Journal of Sound and Vibration (2000) **235**(2), 299–316
doi:10.1006/jsvi.2000.2919, available online at http://www.idealibrary.com on IDEAL®

THE ROLE OF EIGENPARAMETER GRADIENTS IN THE DETECTION OF PERTURBATIONS IN DISCRETE LINEAR SYSTEMS

G. M. LLOYD AND M. L. WANG

Department of Civil and Materials Engineering, University of Illinois at Chicago, 842 West Taylor Street, Chicago, ILL 60607, U.S.A. E-mail: lloydg@asme.org

AND

D. SATPATHI

S&C Electric Company, R&D Division, 6601 North Ridge Blvd., Chicago, ILL 60626, U.S.A.

(*Received* 10 *May* 1999, *and in final form* 21 *January* 2000)

Continuous mechanical systems can be characterized through analysis of discrete linear models. Such models can provide approximations for the eigenvalues and eigenvectors (collectively, "eigenparameters"). Although the eigenparameters do not qualify as measures for the state of the system in response to specific loading and boundary conditions, they do reflect the identity of the system, and this in itself has important applications. In particular, the eigenparameters can be used to study the sensitivity of a system to perturbations, due perhaps to damage incurred by one or more discrete elements. These studies can rationalize the choice and weighting of eigenparameters for system identification strategies, damage detection algorithms, and damage assessment methods. To this end, this paper develops a set of sensitivity coefficients based on gradients of the eigenparameters. Sensitivities are normalized with respect to that of the harmonic oscillator, and generalized to include the mode vectors through the definition of a figure of merit. Analytical and numerical examples based on appropriate elements are used to illustrate the utility of the approach.

1. INTRODUCTION

The ability to diagnose changes in a mechanical system is a key part of maintaining the original performance of the system. An important part of the diagnosis procedure is the selection of an appropriate mathematical model and of measurables of the model which are sensitive to changes deemed significant with respect to some design criterion [1–3]. Even in the context of linear models, the selection of such measures, and the evaluation of their sensitivities, is a difficult and largely unresolved problem. For example, modal frequencies [2] and modal vectors [3] are often assumed (or dismissed) as the basis for system measures. This is despite the fact that in many cases the *sensitivities* of these parameters (or objective functions based upon them) have not been investigated in systematic detail, although scattered theoretical and computational results have been reported. These will be reviewed in section 3. The purpose of this paper will be to propose a sensitivity coefficient based upon gradients of the eigenparameters, and to demonstrate its use for both eigenvalues and eigenvectors.

0022-460X/00/320299 + 18 $35.00/0

It will be insightful to begin with a one-degree-of-freedom (d.o.f.) system—the simple harmonic oscillator—for which the key measurable is the natural frequency, ω. The sensitivity of ω can be established in an unambiguous manner as follows:

$$\left(\frac{\partial \omega}{\partial k}\right)_m = \frac{1}{2m\omega}. \tag{1}$$

Using this sensitivity with a first order Taylor expansion about ω_0 leads to the relative frequency sensitivity (to stiffness), which will be denoted by r^{ω}:

$$r^{\omega} \stackrel{\text{def}}{=} \frac{\Delta\omega}{\omega_0} = \frac{1}{2}\frac{\Delta k}{k_0}, \tag{2}$$

i.e., a 1% change in stiffness is reflected by a 0·5% change in frequency. If a *figure of merit* is defined as the ratio

$$\beta \stackrel{\text{def}}{=} \frac{2r^{\omega}}{\Delta k/k}, \tag{3}$$

it follows that $\beta = 1$ for the simple harmonic oscillator.

Realistic systems have a large number of degrees of freedom, or are continuous systems which can be analyzed experimentally and computationally as such [4–6]. In order to generalize the previous concepts to such systems, consider a linear, time-invariant, undamped system described as [7]

$$\boldsymbol{M}\cdot\mathbf{x} + \boldsymbol{K}\cdot\mathbf{x} = \mathbf{0}. \tag{4}$$

$\boldsymbol{M}$ and $\boldsymbol{K}$ denote the mass and stiffness tensors. For many systems of interest, the components of $\boldsymbol{K}$ will form a symmetric matrix, and the number of independent parameters will be substantially less than n^2. Let $\mathbf{k}$ denote the vector which parameterizes the eigenproblem [8]. An assumed modes analysis of equation (4) leads to the well-known eigenvalue problem

$$[\lambda\boldsymbol{M} - \boldsymbol{K}(k)]\cdot\mathbf{x} = \boldsymbol{K}^d\cdot\mathbf{x} = \mathbf{0}, \tag{5}$$

where $\{\lambda_i, \mathbf{x}^{(i)}\}$ denotes the set of eigenvalues and eigenvectors, respectively, and $\boldsymbol{K}^d$ is the dynamic stiffness tensor. It is satisfied only for certain values of λ_i (the eigenvalues) and $\mathbf{x}^{(i)}$ (the eigenvectors). It is assumed that the eigenvalues are real and distinct in this paper. The importance of the eigenproblem is that it is the basis for a separated space–time solution using the modal tensor constructed from the eigenvectors and defined by

$$\hat{\boldsymbol{X}} = \hat{\mathbf{x}}^{(j)}\otimes\mathbf{e}_j = \hat{x}_i^{(j)}\mathbf{e}_i\otimes\mathbf{e}_j. \tag{6}$$

This is the basis for the linear transformation for an arbitrary displacement vector $\mathbf{x}$: $\mathbf{x} = \hat{\boldsymbol{X}}\cdot\boldsymbol{\eta}$ through which the kinetic and potential energies may be expressed as quadratic forms: $\mathcal{T} = \frac{1}{2}\,\dot{\boldsymbol{\eta}}\cdot\dot{\boldsymbol{\eta}}$, and $\mathcal{V} = \frac{1}{2}\,\boldsymbol{\eta}\cdot\boldsymbol{\Lambda}\cdot\boldsymbol{\eta}$ in the generalized coordinates η. Lagrange's equations then lead to a set of uncoupled oscillators for which n sensitivity coefficients are well defined in principle, similar to equation (1).

2. EIGENVALUE AND EIGENVECTOR GRADIENTS

The natural avenue for examining perturbations of parameterized tensors is to calculate an appropriate gradient and then to linearize using a Taylor series. Research in the area of eigensolutions of equation (5) has progressed along two lines. The first has been devoted toward rigorously establishing differentiability conditions. Lancaster, 1964 [9] examined the properties of the eigenvalue problem (with $\boldsymbol{M} = \boldsymbol{I}$). Andrew, 1979 [10] extended the analysis in examining an iterative method of Rudisill and Chu, 1975 [11]. More recently, Gollan, 1987 [12] examined the parameterized eigenvalue problem for symmetric matrices and established differentiability of simple eigenvalues about vector-valued parameters. Over ton, 1992 [13] has addressed problems related to min–max optimization of the eigenvalues. Nylen and Uhlig, 1997 [14] examined a different, but related problem — that of an inverse perturbative problem. These studies address important foundational issues, but generally have not concerned themselves with applications to specific problems.

The second line of research has been largely applied, with a particular emphasis on methods for efficient calculation of the eigenparameter gradients appropriate for large systems (when truncation of modes is usually the case), or for particular degeneracies. This body of literature is quite extensive. Fox and Kapoor, 1968 [15] provided several expressions for the derivatives of the eigenvalues and eigenvectors, and applied the former to two very simple structures. More importantly, they popularized the modal expansion method, which could be applied (albeit, with a degree of approximation) to large systems for which incomplete modal information was available. Taylor, 1975 [16] presented analytic results for the quadratic eigenvalue problem. Nelson, 1976 [17] derived a direct method for the first eigenvector derivatives valid for the case of non-repeated eigenvalues. Yuen, 1985 [18] numerically studied changes in the eigenparameters as a function of location for perturbations in the elasticity modulus and area moment of a cantilever. Adelman, 1986 [19] provided a comprehensive survey encompassing several disciplines, dating back to work by J. Jacobi in 1846. Lim *et al*., 1987 [20] provided a comprehensive list of expressions for the general non-self-adjoint eigenvalue problem which clarified misconceptions regarding the eigenspaces associated with eigenvector derivatives. Bernard and Bronowicki, 1994 [21] extended the modal expansion method for cases associated with repeated roots of self-adjoint systems. The accuracy of the method was evaluated on 18 modes (out of 30) of a test structure; eigenparameter derivatives compared favourably with finite difference estimates. Chen *et al*., 1994 [7] proposed a second order perturbation method to examine the changes of the eigenpairs of structures. They provided numerical examples for a 15-element truss structure. They found increased accuracy against an alternative method, particularly for eigenvalues. Chen, 1995 [22] extended the method of Fox and Kapoor, 1968 [15] for eigenvector derivatives for doubly repeated eigenvalues under restrictions on the stiffness perturbations. Alvin, 1997 [23] examined numerical estimates of eigenvector sensitivity calculations for distinct roots using a preconditioned conjugate projected gradient algorithm, and evaluated the accuracy for a large system ($> 10^2$ d.o.f.) and found improved accuracy, particularly for higher modes.

Despite the extent of these results, there has been little discussion regarding the relative sensitivities of the different eigenparameters to multiple parameters, as occurs in designing measurement systems for damage detection studies, and in interpreting the resulting data. The brevity of discussion in this area may reflect prior emphasis on optimization problems, where a small subset of easily modified design parameters is selected beforehand. In order to proceed in this direction, take the inner product of equation (5) with $\mathbf{x}^{(i)}$, and differentiate with respect to the independent components of $\boldsymbol{K}$ as follows:

$$\mathrm{d}_{,\mathbf{k}}\{\mathbf{x}^{(i)} \cdot [\lambda_i \boldsymbol{M} - \boldsymbol{K}(\mathbf{k})] \cdot \mathbf{x}^{(i)}\} = 0. \tag{7}$$

It is assumed that $\boldsymbol{M}$ is constant, which is a reasonable assumption when examing changes for most mechanical systems. It is straightforward to show that the eigenvalue gradients can be expressed in the form [15, 12, 20]

$$\lambda_i \otimes \overleftarrow{\nabla}^{\mathbf{k}} = \frac{\mathbf{x}^{(i)} \otimes \mathbf{x}^{(i)} : \boldsymbol{K} \otimes \overleftarrow{\nabla}^{\mathbf{k}}}{\mathbf{x}^{(i)} \otimes \mathbf{x}^{(i)} : \boldsymbol{M}}, \tag{8}$$

where $\boldsymbol{K} \otimes \overleftarrow{\nabla}^{\mathbf{k}}$ denotes the linear transformation relating $\mathrm{d}_{,\mathbf{k}}\boldsymbol{K}$ and $\mathrm{d}\mathbf{k}$. In simple problems, one usually has analytical expressions for the λ_i, and the gradients can be calculated directly. If the customary normalization of the eigenvectors to $\boldsymbol{M}$ is performed,

$$\hat{\mathbf{x}}^{(i)} \cdot \boldsymbol{M} \cdot \hat{\mathbf{x}}^{(i)} = 1, \tag{9}$$

the gradient in equation (8) can be simplified as

$$\omega_i \otimes \overleftarrow{\nabla}^{\mathbf{k}} = \frac{\hat{\mathbf{x}}^{(i)} \otimes \hat{\mathbf{x}}^{(i)} : \boldsymbol{K} \otimes \overleftarrow{\nabla}^{\mathbf{k}}}{2\omega_i}. \tag{10}$$

This is the analogous expression to equation (1) for higher d.o.f. systems. Equation (10) can be used to develop an eigenvalue sensitivity through a Taylor series expansion in $\mathbf{k}$:

$$\omega_p - \omega_i(\mathbf{k}_0) = \omega_i \otimes \overleftarrow{\nabla}^{\mathbf{k}=\mathbf{k}_0} \cdot \Delta\mathbf{k} + \mathcal{O}(\|\mathbf{k}\|^2), \tag{11}$$

where $\omega_p \equiv \omega_i(\mathbf{k}_0 + \Delta\mathbf{k})$. Reference [9] provides properties of the coefficient derivatives when $\boldsymbol{M}$ is the identity tensor. Dividing by ω_i gives the relative change due to a perturbation in $\mathbf{k}$ [2]:

$$r^{\omega} = \frac{\hat{\mathbf{x}}^{(i)} \otimes \hat{\mathbf{x}}^{(i)} : \boldsymbol{K} \otimes \overleftarrow{\nabla}^{\mathbf{k}=\mathbf{k}_0} \cdot \Delta\mathbf{k}}{2\omega_i^2}. \tag{12}$$

The quantity r^{ω} in equation (12) becomes the appropriate measure for analysis of the eigenvalue sensitivities when changes in stiffness are scaled:

$$r^{\omega} = \|\mathbf{k}\| \frac{\hat{\mathbf{x}}^{(i)} \otimes \hat{\mathbf{x}}^{(i)} : \boldsymbol{K} \otimes \overleftarrow{\nabla}^{\mathbf{k}=\mathbf{k}_0}}{2\omega_i^2} \cdot \frac{\Delta\mathbf{k}}{\|\mathbf{k}\|} = \|\mathbf{k}\| \frac{\omega_i \otimes \overleftarrow{\nabla}^{\mathbf{k}=\mathbf{k}_0}}{\omega_i} \cdot \frac{\Delta\mathbf{k}}{\|\mathbf{k}\|}. \tag{13}$$

Equation (13) is the desired generalization of equation (2), for which we have been unable to find a systematic analysis. The quantity r^{ω} is an estimate of the directional derivative in the direction $\boldsymbol{\varepsilon}_{\mathbf{k}} = \Delta\mathbf{k}/\|\mathbf{k}\|$, which is one independent parameter of interest.

A similar analysis can be applied to the eigenvectors [15, 24–27], although it will be more convenient to take the derivatives of the eigenvectors directly. The eigenvector gradients can be used to approximate the ith perturbed eigenvector, again using a Taylor series:

$$\hat{\mathbf{x}}_p^{(i)} = \hat{\mathbf{x}}_0^{(i)} + \hat{\mathbf{x}}^{(i)} \otimes \overleftarrow{\nabla}^{\mathbf{k}=\mathbf{k}_0} \cdot \Delta\mathbf{k} + \frac{1}{2!} \hat{\mathbf{x}}^{(i)} \otimes \overleftarrow{\nabla} \overleftarrow{\nabla}^{\mathbf{k}=\mathbf{k}_0} : \Delta\mathbf{k} \otimes \Delta\mathbf{k} + \cdots. \tag{14}$$

Relative comparisons among eigenvectors requires an inner product norm. The obvious choice is

$$\gamma = \hat{\mathbf{x}}_0^{(i)} \cdot \boldsymbol{M} \cdot \hat{\mathbf{x}}_p^{(i)}. \tag{15}$$

The norm in equation (15) is unity for $\hat{\mathbf{x}}_p^{(i)} = \hat{\mathbf{x}}_0^{(i)}$, and zero for $\hat{\mathbf{x}}_p^{(i)} = \hat{\mathbf{x}}_p^{(j)}$, $i \neq j$. This may be compared to a common heuristic called the modal assurance criterion [3] which is sometimes used to access the "correlations" among eigenvectors:

$$\mathrm{MAC} = \frac{(\mathbf{x}^{(i)} \cdot \mathbf{x}^{(j)})^2}{(\mathbf{x}^{(i)} \cdot \mathbf{x}^{(i)})(\mathbf{x}^{(j)} \cdot \mathbf{x}^{(j)})}. \tag{16}$$

Here, $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$ denote "different" modes. However, the normalization criterion equation (9) implies the following relation [10, 20]:

$$\hat{\mathbf{x}}^{(i)} \cdot \boldsymbol{M} \cdot \hat{\mathbf{x}}^{(i)} \otimes \overleftarrow{\nabla} = \mathbf{0}. \tag{17}$$

As a result, estimating $\hat{\mathbf{x}}_p^{(i)}$ requires at minimum a second order calculation. We have not encountered an analysis where this information is used in the context of classifying the sensitivities of the eigenparameters. Using equation (14), it is straightforward to show

$$r^x \stackrel{\text{def}}{=} \frac{\gamma - \gamma_0}{\gamma_0} = \frac{1}{2} \hat{\mathbf{x}}_0^{(i)} \cdot \boldsymbol{M} \cdot \hat{\mathbf{x}}^{(i)} \otimes \overleftarrow{\nabla} \overleftarrow{\nabla}^{\mathbf{k} = \mathbf{k}_0} : \Delta\mathbf{k} \otimes \Delta\mathbf{k}, \tag{18}$$

since $\gamma_0 = 1$. This is the desired generalization of equation (12). The quantities r^ω and r^x will be referred to as eigensensitivities. A figure of merit for the eigenvalues similar to equation (3) may also be defined using equation (18).

3. NUMERICAL EXAMPLE

The application of equations (12) and (18) to an arbitrary system described by equation (4) can be daunting due to algebraic and computational complexities. In this section, the calculation of eigenparameter sensitivity coefficients is demonstrated using a two-d.o.f. model, and the directional sensitivities are restricted to variations in one stiffness element at a time. Higher order systems are considered in the following section. The system, which may be considered the discrete model for longitudinal vibrations of a bar, is shown in Figure 1. The mass and stiffness matrices for this system are

$$[\boldsymbol{M}] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad [\boldsymbol{K}] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}. \tag{19}$$

The generalization for higher order systems is obvious. The first eigenpair is

$$\lambda_1 = \frac{(k_1 + 2k_2 + k_3) - \sqrt{k_1^2 + 4k_2^2 + k_3^2 - 2k_1 k_3}}{2m},$$

$$\mathbf{x}^{(1)} = \frac{(k_2 + k_3) - \lambda_1 m}{k_2} \mathbf{e}_1 + \mathbf{e}_2 = \frac{k_3 - k_1 + \sqrt{k_3^2 - 2k_1 k_3 + 4k_2^2 + k_1^2}}{2k_2} \mathbf{e}_1 + \mathbf{e}_2. \tag{20}$$

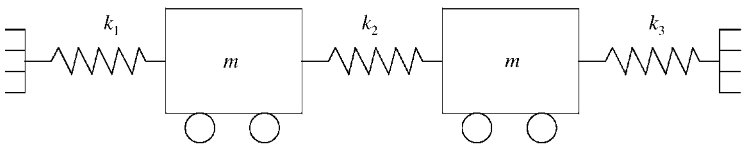

Figure 1. A two degree of freedom system.

The second eigenpair is

$$\lambda_2 = \frac{(k_1 + 2k_2 + k_3) - \sqrt{k_1^2 + 4k_2^2 + k_3^2 - 2k_1k_3}}{2m},$$

$$\mathbf{x}^{(2)} = \frac{(k_2 + k_3) - \lambda_2 m}{k_2}\mathbf{e}_1 + \mathbf{e}_2 = \frac{k_3 - k_1 - \sqrt{k_3^2 - 2k_1k_3 + 4k_2^2 + k_1^2}}{2k_2}\mathbf{e}_1 + \mathbf{e}_2. \quad (21)$$

In order to investigate the eigensensitivities, it is reasonable to focus upon two limiting parameterizations of $\boldsymbol{K}$, corresponding to a symmetric stiffness perturbation, and an asymmetric perturbation. These cases will be referred to as cases A and B, respectively.
(A) $k_1 = k_3$ (center symmetry); $\mathbf{k} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2$.
(B) $k_1 = k_2$; $\mathbf{k} = k_1\mathbf{e}_1 + k_3\mathbf{e}_2$.
The corresponding stiffness tensors are

$$\boldsymbol{K} \otimes \overleftarrow{\nabla} = \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2$$
$$+ \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2,$$
$$\boldsymbol{K} \otimes \overleftarrow{\nabla} = 2\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1$$
$$+ \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \quad (22)$$

respectively. For case A, the eigenparameters simplify to the expression

$$\{(\lambda_{Ai}, \hat{\mathbf{x}}^{(Ai)})\} = \left\{\left(\frac{k_1}{m}, \frac{\mathbf{e}_1 + \mathbf{e}_2}{\sqrt{2m}}\right)\left(\frac{k_1 + 2k_2}{m}, \frac{\mathbf{e}_1 - \mathbf{e}_2}{\sqrt{2m}}\right)\right\}, \quad (23)$$

with the smaller eigenvalue corresponding to the in-phase mode. The eigenvalue gradients are

$$\lambda_{A1} \otimes \overleftarrow{\nabla}^{\mathbf{k}} = \mathbf{e}_1/m, \quad \lambda_{A2} \otimes \overleftarrow{\nabla}^{\mathbf{k}} = (\mathbf{e}_1 + 2\mathbf{e}_2)/m. \quad (24)$$

The eigenvector gradients, calculated directly, vanish identically:

$$\hat{\mathbf{x}}^{(A1)} \otimes \overleftarrow{\nabla}\overleftarrow{\nabla}^{\mathbf{k}} = \mathbf{0}, \quad \hat{\mathbf{x}}^{(A2)} \otimes \overleftarrow{\nabla}\overleftarrow{\nabla}^{\mathbf{k}} = \mathbf{0}. \quad (25)$$

Using the results from equation (24) with $\Delta\mathbf{k} = \Delta k_1\mathbf{e}_1 + \Delta k_2\mathbf{e}_2$, the eigenvalue sensitivities can be expressed as

$$r^{\omega}_{A1} = \frac{\Delta k_1}{2k_1}, \quad r^{\omega}_{A2} = \frac{\frac{1}{2}\Delta k_1 + k_2}{k_1 + 2k_2}, \quad (26)$$

using either of equation (13). For case A, perturbations satisfy $\Delta k_1 = 0$, and all change in $\boldsymbol{K}$ arises from the center spring element, k_2. If the sensitivity coefficient for this situation is denoted by r^{ω}_{A1,k_2}, then clearly from equation (26),

$$r^{\omega}_{A1,k2} = 0. \quad (27)$$

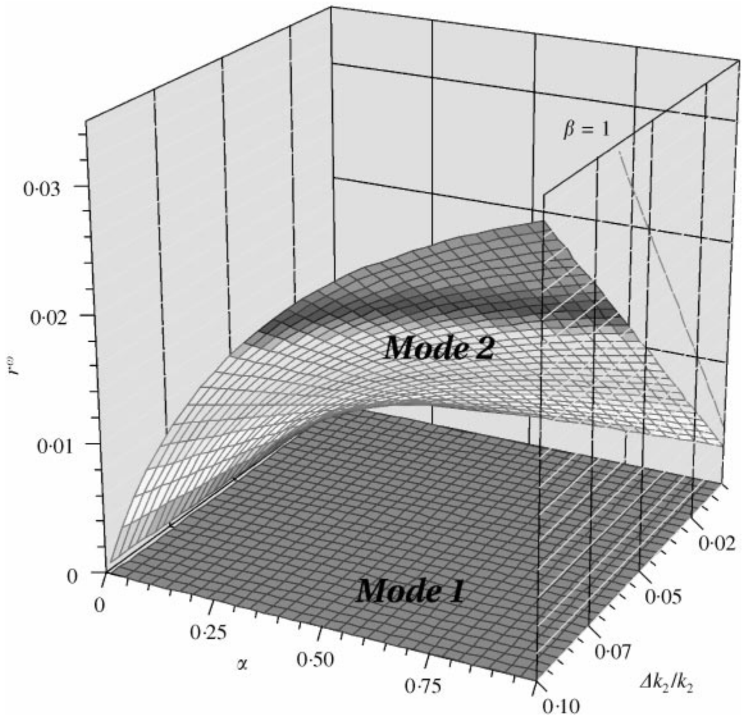

Figure 2. Eigenfrequency sensitivities $r^{\omega}_{A1,k2}$ in equation (28), and $r^{\omega}_{A2,k2}$. The line $\beta_{\alpha=1} = 1$ is the sensitivity for the harmonic oscillator.

Hence, the lowest frequency sensitivity is uncoupled from k_2. The eigenvalue sensitivity of the second mode can be written as

$$r^{\omega}_{A2,k2} = \frac{k_2}{k_1 + 2k_2}\frac{\Delta k_2}{k_2} = \frac{\alpha}{1 + 2\alpha}\frac{\Delta k_2}{k_2}, \tag{28}$$

where $k_2 = \alpha k_1$, and α is the value of the original stiffness of k_2 expressed as a fraction of k_1 (and k_3). α is adopted as the second independent variable for this study. It is assumed here that $\alpha \in (0, 1]$, so that "degenerative" perturbations are considered, and separate elements remain coupled, however weakly. However, in principle α can be any positive real quantity.

The behaviours of r^{ω}_{A1,k_2} and r^{ω}_{A1,k_2} is shown in Figure 2 as a function of $\Delta k_2/k_2$ and α. For $r^{\omega}_{A2,k2}$, the greatest sensitivity occurs when all springs have comparable stiffnesses ($\alpha = 1$), and is a monotonic function of $\Delta k_2/k_2$. One can observe from Figure 2 that when $k_1 = k_3 \approx k_2$, a 10% change in k_2 results in a 3% change in ω_{A2}. Quantitatively, the figure of merit is given by

$$\beta^{\omega}_{A2,k_2} = \frac{2\alpha}{1 + 2\alpha}, \tag{29}$$

from which it follows that max $\beta_{A2,k_2} = \frac{2}{3}$. The eigenvector sensitivities for case A are, of course, zero in view of equation (25), regardless of α. This is a reflection of the high degree of symmetry, and small value of n.

This analysis can be repeated for case B. The eigenpairs for this case are

$$\lambda_{B1,B2} = \frac{3k_1 + k_3 \mp \sqrt{5k_1^2 + k_3^2 - 2k_1k_3}}{2m},$$

$$\hat{\mathbf{x}}^{(B1)} = \frac{a\mathbf{e}_1 + \mathbf{e}_2}{\sqrt{m(1 + a^2)}}, \quad a = \frac{k_3 - k_1 + \sqrt{5k_1^2 + k_3^2 - 2k_1k_3}}{2k_1}. \tag{30}$$

The eigenvalue gradients are

$$\omega_{B1} \otimes \overleftarrow{\nabla} = \frac{1}{4m\omega_{B1}}\left[\left(3 - \frac{10k_1 - 2k_3}{2\sqrt{5k_1^2 + k_3^2 - 2k_1k_3}}\right)\mathbf{e}_1 + \left(1 - \frac{k_3 - k_1}{\sqrt{5k_1^2 + k_3^2 - 2k_1k_3}}\right)\mathbf{e}_2\right]. \tag{31}$$

In case B, the interest is stiffness reduction at the end, i.e., $k_3 < k_1$, and $\Delta\mathbf{k} = \Delta k_3\mathbf{e}_2$. The sensitivity coefficient becomes

$$r^{\omega}_{B1,k_3} = \frac{1}{8m\omega_{B1}^2}\left[1 - \frac{k_3 - k_1}{\sqrt{5k_1^2 + k_3^2 - 2k_1k_3}}\right]\Delta k_3$$

$$= \frac{\alpha}{(2\alpha - 4)\sqrt{\alpha^2 - 2\alpha^2 + 5} + 2\alpha^2 - 4\alpha + 10}\frac{\Delta k_3}{k_3} \tag{32}$$

The behaviour of r^{ω}_{B1,k_3} as well as r^{ω}_{B2,k_3}, which is given by

$$r^{\omega}_{B2,k_3} = \frac{\alpha\sqrt{\alpha^2 - 2\alpha + 5} + \alpha^2 - \alpha}{(2\alpha + 6)\sqrt{\alpha^2 - 2\alpha + 5} + 2\alpha^2 - 4\alpha + 10}\frac{\Delta k_3}{k_3}, \tag{33}$$

are shown together in Figure 3.

It can be shown from equation (32) that $\beta^{\omega}_{B1,k3} = \frac{1}{2}$ and from equation (33), that $\beta^{\omega}_{B2,k3} = \frac{1}{6}$. The maximum sensitivity occurs for $k_3 \approx k_1$ ($\alpha = 1$), which is the same behaviour seen for ω_{A2}. As k_3 becomes smaller relative to k_1 and k_2, the sensivity coefficient r^{ω} in both ω_{B1} and ω_{B2} tends to zero.

The eigenvector sensitivities r^x for case B are algebraically complex. The results were obtained by parameterizing the eigenvectors in terms of α, as follows:

$$\hat{\mathbf{x}}^{(B1)} = -\frac{\sqrt{\alpha^2 - 2\alpha + 5} + \alpha - 1}{\sqrt{[(2 - 2\alpha)\sqrt{\alpha^2 - 2\alpha + 5} + 2\alpha^2 - 4\alpha + 10]\,m}}\mathbf{e}_1$$

$$+ \frac{2}{\sqrt{[(2 - 2\alpha)\sqrt{\alpha^2 - 2\alpha + 5} + 2\alpha^2 - 4\alpha + 10]m}}\mathbf{e}_2$$

$$\hat{\mathbf{x}}^{(B2)} = -\frac{\sqrt{\alpha^2 - 2\alpha + 5} - \alpha + 1}{\sqrt{[(2 - 2\alpha)\sqrt{\alpha^2 - 2\alpha + 5} + 2\alpha^2 - 4\alpha + 10]\,m}}\mathbf{e}_1$$

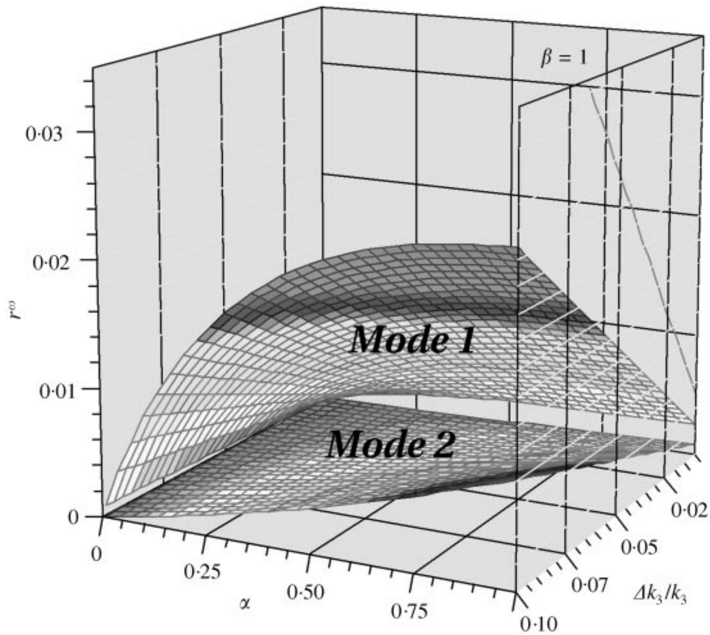

Figure 3. Eigenvalue sensitivities $r^{\omega}_{B1,k3}$ and $r^{\omega}_{B2,k3}$ for case B.

$$+\frac{2}{\sqrt{[(2-2\alpha)\sqrt{\alpha^2-2\alpha+5}+2\alpha^2-4\alpha+10]m}}\mathbf{e}_2$$

$$=\frac{-\mathbf{e}_1+\mathbf{e}_2}{\sqrt{2m}},\ (\alpha\to 1). \tag{34}$$

The eigenvectors reduce to those of the homogeneous system with $\alpha \to 1$, as expected. The eigenvector sensitivities defined by equation (18) then become

$$r^{x}_{B1,k3}=\left(\frac{\Delta k_3}{k_3}\right)^2 \frac{\begin{array}{r}\sqrt{\alpha^2-2\alpha+5}(-49\alpha^8+294\alpha^7-1055\alpha^6+2260\alpha^5-3071\alpha^4+2406\alpha^3-785\alpha^2)\\ +(\alpha^2-2\alpha+5)^{3/2}(5\alpha^6-20\alpha^5-98\alpha^4+236\alpha^3-59\alpha^2)+(\alpha^2-2\alpha+5)^{5/2}\\ \times(41\alpha^4-82\alpha^3+9\alpha^2)+3\alpha^2(\alpha^2-2\alpha+5)^{7/2}-96\alpha^7+480\alpha^6-1472\alpha^5\\ +2496\alpha^4-2528\alpha^3+1120\alpha^2\end{array}}{\begin{array}{l}32\alpha^9-288\alpha^8+1632\alpha^7+(\alpha^2-2\alpha+5)^{3/2}(8\alpha^6-48\alpha^5+216\alpha^4-544\alpha^3\\ +1080\alpha^2-1200\alpha+1000)-6048\alpha^6+16800\alpha^5+(\alpha^2-2\alpha+5)^{5/2}(24\alpha^4-96\alpha^3\\ +240\alpha^2-288\alpha+120)-34272\alpha^4+53024\alpha^3-58080\alpha^2+43200\alpha-16000\end{array}}$$

$$r^x_{B2,k3} = \left(\frac{\Delta k_3}{k_3}\right)^2 \frac{\begin{gathered}\sqrt{\alpha^2-2\alpha+5}(-49\alpha^8+294\alpha^7-1055\alpha^6+2260\alpha^5-3071\alpha^4+2406\alpha^3-785\alpha^2)\\ +(\alpha^2-2\alpha+5)^{3/2}(5\alpha^6-20\alpha^5-98\alpha^4+236\alpha^3-59\alpha^2)+(\alpha^2-2\alpha+5)^{5/2}\\ \times(41\alpha^4-82\alpha^3+9\alpha^2)+3\alpha^2(\alpha^2-2\alpha+5)^{7/2}+96\alpha^7-480\alpha^6+1472\alpha^5-2496\alpha^4\\ -2528\alpha^3-1120\alpha^2\end{gathered}}{\begin{gathered}32\alpha^9+288\alpha^8-1632\alpha^7+(\alpha^2-2\alpha+5)^{3/2}(8\alpha^6-48\alpha^5+216\alpha^4-544\alpha^3\\ +1080\alpha^2-1200\alpha+1000)-6048\alpha^6+16800\alpha^5+(\alpha^2-2\alpha+5)^{5/2}\\ \times(24\alpha^4-96\alpha^3+240\alpha^2-288\alpha+120)-34272\alpha^4-53024\alpha^3+58080\alpha^2\\ -43200\alpha+16000\end{gathered}} \quad (35)$$

The gradients were evaluated using the *outermap* facility in *Macsyma* [28]. These results imply that, if β^x is defined at $\alpha = 1$, the eigenvector figures of merit for the two modes are

$$\beta^x_{B1} = 1/16, \qquad \beta^x_{B2} = 1/16. \quad (36)$$

The definition of β^x has included a factor of 2, to be consistent with the definition of β^ω. A negative sign is also necessary to account for the second order nature of the sensitivity

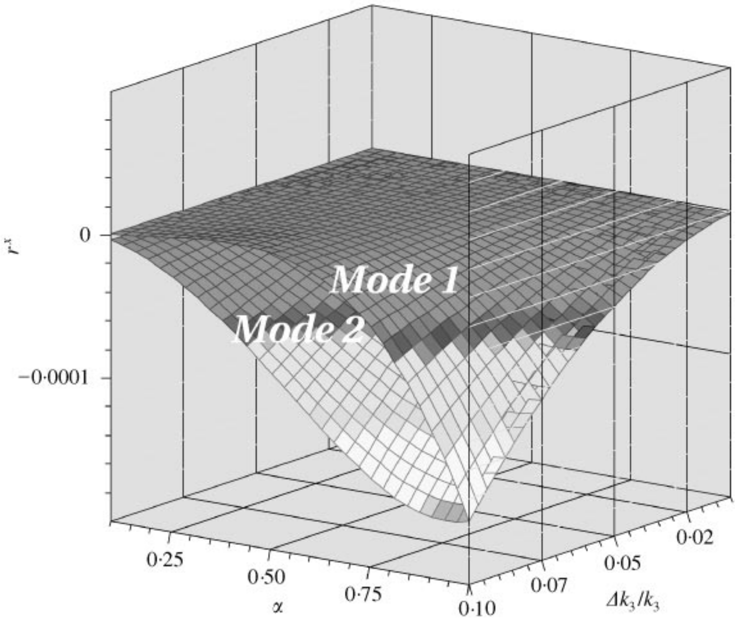

Figure 4. Eigenvector sensitivities $r^x_{B1,k3}$ and $r^x_{B2,k3}$.

TABLE 1

β for 2-d.o.f. model

Model	ω_1	ω_2	$\hat{\mathbf{x}}^{(1)}$	$\hat{\mathbf{x}}^{(2)}$
A	0	2/3	0	1/16
B	1/2	1/6	0	1/16

and the behaviour of the inner product. The final results are plotted in Figure 4. The eigenvector sensitivity is seen to increase with mode number.

The FOMs are summarized in Table 1, and this presentation is especially useful for gaging all the eigensensitivities *in toto*. Perhaps the most useful observation is that certain eigenparameter sensitivities (both eigenvalue and eigenvector) may vanish identically, which is fairly well known. On the other hand, despite what may be called conventional wisdom, there is evidently no basis to favour the eigenvector inner product over the more sensitive eigenvalue measures, particularly when the goal is to identify the onset of small $\Delta\mathbf{k}$'s.

4. DISCUSSION

The small number of elements in the previous example enforces symmetry to a certain extent. The analysis can be extended to $n = 4$ in order to generalize the results. Again, the parameter k_2 will be the element varied, and eigenvalue gradients will be calculated for $\mathbf{k} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2$, and evaluated at selected values of α and $\Delta\mathbf{k} = \Delta k_2/k_2\mathbf{e}_2$. For $n = 4$, there are three cases of interest: perturbations of the centre element, intermediate element, and an end element. The stiffness components for these cases are

$$\begin{bmatrix} 2k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & k_2 & 0 \\ 0 & -k_2 & k_1+k_2 & -k_1 \\ 0 & 0 & -k_1 & 2k_1 \end{bmatrix}, \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_1+k_2 & -k_1 & 0 \\ 0 & -k_1 & 2k_1 & -k_1 \\ 0 & 0 & -k_1 & 2k_1 \end{bmatrix},$$

$$\begin{bmatrix} k_1+k_2 & -k_1 & 0 & 0 \\ -k_1 & 2k_1 & -k_1 & 0 \\ 0 & -k_1 & 2k_1 & -k_1 \\ 0 & 0 & -k_1 & 2k_1 \end{bmatrix}. \tag{37}$$

The problem is simplified somewhat by defining a dimensionless eigenvalue, $\lambda^* = \lambda/\lambda_0$. The modified eigenproblem becomes $\lfloor 2\lambda^*\boldsymbol{I} - \boldsymbol{K}^* \rfloor = 0$, where the dimensionless $\boldsymbol{K}^*$ are given by

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 1+\alpha & -\alpha & 0 \\ 0 & -\alpha & 1+\alpha & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 1+\alpha & -\alpha & 0 & 0 \\ -\alpha & 1+\alpha & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 1+\alpha & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \tag{38}$$

respectively. The eigenvalues for case A are

$$\lambda_i^* \doteq \left\{ \frac{(3-\sqrt{5})}{4}, -\frac{\sqrt{4\alpha^2 - 4\alpha + 5} - 2\alpha - 3}{4}, \frac{(3+\sqrt{5})}{4}, \frac{\sqrt{4\alpha^2 - 4\alpha + 5} + 2\alpha + 3}{4} \right\}. \tag{39}$$

For $\alpha = 1$, all cases reduce to $\lambda_i^* \doteq \{0{\cdot}19098, 0{\cdot}69098, 1{\cdot}30902, 1{\cdot}80902\}$, which split symmetrically about unity, consistent with $n = 4$. The sensitivity coefficient β^ω can be

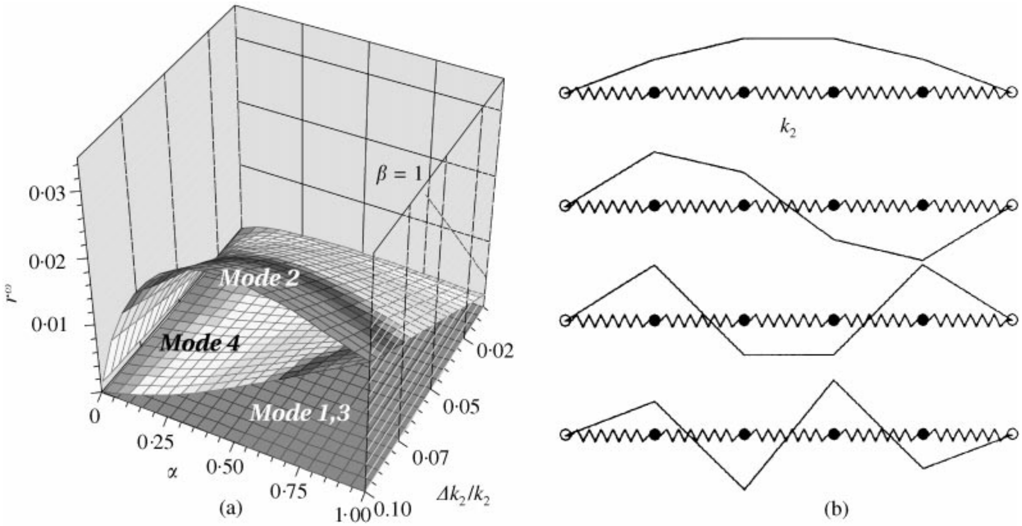

Figure 5. (a) Eigenfrequency sensitivities $r^{\omega^*}_{A1,k2}$, $r^{\omega^*}_{A2,k2}, r^{\omega^*}_{A3,k2}$, and $r^{\omega^*}_{A4,k2}$. (b) Discrete model and eigenvectors—k_2 is the centre spring.

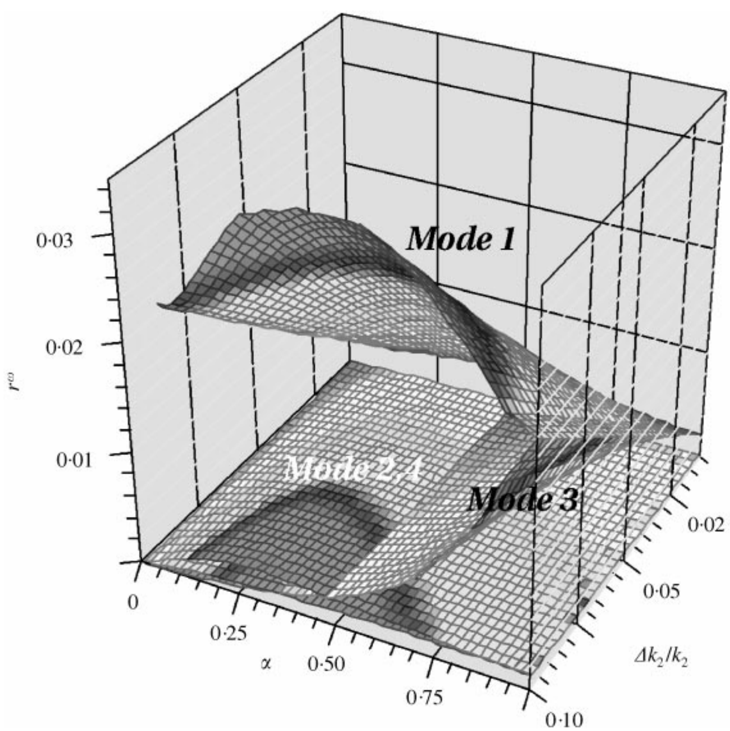

Figure 6. Eigenfrequency sensitivities $r^{\omega^*}_{B1,k2}$, $r^{\omega^*}_{B2,k2}$, $r^{\omega^*}_{B3,k2}$, and $r^{\omega^*}_{B4,k2}$.

obtained using the direct procedure via the chain rule

$$r^{\omega^*} = \frac{\alpha}{2\lambda^*}\frac{\mathrm{d}\lambda^*}{\mathrm{d}\alpha}\frac{\Delta k_2}{k_2}. \tag{40}$$

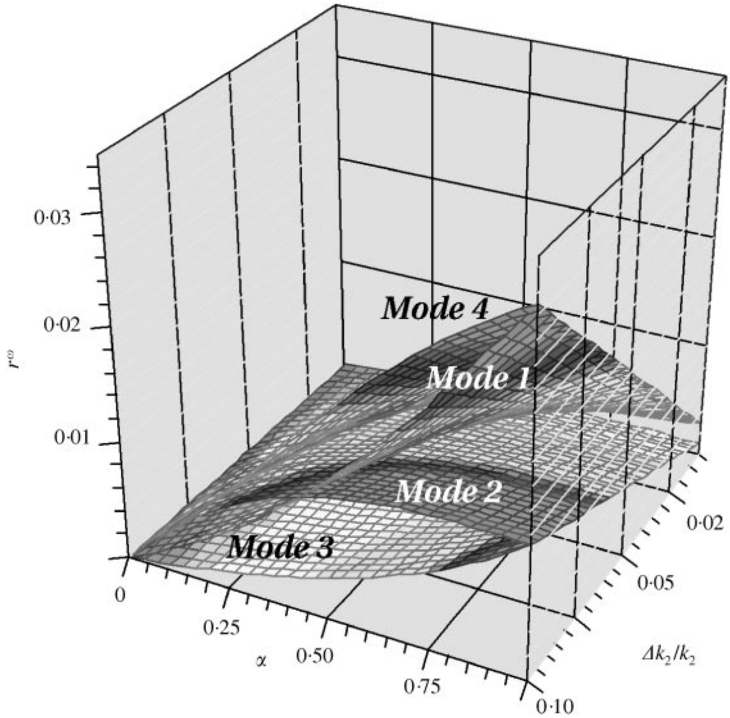

Figure 7. Eigenfrequency sensitivities $r^{\omega^*}_{C1,k2}$, $r^{\omega^*}_{C2,k2}$, $r^{\omega^*}_{C3,k2}$, and $r^{\omega^*}_{C4,k2}$.

TABLE 2

β^* for 4-d.o.f. model

Model	ω_1^*	ω_2^*	ω_3^*	ω_4^*
Case A (112)	0	0·4	0	0·4
Case B (121)	0·45212	0·0276	0·45212	0·0276
Case C (211)	0·62666	0·23936	0·23936	0·62666

This incidentally provides an interpretation for β, $\beta = \mathrm{d}\ln\lambda^*/\mathrm{d}\ln\alpha$. The eigenvalue sensitivity coefficients for the three cases are plotted in Figures 5–7. The FOMs are summarized in Table 2.

The sensitivity coefficients show a complicated behaviour, the most notable feature of which is perhaps the vanishing of β^ω for certain cases. Insight into this behaviour is provided by examining the homogeneous eigenvectors (which are sketched in Figure 5):

$$\hat{\mathbf{x}}^{(1)} \doteq \{0{\cdot}37175, 0{\cdot}6015, 0{\cdot}6015, 0{\cdot}37175\}\, m^{-1/2},$$

$$\hat{\mathbf{x}}^{(2)} \doteq \{0{\cdot}6015, 0{\cdot}37175, -0{\cdot}37175, -0{\cdot}6015\}\, m^{-1/2},$$

$$\hat{\mathbf{x}}^{(3)} \doteq \{0{\cdot}6015, -0{\cdot}37175, -0{\cdot}37175, 0{\cdot}6015\}\, m^{-1/2},$$

$$\hat{\mathbf{x}}^{(4)} \doteq \{0{\cdot}37175, -0{\cdot}6015, 0{\cdot}6015, -0{\cdot}37175\}\, m^{-1/2}. \qquad (41)$$

Evidently, at least for case A, $\beta = 0$ for those modes which are symmetric about k_2 (modes 1 and 3). In all cases, there is curious pairing of $\beta^\omega_{\alpha=1}$.

The discrete d.o.f. correspond to spatially distributed locations in a continuous system, and this observation leads to a quantification of an important result. Simply, the sensitivity f eigenparameters (and objective comparison functions based upon them) to changes in stiffness demonstrate a pronounced spatial dependence. Moreover, the sensitivity is also a function of the original stiffness α. This later observation is important if one is searching for changes in an already "damaged" structure, for instance.

In general, eigenparameter sensitivities are strong functions of the form of the stiffness matrix, and we briefly present a contrasting example to the bar model intended to guide analysis of the structure (a prestressed multispan segmental concrete bridge) shown in Figure 8. Sensitivities were examined by simulating stiffness perturbations through reductions of the flexural rigidity using the global mass and stiffness matrices from a discrete finite element model. These are given in equations (42) and (43) for a three-node (two element) model based upon Hermite nodal basis functions [30]. The model is easily extended to a larger number of elements by inspection:

$$[K] = \left(\begin{array}{cc} \begin{pmatrix} \frac{12EI_1}{h_1^3} & -\frac{12EI_1}{h_1^3} & 0 \\ -\frac{12EI_1}{h_1^3} & \frac{12EI_1}{h_1^3} + \frac{12EI_2}{h_2^3} & -\frac{12EI_2}{h_2^3} \\ 0 & -\frac{12EI_2}{h_2^3} & \frac{12EI_2}{h_2^3} \end{pmatrix} & \begin{pmatrix} \frac{6EI_1}{h_1^2} & \frac{6EI_1}{h_1^2} & 0 \\ -\frac{6EI_1}{h_1^2} & -\frac{6EI_1}{h_1^2} + \frac{6EI_2}{h_2^2} & \frac{6EI_2}{h_2^2} \\ 0 & -\frac{6EI_2}{h_2^2} & -\frac{6EI_2}{h_2^2} \end{pmatrix} \\ \begin{pmatrix} \frac{6EI_1}{h_1^2} & -\frac{6EI_1}{h_1^2} & 0 \\ \frac{6EI_1}{h_1^2} & -\frac{6EI_1}{h_1^2} + \frac{6EI_2}{h_2^2} & -\frac{6EI_2}{h_2^2} \\ 0 & \frac{6EI_2}{h_2^2} & -\frac{6EI_2}{h_2^2} \end{pmatrix} & \begin{pmatrix} \frac{4EI_1}{h_1} & \frac{2EI_1}{h_1} & 0 \\ \frac{2EI_1}{h_1} & \frac{4EI_1}{h_1} + \frac{4EI_2}{h_2} & -\frac{2EI_2}{h_2} \\ 0 & -\frac{2EI_2}{h_2} & \frac{4EI_2}{h_2} \end{pmatrix} \end{array} \right) \tag{42}$$

$$[M] = \bar{\rho} \left[\begin{array}{cc} \begin{pmatrix} \frac{13}{35}h_1 & \frac{9}{70}h_1 & 0 \\ \frac{9}{70}h_1 & \frac{13}{35}h_1 + \frac{9}{70}h_2 & \frac{9}{70}h_2 \\ 0 & \frac{9}{70}h_2 & \frac{13}{35}h_2 \end{pmatrix} & \begin{pmatrix} \frac{11}{210}h_1^2 & -\frac{13}{420}h_1^2 & 0 \\ \frac{13}{420}h_1^2 & -\frac{11}{210}h_1^2 + \frac{11}{210}h_2^2 & -\frac{13}{420}h_2^2 \\ 0 & \frac{13}{420}h_2^2 & -\frac{11}{210}h_2^2 \end{pmatrix} \\ \begin{pmatrix} \frac{11}{210}h_1^2 & \frac{13}{420}h_1^2 & 0 \\ -\frac{13}{420}h_1^2 & -\frac{11}{210}h_1^2 + \frac{11}{210}h_2^2 & \frac{13}{420}h_2^2 \\ 0 & -\frac{13}{420}h_2^2 & -\frac{11}{210}h_2^2 \end{pmatrix} & \begin{pmatrix} \frac{1}{105}h_1^2 & -\frac{1}{140}h_1^3 & 0 \\ -\frac{1}{140}h_1^3 & \frac{1}{105}h_1^3 + \frac{1}{105}h_2^3 & -\frac{1}{140}h_2^3 \\ 0 & -\frac{1}{140}h_2^3 & \frac{1}{105}h_2^3 \end{pmatrix} \end{array} \right] \tag{43}$$

Figure 8. Views of (a) the Kiswaukee bridge and (b) top deck.

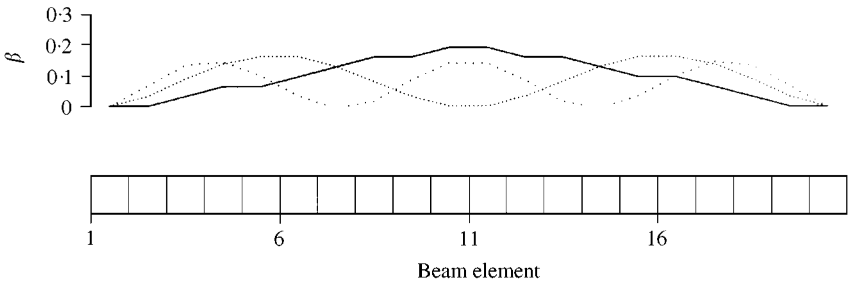

Figure 9. Figure of merit β as a function of location of damage for $EI = 0.5EI|_0$ for a hinged–hinged beam.

The corresponding natural frequencies for the continuous system are given by

$$\omega_i = \sqrt{\frac{EI}{\bar{\rho}}}\left(\frac{i\pi}{L}\right)^2, \quad i = 1, 2, \ldots. \tag{44}$$

For $L = 76.2$ m, $EI|_0 = 1.52 \times 10^{12}$ N m^2, and $\bar{\rho} = 17564.2$ kg/m, the natural frequencies are 2.517, 10.067, and 22.650 Hz. The parameters are based upon a single span of the structure shown in Figure 8. The natural frequencies of the discrete model are 2.515, 9.960, and 21.697 Hz. The sensitivities for a 20-element model are shown in Figure 9. In contrast to bar-like structures, it is the asymmetric modes which show no response to symmetric damage.

The results from the 20-element model were compared to a much more sophisticated FEM model developed from design drawings, using 16 092 elements with the *ABAQUS* code [29]. The general nature of the spatial dependence of the frequency sensitivities agreed with those in Figure 9.

Accurate estimates of the (mass normalized) eigenvectors were available from ABAQUS. These were used to study the behaviour of the eigenvector inner product to reductions of the flexural rigidity in the middle of a span. The results are shown in Figure 10. The behaviour of the MAC agrees with the analysis presented earlier in this paper.

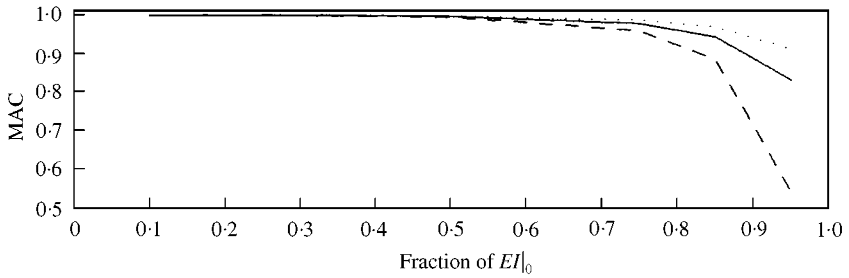

Figure 10. Inner product between $\hat{\mathbf{x}}_0$ and $\hat{\mathbf{x}}_p$ for perturbation at midspan ——, first mode, – – –, second mode, · · · ·, third mode.

The decision to concentrate upon modal updating procedures using the (first order sensitive) modal frequencies was based upon the previous observations, and to initiate continuous monitoring of ambient-excited modal frequencies. The sensitivity calculations provide the basis for selecting the appropriate record lengths required to resolve small frequency perturbations at selected locations. (While the measurements are well within current technology, it should be noted that the dependence of the modal frequencies to temperature must be removed [31, 32].) As a cautionary note, we emphasize in addition that the results from both mechanical systems discussed in this paper indicate that the loss of sensitivity of the eigenparameters does not constitute a sufficient condition for concluding an absence of alterations in element stiffnesses.

5. SUMMARY

This paper has addressed the quantification of eigenparameter sensitivities for discrete linear systems from a very basic perspective. A sensitivity coefficient β based on a normalization to the sensitivity of a single-degree-of-freedom system was formulated for eigenvalues and extended to eigenvectors in a simple way. The use of these coefficients was examined using a discrete model of a bar. It was found that the normalized sensitivity depends upon the location incurring a perturbation. The maximum sensitivity generally, but not exclusively, occurs when all elements have the same stiffness. Bars with existing "damage" show greatly reduced sensitivities. An inner product for the eigenvectors, similar to the MAC, has a second order dependence on stiffness perturbations; numerical results suggested sensitivity an order of magnitude smaller than the eigenvalues. A numerical eigenvalue analysis was applied to models of a beam. While specific differences were identified, the same general cautionary statements apply unchanged. The choice of suitable eigenparameters for system identification and damage assessment should begin by identifying location and mode sensitivities.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the support of this work by the National Science Foundation through Grant NSF9622576. The FEM element calculations presented in Figure 10 were conducted by the doctoral student Zhenglei Chen, whose assistance is greatly appreciated.

REFERENCES

1. M. I. Friswell, J. E. T. Penny and S. D. Garvey 1998 *Computers and Structures* **69**, 547–556. A combined genetic and eigensensitivity algorithm for the location of damage in structures.
2. A. Messina, E. J. Williams and T. Contursi, 1998 *Journal of Sound and Vibration* **216**, 791–808. Structural damage detection by a sensitivity and statistical-based method.
3. S. W. Doebling, F. M. Hemez, L. D. Peterson and C. Farhat 1979 *AIAA Journal* **35**, 693–699. Improved damage location accuracy using strain energy-based mode selection criteria.
4. I. A. Bolgrabskaya 1996 *Journal of Applied Mathematics Mechanics* **60**, 343–347. A justification for investigating the dynamic properties of an elastic bar using a model of a system of coupled rigid bodies.
5. M. A. El-Kady and A. A. Al-Ohaly 1997 *Journal of Sound and Vibration* **199**, 464–471. Fast eigenvalue sensitivity calculations for special structures of system derivatives.
6. M. A. Norris and L. Meirovitch 1988 *NASA CR*-182739. On the problem of modeling for parameter identification in distributed structures.
7. S. Chen, D. Song and A. Ma 1994 *Communications on Numerical Methods in Engineering* **10**, 112–119. Eigensolution reanalysis of modified structures using perturbations and Rayleigh quotients.
8. U. Prells and M. I. Friswell 1997 *AIAA Journal* **35**, 1363–1368. Partial derivatives of repeated eigenvalues and their derivatives.
9. P. Lancaster 1964 *Numerische Mathematik* **6**, 377–387. On eigenvalues of matrices dependent on a parameter.
10. A. L. Andrew 1979 *Journal of the Institute of Mathematics and its Applications* **24**, 209–218. Iterative computation of derivatives of eigenvalues and eigenvectors.
11. C. S. Rudisill and Y. Y. Chu 1975 *AIAA Journal* **13**, 834–837. Numerical methods for evaluating the derivatives of eigenvalues and eigenvectors.
12. B. Gollan 1987 *Mathematical Programming* **30**, 67–81. Eigenvalue perturbations and nonlinear parametric optimization.
13. M. L. Overton 1992 *SIAM Journal of Optimization* **2**, 88–120. Large-scale optimization of eigenvalues.
14. P. Nylen and F. Uhlig 1997 *Linear Algebra and Its Applications* **254**, 409–425. Inverse problems associated with spring–mass systems.
15. R. L. Fox and M. P. Kapoor 1968 *AIAA Journal* **6**, 2426–2429. Rates of change of eigenvalues and eigenvectors.
16. D. L. Taylor and T. R. Kane 1975 *Journal of Applied Mechanics* **42**, 478–483. Multi-parameter quadratic eigenproblems.
17. R. B. Nelson 1976 *AIAA Journal* **14**, 1201–1205. Simplified calculation of eigenvector derivatives.
18. M. M. Yuen 1985 *Journal of Sound and Vibration* **103**, 301–310. A numerical study of the eigenparameters of a damaged cantilever.
19. H. M. Adelman 1986 *AIAA Journal* **24**, 823–832. Sensitivity analysis of discrete structural systems.
20. K. B. Lim, J. L. Junkins and B. P. Wang 1987 *Journal of Guidance* **10**, 581–587. Re-examination of eigenvector derivatives.
21. M. L. Bernard and A. J. Bronowicki 1994 *AIAA Journal* **32**, 1500–1506. Modal expansion method for eigensensitivity with repeated roots.
22. T. Y. Chen 1995 *AIAA Journal* **34**, 1531–1533. Eigenvector derivatives for doubly repeated eigenvalues.
23. K. F. Alvin 1997 *AIAA Journal* **35**, 1760–1766. Efficient computation of eigenvector sensitivities for structural dynamics.
24. Y. Kim, S. Lee and J. L. Junkins 1995 *Journal of Guidance, Control, and Dynamics* **18**, 899–906. Eigenvector derivatives for mechanical second order systems.
25. S. J. Cox 1995 *Journal of Functional Analysis* **133**, 30–40. The generalized gradient at a multiple eigenvalue.
26. De-Wen Zhang and Fu-Shang Wei 1997 *AIAA Journal* **35**, 712–718. Effecient computation of many eigenvector derivatives using dynamic flexibility method.
27. M. I. Friswell 1994 *Journal of Guidance* **18**, 919–921. Calculation of second and higher order eigenvector derivatives.
28. 1996 *Macsyma Mathematics and System Reference Manual*. Macsyma Inc. 16th edition, 20 Academy Street, Arlington, MA 02476, USA.

29. Z. CHEN, D. SATPATHI and M. L. WANG 1999 *ABAQUS User's Conference*, *May* 26–28, *Chester* U.K., 135. Numerical investigation of cracking mechanisms of the Kiswaukee bridge using ABAQUS.
30. W. X. REN, T. XIANGGUANG and Z. ZHENG 1999. *Computers and Structures* **71**, 105–119. Nonlinear analysis of plane frames using rigid body-spring discrete element method.
31. P. M. DIXIT 1999 *M.S. thesis*, *Department of Civil and Materials Engineering*, *University of Illinois at Chicago*. The detection of perturbations of modal parameters using a probabilistic comparison technique.
32. M. A. WAHAB and G. D. ROECK 1997 *Structural Engineering International* **7**, 266–270. Effect of temperature on dynamic system parameters of a highway bridge.

APPENDIX A: NOMENCLATURE

$\hat{}$	denotes a mass-normalized eigenvector
$\mathbf{x}^{(i)}$	ith eigenvector
$\hat{\mathbf{X}}$	modal tensor
λ_i	ith eigenvalue
λ_0	eigenvalue associated with a single mass and two spring elements ($= 2\lambda_f$)
λ_f	eigenvalue for free oscillator (k/m)
λ^*	dimensionless eigenvalue
$\boldsymbol{M}$	mass tensor, dim $\boldsymbol{M} = n \times n$
$\boldsymbol{K}$	stiffness tensor, dim $\boldsymbol{K} = n \times n$
$\boldsymbol{K}^d$	dynamic stiffness tensor
$\mathbf{k}$	parameterization of $\boldsymbol{K}$
$\boldsymbol{\Lambda}$	eigenvalue tensor
r^{ω}	frequency sensitivity coefficient
r^{x}	eigenvector sensitivity coefficient
MAC	modal assurance criterion
β	figure of merit
α	ratio of stiffness elements
Δ	denotes a small perturbation
ω	angular frequency
$\overleftarrow{\boldsymbol{\nabla}}^{\mathbf{k}}$	left gradient operator w.r.t. stiffness parameters $\mathbf{k}$
$\mathbf{e}_i$	Cartesian base vector