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad 0 < x < \pi, \tag{4}$$

where $F(t) = u(\pi, t) - u(0, t)$ and ε a small dimensionless parameter. For the derivation of this problem we refer to section 2.

In this paper, formal approximations, i.e., functions that satisfy the differential equation and the initial and boundary values up to some order in ε, will be constructed for the initial–boundary value problem (1)–(4), using a Fourier-mode expansion and a two-time-scales perturbation method. The interaction (energy exchange) between the different oscillation modes will be considered for the cases $F(t) = 0$ (no external forcing, that is, the ends of the beams are fixed in horizontal direction, i.e., the case of free vibrations) and $F(t) = C\cos(\omega t)$ (external forcing). It will be shown that in the case $F(t) = 0$ the amplitudes of the different modes are constant and the only interaction between the modes occurs in the phases of the different oscillations modes (for example, mode n causes a phase shift of the phases of all other modes $m \neq n$). No internal resonances occur. In this paper, we mean by internal resonance that there is an energy transfer from one oscillation mode to another oscillation mode. So by no internal resonance we mean no energy transfer occurs between the different oscillation modes (up to $\mathcal{O}(\varepsilon)$ on a time-scale of order ε^{-1}). The case $F(t) = C\cos(\omega t)$ is more complicated. It will be shown that for most values of ω the analysis is similar to the case $F(t) = 0$. The influence of $F(t)$ in that case is of $\mathcal{O}(\varepsilon)$ on a time-scale of order ε^{-1}, and extra terms appear in the $\mathcal{O}(\varepsilon)$-approximation. However, for specific values of ω, i.e., $\omega \approx 2\omega_{k_p}$, where ω_{k_p} is an eigenfrequency of the linearized system ($\varepsilon = 0$), the influence of $F(t)$ is of $\mathcal{O}(1)$ on a time-scale of order ε^{-1}. The amplitude of mode k is no longer constant, but the amplitudes of all other modes remain constant. The mode interactions remain restricted to phase shifts of the phases of the different oscillation modes. Similar mode interactions have been studied for example in references [11, 12, 20, 21], but to our knowledge these mode interactions for weakly non-linear beam equations have not yet been studied thoroughly. The analysis presented in this paper hold for all p^2-values, which is different from the analysis in references [11] or [12], where mode interactions and internal resonances occur for specific p^2-values.

The outline of the paper is as follows. In section 2, the initial–boundary value problem (1)–(4) will be derived. In section 3, we apply a two-time-scales perturbation method to the initial–boundary value problem (1)–(4). We show that for most values of ω the amplitudes of the different oscillation modes remain constant. For specific ω-values the oscillation of specific modes changes and the amplitudes of certain modes are no longer constant. We construct a formal approximation of $\mathcal{O}(\varepsilon)$ for solutions of the initial–boundary value problem for the cases $F(t) = 0$, $F(t) = C\cos(\omega t)$ with $\omega \neq 2\omega_{k_p} + \varepsilon\alpha$ and $F(t) = C\cos(\omega t)$ with $\omega = 2\omega_{k_p} + \varepsilon\alpha$, where $\alpha \in \mathbb{R}$ is a detuning parameter. In section 4, the mode interactions between the different oscillation modes will be studied in detail for the three cases mentioned above. In Section 5, some conclusions and general remarks will be given.

2. A MATHEMATICAL FORMULATION OF THE PROBLEM

To derive the equations of motion for an elastic beam we will follow part of the analysis given in reference [22]. We consider an elastic beam of length l, simply supported in a vertical direction. An external force will be applied at the ends of the beam such that the ends of the beam can move in horizontal direction only. Oscillations are possible due to the strain of the beam. The x-axis is defined to be the horizontal axis. The z-axis is defined to be the vertical axis. The y-axis is perpendicular to the (x, z)-plane. We introduce the following symbols: μ is the mass of the beam per unit length, ρ the mass density of the beam, A the area of the cross-section Q of the beam perpendicular to the x-axis (so $\mu = \rho A$), E the elasticity modulus (Young's modulus), I the axial moment of inertia of the cross-section. The inertial axes of the cross-section Q are the y- and z-axes, so $I = \iint_Q z^2\,dy\,dz$. We assume that the beam can move in the x- and z-directions only. The vertical displacement of the beam from rest is $w = w(x, t)$, the horizontal displacement of the beam is $u = u(x, t)$. The curvature of the beam in the (x, z)-plane can be approximated by w_{xx} as follows. From Figure 1 we can see that the radius $\mathscr{R}$ of the curvature is given by $\mathscr{R}\Delta\varphi \approx \Delta s$, where $\Delta\varphi$ and Δs are defined in Figure 1. Furthermore, $\tan\Delta\varphi \approx \Delta w/\Delta x$ and $\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta w)^2}$. For $\Delta x \to 0$ this gives us $\mathscr{R} = (1 + w_x^2)^{3/2}/w_{xx}$. Assuming that w_x is small with respect to 1, we can approximate the curvature, which is equal to $1/\mathscr{R}$, by w_{xx}. Using this, the strain ε_{xx} due to "pure" bending of a line-element of the beam at a distance z from the line of centroids (the x-axis) is given by

$$\varepsilon_{xx} = \frac{(\mathscr{R} - z)\Delta\varphi - \pi\Delta\varphi}{\mathscr{R}\Delta\varphi} = -\frac{z}{\mathscr{R}} \approx -zw_{xx}.$$

Furthermore, the strain ε_{x0} due to stretching of the line of centroids of a line-element of the beam can be approximated by $u_x + \frac{1}{2}w_x^2$ as follows. From Figure 2 and the definition of strain due to stretching, which can be found in any standard textbook on mechanics (see, for example, reference [23]) we have the following expression for ε_{x0}:

$$\varepsilon_{x0} = \frac{\sqrt{(\Delta x + \Delta u)^2 + (\Delta w)^2} - \Delta x}{\Delta x}.$$

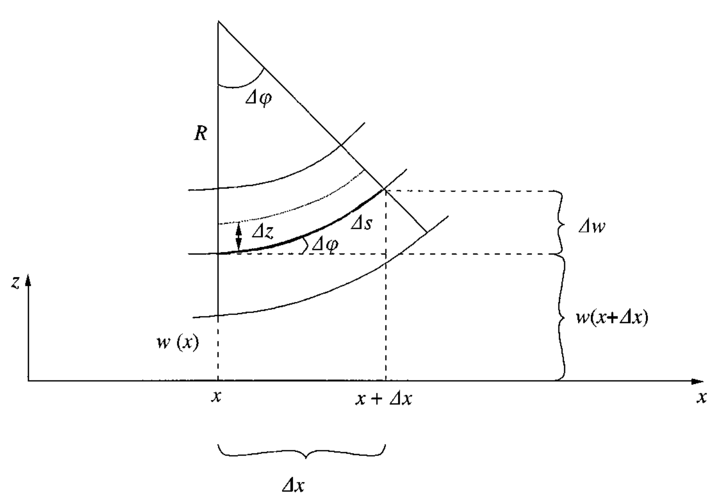

Figure 1. The bending of a line-element Δx.

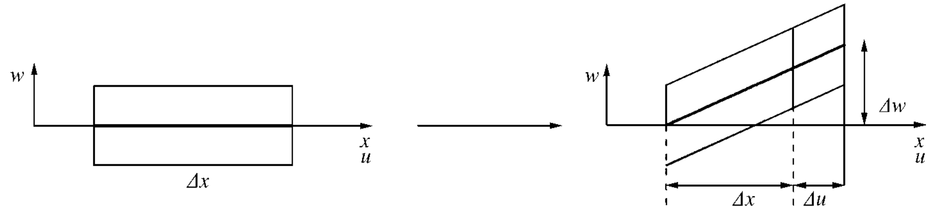

Figure 2. The stretching of a line-element Δx.

For $\Delta x \to 0$ this gives us $\varepsilon_{x0} = \sqrt{1 + 2u_x + u_x^2 + w_x^2} - 1$. By assuming that u_x^2 is small with respect to u_x, and by expanding the square root as a Taylor series, we have $\varepsilon_{x0} \approx u_x + \frac{1}{2}w_x^2$. The total strain of a line-element of the beam at a distance z from the x-axis is given by $\varepsilon_x = \varepsilon_{x0} + \varepsilon_{xx} = u_x + \frac{1}{2}w_x^2 - zw_{xx}$. It is shown in reference [22] that, using Hooke's Law, the work performed to deflect the beam from its initial position, is

$$\mathscr{A}(t) = \frac{1}{2}EA\int_0^l \left[u_x + \frac{1}{2}w_x^2\right]^2 \mathrm{d}x + \frac{1}{2}EI\int_0^l (w_{xx})^2\,\mathrm{d}x. \tag{5}$$

The kinetic energy of the beam is given by

$$\mathscr{E}_k(t) = \frac{1}{2}\mu\int_0^l [u_t^2 + w_t^2]\,\mathrm{d}x. \tag{6}$$

Using equations (5) and (6) the Hamiltonian integral is

$$\begin{aligned}\mathscr{F} = \mathscr{F}(t_2) - \mathscr{F}(t_1) &= \int_{t_1}^{t_2} (\mathscr{A}(t) - \mathscr{E}_k(t))\,\mathrm{d}t \\ &= \frac{1}{2}\int_{t_1}^{t_2}\int_0^l \left\{EA\left[u_x + \frac{1}{2}w_x^2\right]^2 + EI(w_{xx})^2 - \mu[\mu_t^2 + w_t^2]\right\}\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{7}$$

Using Hamilton's Principle, which states that the variation of $\mathscr{F}$ is equal to 0, the Euler equations for this problem are

$$\mu u_{tt} - EA\frac{\partial}{\partial x}\left[u_x + \frac{1}{2}w_x^2\right] = 0, \tag{8}$$

$$\mu w_{tt} + EIw_{xxxx} - EA\frac{\partial}{\partial x}\left\{w_x\left[u_x + \frac{1}{2}w_x^2\right]\right\} = 0. \tag{9}$$

The system given by equations (8)–(9) can be simplified by the following assumption, introduced by Kirchhoff (see reference [24]): the velocity of the beam in x-direction, u_t, is small compared to w_t and can be neglected in equation (7), so $\mathscr{F} = \frac{1}{2}\int_{t_1}^{t_2}\int_0^l\{EA[u_x + \frac{1}{2}w_x^2]^2 + EI(w_{xx})^2 - \mu[w_t^2]\}\,\mathrm{d}x\mathrm{d}t$. The system given by equations (8)–(9) can now be simplified to

$$EA\frac{\partial}{\partial x}\left[u_x + \frac{1}{2}w_x^2\right] = 0, \tag{10}$$

$$\mu w_{tt} + EIw_{xxxx} - EAw_{xx}\left[u_x + \frac{1}{2}w_x^2\right] = 0. \tag{11}$$

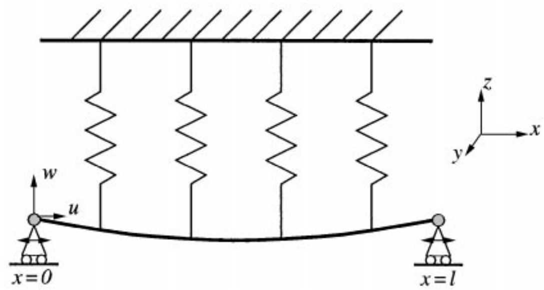

Figure 3. A simple model of a suspension bridge.

From equation (10) we get $u_x + \frac{1}{2} w_x^2 = \varepsilon_{x0}$ is a function of t only. Integrating ε_{x0} with respect to x from 0 to l gives us $\int_0^l (u_x + \frac{1}{2} w_x^2) \mathrm{d}x = \varepsilon_{x0} l$, which means $u(l,t) - u(0,t) + \frac{1}{2}\int_0^l w_x^2 \mathrm{d}x = \varepsilon_{x0} l = (u_x + \frac{1}{2} w_x^2)\, l$. Substituting this into equation (11) gives us the following equation for the vertical displacement w:

$$\mu w_{tt} + EI w_{xxxx} - \frac{EA}{l}\left[u(l,t) - u(0,t) + \frac{1}{2}\int_0^l w_x^2 \,\mathrm{d}x\right] w_{xx} = 0. \tag{12}$$

If other external forces are considered, the right-hand side of equation (12) becomes non-zero.

In reference [1] a survey of literature on oscillations of suspension bridges is given. Using a similar analysis, we will derive simplified model for non-linear oscillations in suspension bridges, where the vertical displacement of an elastic beam is given by equation (12). We model the suspension bridge as a beam of length l. In this paper, the stays of the bridges are modelled as two-sided springs, as sketched in Figure 3. In reference [12], the stays of the bridge are modelled as two-sided springs with a small non-linearity (εw^2). A next step would be to model the springs using w^+ and w^-, as is done, for example, in reference [1]. The torsional vibration of the beam is not taken into account (that is, is considered to be small compared to the vertical vibration). We introduce the following symbols: κ, the spring constant of the stays of the bridge, and W, the weight of the bridge per unit length, which we consider to be constant, i.e., $W = \mu g$, with g the gravitational acceleration. The equation describing the vertical displacement of the beam then is

$$\mu w_{tt} + EI w_{xxxx} + \kappa w = -\mu g + \frac{EA}{l}\left[u(l,t) - u(0,t) + \frac{1}{2}\int_0^l w_x^2 \,\mathrm{d}x\right] w_{xx}. \tag{13}$$

Equation (13) will be simplified by eliminating the term $-\mu g$ using $w = \tilde{w} + (\mu g/\kappa)s(x)$, where $s(x)$ satisfies the following time-independent linear equation with boundary conditions:

$$s^{(4)}(x) + \frac{\kappa}{EI}s(x) = -\frac{\kappa}{EI}, \quad 0 < x < l,$$

$$s(0) = s(l) = 0, \qquad s^{(2)}(0) = s^{(2)}(l) = 0.$$

It can be shown that with $\beta = (\kappa/4EI)^{1/4}$, $s(x) = \cos(\beta x)(\cosh(\beta x) - 1 + (\sin(\beta l)\sin(\beta x)\cosh(\beta x) - (\sinh(\beta l)\cos(\beta x)\sinh(\beta x))/(\cos(\beta l) + \cosh(\beta l))$. The term $(\mu g/\kappa)s(x)$ represents the deflection of the beam in static state due to gravity.

Using the dimensionless variables

$$\bar{w} = \frac{l}{A}\,\bar{w}, \qquad \bar{x} = \frac{\pi}{l}\,x, \quad \bar{t} = \left(\frac{\pi}{l}\right)^2 \left(\frac{EI}{\mu}\right)^{1/2} t, \qquad \bar{u} = \frac{4}{\pi}\left(\frac{l}{A}\right)^2 \frac{l}{\pi}\,u,$$

equation (13) becomes

$$\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + \left(\frac{l}{\pi}\right)^4 \frac{\kappa}{EI}\,\bar{w}$$

$$= \frac{A}{I}\frac{A}{l}\left(\frac{1}{4}\frac{A}{l}\left(\bar{u}(\pi,\bar{t}) - \bar{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \bar{w}_x^2\,\mathrm{d}\bar{x}\right)\bar{w}_{\bar{x}\bar{x}} + \frac{1}{4}\frac{\mu g}{\kappa}\,\mathscr{H}\right) \tag{14}$$

with

$$\mathscr{H} = \left(\bar{u}(\pi,\bar{t}) - \bar{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \bar{w}_x^2\,\mathrm{d}\bar{x}\right) s^{(2)}\left(\frac{l}{\pi}\,\bar{x}\right)$$

$$+ \frac{4}{\pi}\left[\int_0^\pi \bar{w}_{\bar{x}} s^{(1)}\left(\frac{l}{\pi}\,\bar{x}\right)\mathrm{d}\bar{x}\left(\bar{w}_{\bar{x}\bar{x}} + \frac{\mu g}{\kappa}\frac{l}{A}\,s^{(2)}\left(\frac{l}{\pi}\,\bar{x}\right)\right)\right]$$

$$+ \frac{2}{\pi}\frac{\mu g}{\kappa}\frac{l}{A}\left[\int_0^\pi \left(s^{(1)}\left(\frac{l}{\pi}\,\bar{x}\right)\right)^2 \mathrm{d}\bar{x}\left(\bar{w}_{\bar{x}\bar{x}} + \frac{\mu g}{\kappa}\frac{l}{A}\,s^{(2)}\left(\frac{l}{\pi}\,\bar{x}\right)\right)\right].$$

Assuming that the area A of the cross-section is small compared to the length l, we put $\bar{\varepsilon} = A/l$, with $\bar{\varepsilon}$ a small parameter. We assume w, and therefore $\bar{w}$, to be of $\mathcal{O}(\bar{\varepsilon})$. Furthermore, we assume that the deflection of the beam in a static state due to gravity, $(\mu g/\kappa)s$, is small with respect to the vertical displacement $\bar{w}$, which is of order $\bar{\varepsilon}$. This means we assume $\mu g/\kappa$ is $\mathcal{O}(\bar{\varepsilon}^n)$, with $n > 1$, since $s(x)$ is of order 1, as can be seen from the expression for s which was given above (as well as $s^{(1)}(x)$ and $s^{(2)}(x)$). Since $\mathscr{H} = \mathcal{O}(1)$, equation (14) becomes

$$\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + p^2\bar{w} = \frac{1}{4}\frac{A}{I}\left[\bar{\varepsilon}^2\left(\bar{u}(\pi,\bar{t}) - \bar{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \bar{w}_x^2\,\mathrm{d}\bar{x}\right)\bar{w}_{\bar{x}\bar{x}} + \mathcal{O}(\bar{\varepsilon}^m)\right]$$

with $m > 2$ and $p^2\bar{w} = (l/\pi)^4\kappa/EI$. Setting $\varepsilon = \frac{1}{4}(A/I)\bar{\varepsilon}^2$, we can now introduce the following initial–boundary value problem, which describes, up to $\mathcal{O}(\varepsilon^n)$, $n > 1$, the vertical displacement of an elastic beam with a linear spring force and a constant gravity force acting on it, and with an external force $F(t)$ acting on the ends of the beam in horizontal direction:

$$w_{tt} + \omega_{xxxx} + p^2 w = \varepsilon\left(F(t) + \frac{2}{\pi}\int_0^\pi w_x^2\,\mathrm{d}x\right) w_{xx}, \quad 0 < x < \pi, \quad t > 0, \tag{15}$$

$$w(0,t) = w(\pi,t) = 0, \quad t \geqslant 0, \tag{16}$$

$$w_{xx}(0,t) = w_{xx}(\pi,t) = 0, \quad t \geqslant 0, \tag{17}$$

$$w(x,0) = w_0(x), \qquad w_t(x,0) = w_1(x), \quad 0 < x < \pi, \tag{18}$$

where $F(t) = u(\pi, t) - u(0, t)$ and ε a small dimensionless parameter. In this paper, we are interested in an harmonic excitation of the ends of the beam in horizontal direction, which means we take $F(t) = C\cos(\omega t)$, with C a constant $\neq 0$ which represents the amplitude of the external excitation and ω the frequency of the external excitation. Since we consider u_t small compared to w_t, it can be shown that ω must be of $\mathcal{O}(1)$. Furthermore, ω can be taken positive without loss of generality. Furthermore, ε and p are constants with $0 < \varepsilon \ll 1$ and $p > 0$, $w = w(x, t)$ is the vertical displacement of the beam, x is the co-ordinate along the beam, $w_0(x)$ is the initial displacement of the beam in vertical direction and $w_1(x)$ is the initial velocity of the beam in vertical direction. All functions are assumed to be sufficiently smooth. The first two terms on the left-hand side of equation (15) are the linear part of the beam equation, $p^2 w$ represents the linear restoring force of the spring, $(\int_0^\pi w_x^2 \mathrm{d}x) w_{xx}$ is due to the strain of the beam and $F(t) w_{xx}$ is due to an external force acting on the ends of the beam in horizontal direction. The boundary conditions describe a simply supported beam. As we showed above, the initial–boundary value problem (15)–(18) can be considered as a simple model for non-linear oscillations in suspension bridges. In the next section, a formal approximation of the solution of (15)–(18) will be constructed.

3. THE CONSTRUCTION OF FORMAL APPROXIMATIONS—GENERAL CASE

In this and the next section, we construct a formal approximation of the solution of the initial–boundary value problem (15)–(18). When straightforward ε-expansions are used to approximate solutions, secular terms can occur in the approximations. To avoid these secular terms we use a two-time-scales perturbation method.

The boundary conditions imply that w can be written as a Fourier sine-series in x: $w(x, t) = \sum_{m=1}^{\infty} q_m(t) \sin(mx)$. Substituting this series in equation (15), we obtain the following system of equations:

$$\sum_{k=1}^{\infty} (\ddot{q}_k + (k^4 + p^2) q_k) \sin(kx) = -\varepsilon \left(\sum_{k=1}^{\infty} \left(F(t) + \sum_{m=1}^{\infty} m^2 q_m^2 \right) k^2 q_k \sin(kx) \right).$$

Using orthogonality properties of the sine functions on $[0, \pi]$ it can be shown easily that the equation for each q_n is

$$\ddot{q}_n + (n^4 + p^2) q_n = -\varepsilon \left(F(t) + \sum_{m=1}^{\infty} m^2 q_m^2 \right) n^2 q_n \tag{19}$$

for $n = 1, 2, 3, \ldots$, with $F(t) = C\cos(\omega t)$ and where q_n must satisfy the following initial conditions:

$$q_n(0) = \frac{2}{\pi} \int_0^\pi w_0(x) \sin(nx) \mathrm{d}x, \quad \dot{q}_n(0) = \frac{2}{\pi} \int_0^\pi w_1(x) \sin(nx) \mathrm{d}x.$$

As stated above, terms that give rise to secular terms may occur on the right-hand side of equation (19). To eliminate these terms we introduce two time-scales, $t_0 = t$ and $t_1 = \varepsilon t$, and assume that q_n can be expanded in a formal power series in ε, that is, $q_n(t) = q_{n,0}(t_0, t_1) + \varepsilon q_{n,1}(t_0, t_1) + \varepsilon^2 q_{n,2}(t_0, t_1) + \cdots$. We substitute this into equation (19) and collect equal powers in ε. The $\mathcal{O}(\varepsilon^0)$-problem becomes

$$\frac{\partial^2}{\partial t_0^2} q_{n,0} + w_{n_p}^2 q_{n,0} = 0, \quad t > 0, \tag{20}$$

$$q_{n,0}(0,0) = \frac{2}{\pi} \int_0^\pi w_0(x) \sin(nx) \, \mathrm{d}x, \tag{21}$$

$$\frac{\partial}{\partial t_0} q_{n,0}(0,0) = \frac{2}{\pi} \int_0^\pi w_1(x) \sin(nx) \, \mathrm{d}x \tag{22}$$

for $n = 1, 2, 3, \ldots,$ with $\omega_{n_p} = \sqrt{n^4 + p^2}$. The general solution for equations (20)–(22) is

$$q_{n,0}(t_0, t_1) = A_{n,0}(t_1) \cos(\omega_{n_p} t_0) + B_{n,0}(t_1) \sin(\omega_{n_p} t_0), \tag{23}$$

where $A_{n,0}$, $B_{n,0}$ satisfy the following initial conditions:

$$A_{n,0}(0) = q_{n,0}(0,0), \quad B_{n,0}(0) = \frac{1}{\omega_{n_p}} \frac{\partial}{\partial t_0} q_{n,0}(0,0).$$

Next, we consider the $\mathcal{O}(\varepsilon^1)$-problem

$$\frac{\partial^2}{\partial t_0^2} q_{n,1} + \omega_{n_p}^2 q_{n,1} = -2 \frac{\partial^2}{\partial t_0 \partial t_1} q_{n,0} - \left(C \cos(\omega t_0) + \sum_{m=1}^{\infty} m^2 q_{m,0}^2 \right) n^2 q_{n,0}, \tag{24}$$

$$q_{n,1}(0,0) = 0, \qquad \frac{\partial}{\partial t_0} q_{n,1}(0,0) = -\frac{\partial}{\partial t_1} q_{n,0}(0,0) \tag{25}$$

for $n = 1, 2, 3, \ldots$. We substitute equation (23) into equation (24) and get

$$\begin{aligned} \frac{\partial^2}{\partial t_0^2} q_{n,1} + \omega_{n_p}^2 q_{n,1} &= 2\omega_{n_p} \left(\frac{\mathrm{d}A_{n,0}}{\mathrm{d}t_1} \sin(\omega_{n_p} t_0) - \frac{\mathrm{d}B_{n,0}}{\mathrm{d}t_1} \cos(\omega_{n_p} t_0) \right) \\ &\quad - \left(C \cos(\omega t_0) + \sum_{m=1}^{\infty} m^2 \mathscr{I}_m \right) n^2 (A_{n,0} \cos(\omega_{n_p} t_0) + B_{n,0} \sin(\omega_{n_p} t_0)) \end{aligned} \tag{26}$$

with

$$\mathscr{I}_m = \tfrac{1}{2}(A_{m,0}^2 + B_{m,0}^2) + \tfrac{1}{2}(A_{m,0}^2 - B_{m,0}^2) \cos(2\omega_{m_p} t_0) + A_{m,0} B_{m,0} \sin(2\omega_{m_p} t_0).$$

Since $\cos(\omega_{n_p} t_0)$ and $\sin(\omega_{n_p} t_0)$ are homogeneous solutions of $q_{n,1}$, we want the coefficients of $\cos(\omega_{n_p} t_0)$ and $\sin(\omega_{n_p} t_0)$ on the right-hand side of equation (26) to be equal to zero (elimination of secular terms). This gives us equations that $A_{n,0}$ and $B_{n,0}$ have to satisfy. In appendix A we show that for specific values of ω, the term $C \cos(\omega t_0)$ gives rise to secular terms. This means that for specific values of ω, i.e., $\omega = 2\omega_{k_p} + \mathcal{O}(\varepsilon)$ ($k = 1, 2, 3, \ldots$), extra terms appear in the equations for $A_{k,0}$ and $B_{k,0}$ (for $n \neq k$ the equations remain the same).

For $F(t) \equiv 0$ it can be seen easily from equation (A2) (see Appendix A) that the equations for $A_{n,0}$ and $B_{n,0}$ are

$$\frac{\mathrm{d}A_{n,0}}{\mathrm{d}t_1} = \frac{1}{4}\frac{n^2}{\omega_{n_p}} B_{n,0}\left[\frac{3}{2}n^2(A_{n,0}^2 + B_{n,0}^2) + \sum_{m \neq n} m^2(A_{m,0}^2 + B_{m,0}^2)\right], \tag{27}$$

$$\frac{\mathrm{d}B_{n,0}}{\mathrm{d}t_1} = \frac{1}{4}\frac{n^2}{\omega_{n_p}} A_{n,0}\left[\frac{3}{2}n^2(A_{n,0}^2 + B_{n,0}^2) + \sum_{m \neq n} m^2(A_{m,0}^2 + B_{m,0}^2)\right] \tag{28}$$

for $n = 1, 2, 3, \ldots$. From equations (27) and (28) we see that if $A_{n,0}(0) = B_{n,0}(0) = 0$ then $\forall t_1 > 0\; A_{n,0}(t_1) = B_{n,0}(t_1) \equiv 0$. So, if we start with zero initial energy in the nth mode, there will be no energy present up to $\mathcal{O}(\varepsilon)$ on a time-scale of order ε^{-1}. We say the coupling between the modes is of $\mathcal{O}(\varepsilon)$. This allows truncation to those modes that have non-zero initial energy. In this case, there is an interaction between all modes with non-zero initial energy, but this interaction does not give rise to internal resonances. It will be shown in section 4.1 that all modes oscillate with a constant amplitude and a linearly changing phase, depending on the initial amplitudes of the oscillation modes. We will discuss equations (27) and (28) in more detail in section 4.1.

For $F(t) = C\cos(\omega t)$ with $\omega \neq 2\omega_{k_p} + \varepsilon\alpha$, it can be seen easily from equation (A2) that the equations for $A_{n,0}$ and $B_{n,0}$ are the same as for the case $F(t) = 0$, i.e. the equations are given by equations (27) and (28). The only influence $F(t)$ has is of $\mathcal{O}(\varepsilon)$ on time-scale of order ε^{-1} in the homogeneous solution for $q_{n,1}$ as is shown at the end of this section.

For $F(t) = C\cos(\omega t)$ with $\omega = 2\omega_{k_p} + \varepsilon\alpha$, where $\alpha \in \mathbb{R}$ of $\mathcal{O}(1)$, it can be seen from equation (A2) that the equations for $A_{k,0}$ and $B_{k,0}$ are

$$\frac{\mathrm{d}A_{k,0}}{\mathrm{d}t_1} = \frac{1}{4}\frac{k^2}{\omega_{n_p}} B_{k,0}\left[\frac{3}{2}k^2(A_{k,0}^2 + B_{k,0}^2) + \sum_{m \neq k} m^2(A_{m,0}^2 + B_{m,0}^2)\right]$$
$$- \frac{1}{4}\frac{k^2}{\omega_{k_p}} C(A_{k,0}\sin(\alpha t_1) + B_{k,0}\cos(\alpha t_1)), \tag{29}$$

$$\frac{\mathrm{d}B_{k,0}}{\mathrm{d}t_1} = \frac{1}{4}\frac{k^2}{\omega_{k_p}} A_{k,0}\left[\frac{3}{2}k^2(A_{k,0}^2 + B_{k,0}^2) + \sum_{m \neq k} m^2(A_{m,0}^2 + B_{m,0}^2)\right]$$
$$- \frac{1}{4}\frac{k^2}{\omega_{k_p}} C(A_{k,0}\cos(\alpha t_1) - B_{k,0}\sin(\alpha t_1)). \tag{30}$$

For $n \neq k$ equations (27) and (28) still hold. We see that for $F(t) = C\cos(\omega t)$ with $\omega = 2\omega_{k_p} + \varepsilon\alpha$ the influence of $F(t)$ is of $\mathcal{O}(1)$ on a time-scale of order ε^{-1} and extra terms appear in the equations for $A_{k,0}$, $B_{k,0}$. We see that if $A_{n,0}(0) = B_{n,0}(0) = 0$ then for all $t_1 > 0 A_{n,0}(t_1) = B_{n,0}(t_1) \equiv 0$, which holds for all n. So, if we start with zero initial energy in the nth mode, there will be no energy present up to $\mathcal{O}(\varepsilon)$ on a time-scale of order ε^{-1}. We say the coupling between the modes is of $\mathcal{O}(\varepsilon)$. This again allows truncation to those modes that have non-zero initial energy. In this case, there is an interaction between all modes with non-zero initial energy and this interaction does not give rise to internal resonances. It will be shown in section 4.2.2 that for all modes $n \neq k$ the oscillation has a constant amplitude and a linearly changing phase, depending on the initial values of the oscillation modes.

Mode k, however, oscillates with changing amplitude and phase, due to the influence of $F(t)$. We will discuss equation (29) and (30) in more detail in section 4.2.2.

When $A_{n,0}$ and $B_{n,0}$ have been determined, and thus $q_{n,0}$, we have constructed an approximation v of the exact solution w of the initial–boundary value problem (15)–(18):

$$v(x,t;\varepsilon) = \sum_{n=1}^{\infty} (q_{n,0}(t_0,t_1) + \varepsilon q_{n,1}(t_0,t_1))\sin(nx) \tag{31}$$

with $q_{n,0}(t_0,t_1) = A_{n,0}(t_1)\cos(\omega_{n_p}t_0) + B_{n,0}(t_1)\sin(\omega_{n_p}t_0)$ and $q_{n,1}(t_0,t_1) = q^{inh}_{n,1}(t_0,t_1) + A_{n,1}(t_1)\cos(\omega_{n_p}t_0) + B_{n,1}(t_1)\sin(\omega_{n_p}t_0)$, with $q^{inh}_{n,1}$ an inhomogeneous solution of equation (26). $A_{n,1}(t_1)$ and $B_{n,1}(t_1)$ can be constructed such that secular term in the $\mathcal{O}(\varepsilon^2)$ approximation are eliminated. Since we are interested in the $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ approximations, we consider $A_{n,1}$ and $B_{n,1}$ to be constant functions which depend on the initial values for $q_{n,1}$ which are given in equation (25). From equation (A2) in Appendix A it can be shown elementarily that, for $\omega \neq 2\omega_{k_p} + \varepsilon\alpha$, $q^{inh}_{n,1}$ is of the following form:

$$\begin{aligned} q^{inh}_{n,1} = {} & D_1\cos(3\omega_{n_p}t_0) + D_2\sin(3\omega_{n_p}t_0) + \sum_{m\neq n} E_{1,m}\cos((2\omega_{m_p} - \omega_{n_p})t_0) \\ & + \sum_{m\neq n} E_{2,m}\sin((2\omega_{m_p} - \omega_{n_p})t_0) + \sum_{m\neq n} F_{1,m}\cos((2\omega_{m_p} + \omega_{n_p})t_0) \\ & + \sum_{m\neq n} F_{2,m}\sin((2\omega_{m_p} + \omega_{n_p})t_0) \\ & + G_1 C\cos((\omega - \omega_{n_p})t_0) + G_2 C\sin((\omega - \omega_{n_p})t_0), \end{aligned} \tag{32}$$

where $D_1, D_2, E_{1,m}, E_{2,m}, F_{1,m}, F_{2,m}, G_1, G_2$ can be determined easily as functions of $A_{n,0}$, $B_{n,0}$, $A_{m,0}$, $B_{m,0}$. For $\omega = 2\omega_{k_p} + \varepsilon\alpha$, $q^{inh}_{n,1}$ is given by equation (32) with $G_1 = G_2 = 0$. The approximation v given by equation (31) satisfies equations (15)–(18) up to order ε. In reference [12] an asymptotic theory for a similar problem has been presented. This asymptotic theory implies that approximations v as constructed above are $\mathcal{O}(\varepsilon)$ approximations of the exact solution on a time-scale of order ε^{-1}.

In the next section, we discuss the behavior of the solutions for $A_{n,0}$, $B_{n,0}$ for three different cases: $F(t) = 0$, $F(t) = C\cos(\omega t_0)$ with $\omega \neq 2\omega_{k_p} + \varepsilon\alpha$ and $F(t) = C\cos(\omega t_0)$ with $\omega = 2\omega_{k_p} + \varepsilon\alpha$, with $\alpha = 0$ and $\alpha \neq 0$ (detuning).

4. MODAL INTERACTIONS

4.1. THE CASE $F(t) = 0$

In the previous section, equations (27) and (28) were given for $A_{n,0}$ and $B_{n,0}$. We introduce polar co-ordinates to transform these equations

$$A_{n,0} = r_n\cos(\phi_n), \qquad B_{n,0} = r_n\sin(\phi_n) \tag{33}$$

with the amplitude $r_n = r_n(t_1)$ and the phase of the oscillation $\phi_n = \phi_n(t_1)$. We get the following equations for r_n, ϕ_n, for $n = 1, 2, 3, \dots$:

$$\dot{r}_n = 0, \qquad \dot{\phi}_n = -\frac{1}{4}\frac{n^2}{\omega_{n_p}}\left[\frac{3}{2}n^2 r_n^2 + \sum_{m\neq n} m^2 r_m^2 c_m^2\right], \tag{34, 35}$$

where the dot represents differentiation with respect to t_1. The solution for equations (34) and (35) is

$$r_n = c_{1,n}, \qquad \phi_n = -\frac{1}{4}\frac{n^2}{\omega_{n_p}}\left[\frac{3}{2}n^2 c_{1,n}^2 + \sum_{m \neq n} m^2 c_{1,m}^2\right] t_1 + c_{2,n},$$

for $n = 1, 2, 3, \ldots$, where $c_{1,n}$, $c_{2,n}$ are constants of integration determined by the initial values $A_{n,0}(0)$ and $B_{n,0}(0)$. In the phase space (r_n, ϕ_n) we have the orbits given by $r_n = c_{1,n}$ and $\dot{\phi}_n < 0$. In this case, the interaction between the oscillation modes is restricted to interaction between the phases of the modes. This interaction depends on the initial values. This means the following: if we increase the initial amplitude of mode n, then due to the interaction with for instance mode m the frequency of mode m becomes higher, and mode m then has a shorter period. There are no internal resonances and no oscillation modes with initial energy zero are excited (as was for instance the case in references [11] or [12]).

4.2. THE CASE $F(t) = C\cos(\omega t_0)$ WITH $C \neq 0$

In appendix A it is shown that only for specific values of ω extra interactions occur between the different oscillation modes. These values are $\omega = 2\omega_{k_p}$ with $k = 1, 2, 3, \ldots$. We therefore consider the following cases separately.

4.2.1. *The case* $\omega \neq 2\omega_{k_p} + \varepsilon\alpha$

As stated in the previous section, the equations for $A_{n,0}$, $B_{n,0}$ for all n, are equal to the equations for the case $F(t) = 0$. There is no extra interaction between the different oscillation modes due to the external force $F(t)$.

4.2.2. *The case* $\omega = 2\omega_{k_p} + \varepsilon\alpha$

In the previous section equations (29) and (30) were given for $A_{k,0}$ and $B_{k,0}$. For $A_{n,0}$, $B_{n,0}$, $n \neq k$ equations (27) and (28) hold. We transform these equations using equation (33) and get the following equations for the oscillation modes k and $n(\neq k)$:

$$\dot{r}_k = -\frac{1}{4}\frac{k^2}{\omega_{k_p}} r_k C \sin(2\phi_k + \alpha t_1), \tag{36}$$

$$\dot{\phi}_k = -\frac{1}{4}\frac{k^2}{\omega_{k_p}}\left[\frac{3}{2}k^2 r_k^2 + \sum_{m \neq k} m^2 r_m^2 + C\cos(2\phi_k + \alpha t_1),\right], \tag{37}$$

$$\dot{r}_n = 0, \quad n \neq k, \tag{38}$$

$$\dot{\phi}_n = -\frac{1}{4}\frac{n^2}{\omega_{n_p}}\left[\frac{3}{2}n^2 r_n^2 + \sum_{m \neq n} m^2 r_m^2\right], \quad n \neq k. \tag{39}$$

We start our analysis of equations (36)–(39) by assuming that there is initial energy present in mode k only, which means the initial conditions are such that initially the system oscillates in one mode only (mode k). That is $w(x, 0) = q_k(0)\sin kx$, $w_t(x, 0) = \dot{q}_k(0)\sin kx$ and so $r_n(0) = 0$ $\forall n \neq k$. Furthermore, we introduce $\psi = 2\phi_k + \alpha t_1$. This means we get the

TABLE 1

Critical points for $C > 0$

α-range	No. of critical points	Critical points	Behaviour
$\alpha < -\dfrac{k^2}{2\omega_{k_p}} C$	0	—	—
$\alpha = -\dfrac{k^2}{2\omega_{k_p}} C$	1	$(0, \pi)$	A higher order singularity
$-\dfrac{k^2}{2\omega_{k_p}} C < \alpha < \dfrac{k^2}{2\omega_{k_p}} C$	3	$(0, \bar{\psi})$	A saddle
		$(0, \tilde{\psi})$	A saddle
		$(\bar{r}_k, \pi)$	A centre
$\alpha = \dfrac{k^2}{2\omega_{k_p}} C$	2	$(0, 0)$	A higher order singularity
		$(\bar{r}_k, \pi)$	A centre
$\alpha > \dfrac{k^2}{\omega_{k_p}} C$	2	$(\tilde{r}_k, 0)$	A saddle
		$(\bar{r}_k, \pi)$	A centre

following equations for r_k and ψ:

$$\dot{r}_k = -\frac{1}{4}\frac{k^2}{\omega_{k_p}} r_k C \sin(\psi), \qquad \dot{\psi} = \alpha - \frac{3}{4}\frac{k^4}{\omega_{k_p}} r_k^2 - \frac{1}{2}\frac{k^2}{\omega_{k_p}} C \cos(\psi). \tag{40, 41}$$

We consider the cases $C > 0$ and $C < 0$ separately. We start with $C > 0$. The critical points of equations (40) and (41) are given in Table 1, where $\bar{\psi}, \tilde{\psi}$ are solutions of $\cos(\psi) = (2\omega_{k_p}/k^2 C)\alpha$ and where $\bar{r}_k = \sqrt{4\omega_{k_p}/3k^4(\alpha + (k^2/2\omega_{k_p})C)}$, $\tilde{r}_k = \sqrt{4\omega_{k_p}/3k^4(\alpha - (k^2/2\omega_{k_p})C)}$. The system is 2π-periodic in ψ, so we consider $\psi \in [0, 2\pi]$. We see that it makes a difference (different bifurcation from critical value) whether ω approaches the critical value $2\omega_{k_p}$ from above or below, i.e., a different behaviour for $\alpha < 0$ and $\alpha > 0$. For $C < 0$ the analysis is similar, where ψ is shifted with a factor π.

The behaviour of solutions of equations (40) and (41) in the (r_k, ψ) phase space is given in Fig. 4, for $-10 \leqslant \alpha \leqslant 10$. These phase spaces have been constructed using a numerical integration method. For the sake of convenience, we have taken $p^2 = 0$, $k = 1$, $C = 1$, (i.e., $\omega_{k_p} = 1$). A similar behaviour is obtained for $p^2 > 0$, $k \neq 1$ and $C \neq 1$. It can be shown that the larger C becomes, the larger the range of α in which interaction occurs.

It should be noted that a first integral can be obtained for equations (40) and (41):

$$\cos(2\phi_k + \alpha t_1) = \gamma \frac{1}{r_k^2} + \frac{2\omega_{k_p}}{k^2 C}\alpha - \frac{3k^2}{4C} r_k^2,$$

where γ is a constant of integration depending on $A_{k,0}(0)$, $B_{k,0}(0)$, k^2, ω_{k_p}, C.

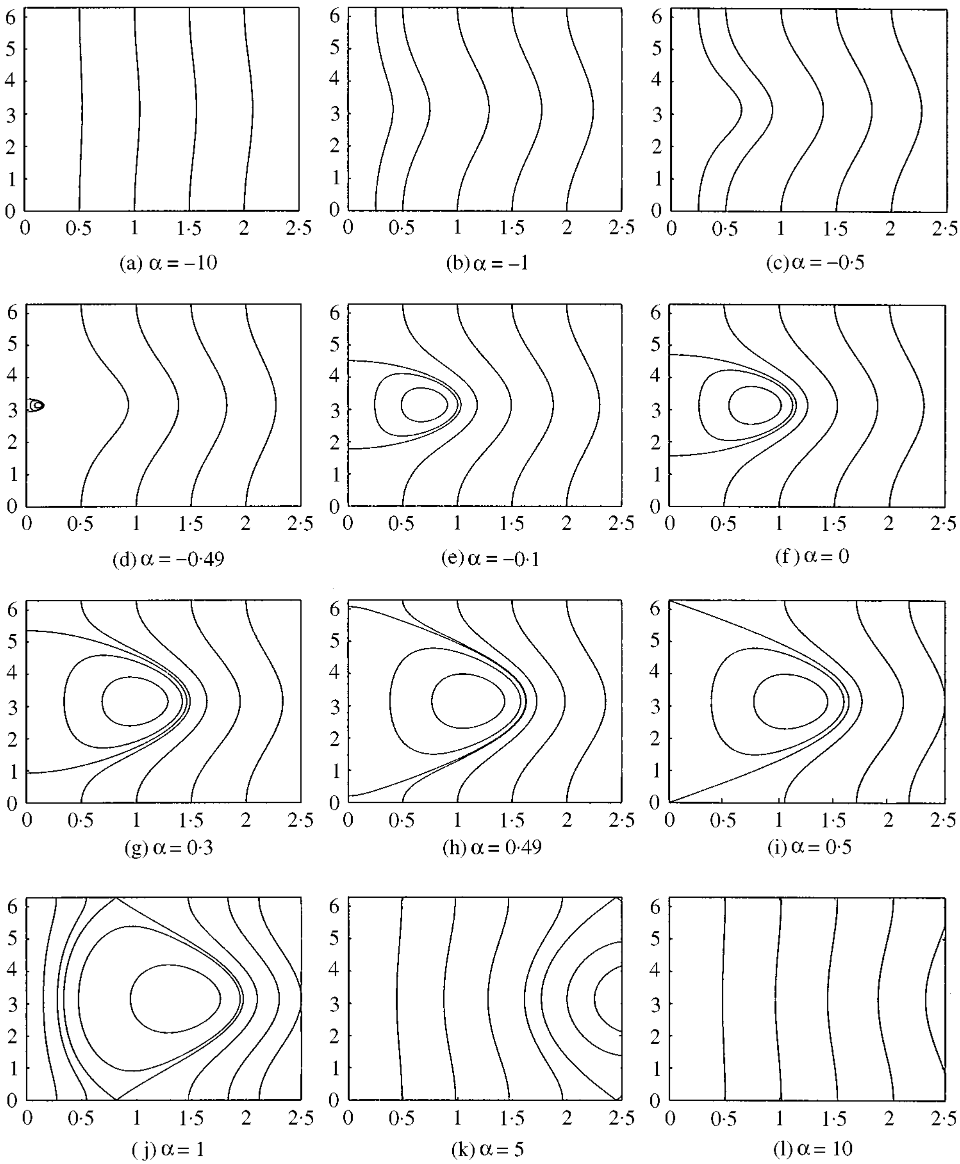

Figure 4. Phase space for $-10 \leqslant \alpha \leqslant 10$, with r_k (horizontal) from 0 to 2·5 and ψ (vertical) from 0 to 2π.

Next, we consider the case with initial energy present in two modes, m and k, which means the initial conditions are such that the system initially oscillates in two modes only (modes m and k). That is, $w(x,0) = q_k(0)\sin kx + q_m(0)\sin mx$, $w_t(x,0) = \dot{q}_k(0)\sin kx + \dot{q}_m(0)\sin mx$ and so $r_n(0) = 0\ \forall n \neq k, m$. We have the following equations (see equations (36)–(39)):

$$\dot{r}_k = -\frac{1}{4}\frac{k^2}{\omega_{k_p}} r_k C \sin(2\phi_k + \alpha t_1), \tag{42}$$

$$\dot{\phi}_k = -\frac{3}{8}\frac{k^4}{\omega_{k_p}} r_k^2 - \frac{1}{4}\frac{k^2}{\omega_{k_p}} m^2 r_m^2 - \frac{1}{4}\frac{k^2}{\omega_{k_p}} C\cos(2\phi_k + \alpha t_1), \tag{43}$$

$$\dot{r}_m = 0, \qquad \dot{\phi}_m = -\frac{3}{8}\frac{m^4}{\omega_{m_p}} r_m^2 - \frac{1}{4}\frac{m^2}{\omega_{m_p}} k^2 r_k^2. \tag{44, 45}$$

From equation (44) we can see that $r_m = c_m$ with c_m a constant. Furthermore, ϕ_m does not appear in the equations for r_k, ϕ_k, so we can analyze the behaviour of solutions of equations (42)–(45) in the (r_k, ψ) phase space (with $\psi = 2\phi_k + \alpha t_1$). The analysis is similar to the analysis for one mode. The only difference is an extra constant term in the equation for ψ, $-\frac{1}{4}(k^2/\omega_{k_p})\, m^2 c_m^2$, which means a phase shift for ψ, which depends on the initial values of mode m. We will not discuss these equations in more detail.

For initial energy present in more than two modes a similar analysis holds. The behaviour of solutions can again be analyzed in the (r_k, ψ) phase space.

5. CONCLUSIONS

In this paper, we consider an initial–boundary value problem for the vertical displacement of a weakly non-linear elastic beam with an external force acting in horizontal direction on the ends of the beam. We have constructed formal approximations of order ε and considered the interaction between different oscillation modes. The analysis presented in this paper holds for all $p \in \mathbb{R}$. In references [11, 12] it has been shown that certain values of p^2 can cause internal resonances. We have shown that in this case this does not occur. We showed that for all cases mode interactions occur only between modes with non-zero initial energy (up to $\mathcal{O}(\varepsilon)$). That is, no modes with zero initial energy are excited up to $\mathcal{O}(\varepsilon)$. We then say the coupling between the modes is of $\mathcal{O}(\varepsilon)$ and truncation is allowed to those modes with non-zero initial energy.

We considered the case with no external forcing ($F(t) = 0$) and the case with external forcing ($F(t) = C\cos(\omega t)$). For $F(t) = 0$ and $C\cos(\omega t)$ for most ω-values, the mode interaction between the modes with non-zero initial energy is restricted to an interaction between the different phases: phase shifts occur due to the interaction. The amplitudes of the oscillating modes remain constant and depend on the initial values only.

We showed that for specific values of ω, i.e., $\omega = 2\omega_{k_p}$ special interactions occur. The mode interactions between the different oscillation modes is still restricted to an interaction between the different phases but the amplitude of mode k is no longer constant: the amplitude of mode k now oscillates around an equilibrium state. This also holds for $\omega = 2\omega_{k_p} + \varepsilon\alpha$ where α is a detuning parameter. The detuning is considered in section 4.2.2. It has been shown how the system detunes from the case $\omega = 2\omega_{k_p}$ to the case $\omega \neq 2\omega_{k_p} + \varepsilon\alpha$.

In this paper, we considered an harmonic external force of the form $F(t) = C\cos(\omega t)$. This analysis can be extended to a more general form of $F(t)$, where F is a T-periodic force, $F(t) = \alpha_0/2 + \sum_n (a_n \cos(v_n t) + b_n \sin(v_n t))$ with $v_n = 2\pi n/T$. This has been discussed in reference [23] for elastic beams or strings. In reference [23], truncation to one or two oscillation modes is applied, without giving a justification. We have shown that in the cases discussed in this paper truncation is valid up to $\mathcal{O}(\varepsilon)$. As can be seen in references [11, 12], truncation to one or two oscillation modes is not valid for all cases. For some cases discussed in those papers mode interactions occur and more modes have to be taken into account. In a way similar to the methods in references [11, 12] the problem with a more general form of $F(t)$ can be studied. The analysis will essentially be the same (depending on the function F); however, the equations will become a bit more complicated. A justification can be given whether truncation is allowed in those cases. This elementary and straightforward analysis is beyond the scope of this paper.

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APPENDIX A. THE DETERMINATION AND ELIMINATION OF SECULAR TERMS

In section 3 we obtained the following equation for each q_n:

$$\ddot{q}_n + (n^4 + p^2) q_n = -\varepsilon\left(F(t) + \sum_{k=1}^{\infty} k^2 q_k^2\right) n^2 q_n$$

for $n = 1, 2, 3, \ldots$. To avoid secular terms in $q_n(t)$ a two-time-scales perturbation method was introduced and $q_n(t)$ was expanded in $q_n(t) = q_{n,0}(t_0, t_1) + \varepsilon q_{n,1}(t_0, t_1) + \cdots$, where $t_0 = t$ and εt. It has been shown that $q_{n,1}$ has to satisfy

$$\frac{\partial^2}{\partial t_0^2} q_{n,1} + \omega_{n_p}^2 q_{n,1} = 2\omega_{n_p}\left(\frac{\mathrm{d}A_{n,0}}{\mathrm{d}t_1}\sin(\omega_{n_p} t_0) - \frac{\mathrm{d}B_{n,0}}{\mathrm{d}t_1}\cos(\omega_{n_p} t_0)\right)$$

$$-\left(C\cos(\omega t_0) + \sum_{m=1}^{\infty} m^2 \mathscr{I}_m\right) n^2 (A_{n,0}\cos(\omega_{n_p} t_0) + B_{n,0}\sin(\omega_{n_p} t_0)) \tag{A1}$$

with

$$\mathscr{I}_m = \tfrac{1}{2}(A_{m,0}^2 + B_{m,0}^2) + \tfrac{1}{2}(A_{m,0}^2 - B_{m,0}^2)\cos(2\omega_{m_p} t_0) + A_{m,0} B_{m,0}\sin(2\omega_{m_p} t_0)$$

and $\omega_{n_p} = \sqrt{n^4 + p^2}$. The equations for the functions $A_{n,0}$ and $B_{n,0}$ will now be determined such that no secular terms occur in $q_{n,1}$. The right-hand side of equation (A1) can be expanded using geometric formula's and becomes

$$2\omega_{n_p}\frac{\mathrm{d}A_{n,0}}{\mathrm{d}t_1}\sin(\omega_{n_p} t_0) - 2\omega_{n_p}\frac{\mathrm{d}B_{n,0}}{\mathrm{d}t_1}\cos(\omega_{n_p} t_0)$$

$$-\frac{1}{4} n^4 [3A_{n,0}(A_{n,0}^2 + B_{n,0}^2)\cos(\omega_{n_p} t_0) + 3B_{n,0}(A_{n,0}^2 + B_{n,0}^2)\sin(\omega_{n_p} t_0)$$

$$+ A_{n,0}(A_{n,0}^2 - 3B_{n,0}^2)\cos(3\omega_{n_p} t_0) + B_{n,0}(3A_{n,0}^2 - B_{n,0}^2)\sin(3\omega_{n_p} t_0)]$$

$$-\sum_{m \neq n} n^2 m^2 \frac{1}{4}[(2B_{n,0}A_{m,0}B_{m,0} - A_{n,0}(B_{m,0}^2 - A_{m,0}^2))\cos(2\omega_{m_p} - \omega_{n_p})t_0)$$

$$-(B_{n,0}(A_{m,0}^2 - B_{m,0}^2) - 2A_{n,0}A_{m,0}B_{m,0})\sin((2\omega_{m_p} - \omega_{n_p})t_0)$$

$$+(A_{n,0}(A_{m,0}^2 - B_{m,0}^2) - 2B_{n,0}A_{m,0}B_{m,0})\cos((2\omega_{m_p} + \omega_{n_p})t_0)$$

$$+(2A_{n,0}A_{m,0}B_{m,0} + B_{n,0}(A_{m,0}^2 - B_{m,0}^2))\sin((2\omega_{m_p} + \omega_{n_p})t_0)$$

$$+ 2A_{n,0}(A_{m,0}^2 + B_{m,0}^2)\cos(\omega_{n_p} t_0) + 2B_{n,0}(A_{m,0}^2 + B_{m,0}^2)\sin(\omega_{n_p} t_0)]$$

$$-n^2\frac{1}{2}C[A_{n,0}\cos((\omega - \omega_{n_p})t_0) - B_{n,0}\sin((\omega - \omega_{n_p})t_0)$$

$$+ A_{n,0}\cos((\omega + \omega_{n_p})t_0) + B_{n,0}\sin((\omega + \omega_{n_p})t_0)]. \tag{A2}$$

As stated in section 3, $\cos(\omega_{n_p} t_0)$ and $\sin(\omega_{n_p} t_0)$ are homogeneous solutions of $q_{n,1}$. We want the coefficients of $\cos(\omega_{n_p} t_0)$ and $\sin(\omega_{n_p} t_0)$ in equation (A2) to be equal to zero in order to eliminate secular terms. This gives us equations for $A_{n,0}$ and $B_{n,0}$. From equation (A2) it can be seen that we have to consider two cases for ω: $\omega \neq 2\omega_{n_p}$ and $\omega = 2\omega_{n_p}$, as is done in sections 3 and 4 respectively. When the secular terms on the right-hand side of equation (A1) have been eliminated, the remaining terms are the inhomogeneous part of the equation for $q_{n,1}$ and an inhomogeneous solution for $q_{n,1}$ can be determined easily. This is discussed further at the end of section 3.