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DAMPING IDENTIFICATION IN MULTI-DEGREE-OF-FREEDOM SYSTEMS VIA A WAVELET-LOGARITHMIC DECREMENT—PART 1: THEORY

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A wavelet-based formula similar to the logarithmic decrement formula is introduced to estimate damping in multi-degree-of-freedom systems from time-domain responses. Both analytical and numerical approaches are investigated.

1. INTRODUCTION

The problem of damping identification very often arises when analyzing dissipative dynamic systems. Generally speaking, damping is associated with a dissipation of vibration energy explained by internal (friction, microstructural effects, etc.) or external (fluid/structure or soil/structure interactions, etc.) mechanisms as recalled in reference [1]. Among the diversity of models characterizing structural damping (modal damping, Rayleigh damping [2], equivalent viscous damping, etc.), the study is focusing on the identification of damping ratio aimed at describing viscous damping. From a practical point of view, the main stake is to obtain relevant damping estimates of multi-degree-of-freedom (m.d.o.f.) systems proceeding to several damping measurements [2, 3] that are very sensitive to noise or excitation defaults.

In the frame of applications, the study of continuous systems modelled by partial differential equations (PDE) with boundary and initial conditions for both displacements and stresses is performed using a discretization of the PDE, via a Rayleigh–Ritz procedure for instance. So, systems with N d.o.f. may be obtained, governed by the equation

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = f(t), \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, respectively, denote mass, damping and stiffness matrices, $f(t)$ stands for the external forcing: let us assume that it does not depend on the displacement vector $\mathbf{X}$ and the velocity vector $\dot{\mathbf{X}}$. The term $\mathbf{C}\dot{\mathbf{X}}$ corresponds to a mathematical expression of viscous damping proportional to velocity. Basile's hypothesis [4] is very often verified: real modal decoupling is realized either because $\mathbf{C}$ is given by a sum of the matrices $\mathbf{M}$, $\mathbf{K}$ and powers or because the special Basley's condition [5, 6] is verified. In the modal basis, one is lead to study the dynamic response of N uncoupled s.d.o.f. oscillations governed by the equation

$$\ddot{q}_j + 2\xi_j \Omega_j \dot{q}_j + \Omega_j^2 q_j = f(t), \quad 1 \leqslant j \leqslant N \tag{2}$$

whose general solution is given by

$$q_j(t) = q_{j0}\mathrm{e}^{-c_j t}\cos(\omega_j t + \varphi_j)\cdot \mathrm{H}_y(t), \quad 1 \leqslant j \leqslant N, \tag{3}$$

where $c_j = \xi_j \Omega_j$, $\omega_j = \Omega_j\sqrt{1-\xi^2}$, φ_j, respectively, stand for the damping, the pseudo-pulsation and the phase of the jth mode, $\mathrm{H}_y(t)$ denoting the Heaviside function. Theoretically, the free solution of equation (1) in the physical basis is written as a linear combination of modal responses $(q_j)_{1\leqslant j \leqslant N}$ given by

$$X_i(t) = \sum_{i=1}^{N} A_{ij}\cdot q_j(t), \tag{4}$$

where A_{ij} stands for the jth coordinate of the ith eigenmode.

Here, we assume that the experimental process consists in moving the system away from its equilibrium position and letting it move in free vibrations or exciting the system with a forced vibration monitoring. This first procedure is often required to identify viscous damping using time-domain techniques such as the logarithmic decrement [7, 8]. In practice, the response is not recorded in the modal basis but in the physical basis and is perturbed by noise measurement: eigenfrequencies associated with these responses are assumed to be easily separated from the main studied eigenfrequency. In such a case, common identification methods cannot be used because the "pure" signal cannot be distinguished from perturbations. If modal decoupling is not possible, signal components recordings of the type (4) are considered in the following study; though modal decoupling remains a major drawback when identifying viscous damping, the case of an s.d.o.f. oscillator governed by the equation

$$\ddot{x} + 2c\dot{x} + \omega_0^2 x = f(t) \tag{5}$$

is nevertheless interesting to understand and test the efficiency of an identification procedure; analysis of either free vibrations, or Green kernel or response to an external sinusoidal forcing of system (5) provides theoretical expressions for c via the theory of ordinary differential equations, Fourier or Laplace transforms, or any other classical theoretical tool.

During the last decade, numerous applications in mechanics have been extensively using wavelets. Coca and Billings [9] performed a continuous-time system identification using wavelet patterns. Staszewski [10] developed a wavelet-based feature selection procedure to defect faults in vibratory problems; Staszewski and Worden analyzed chaotic behavior using wavelets in reference [11]; Lamarque and Malasoma [12] introduced an extension of Lyapunov exponents based on continuous wavelet transform. According to damping identification, Ruzzene *et al.* [13] and Staszewski [1] performed a continuous wavelet analysis of the free response of dynamic systems to extract modal features: they separated modal contributions owing to the rapidly decreasing properties of the progressive Morlet wavelet. In reference [1], Staszewski also introduced a wavelet reconstruction formula to approximate the impulse response of m.d.o.f. systems. The main purpose of this paper is to analyze a damped signal recorded in the physical basis with wavelet patterns optimally localized both in time and frequency domains and to extract "local" information to estimate the damping ratio. As a result, a wavelet logarithmic decrement formula is introduced.

The paper is organized as follows: in Section 2, a non-exhaustive list of standard identification techniques is recalled. In Section 3, we investigate the determination of viscous damping by using a continuous wavelet transform. In Section 4, the latter method is adapted in the frame of a multiresolution analysis more suitable to analyze sampled signals than continuous analysis. In Section 5, we perform an improved determination of damping via a superabundant analysis. Then we draw some conclusions according to the wavelet logarithmic decrement formula. Part II will be dedicated to the validation of the new wavelet procedure on real time-series responses borrowed from *in situ* experiments carried out on a civil engineering building.

2. QUICK OVERVIEW OF IDENTIFICATION TECHNIQUES

The simplest procedure that is commonly used to identify viscous damping is the logarithmic decrement formula (6) depicted by

$$c = \frac{1}{T} \ln \left| \frac{x(t)}{x(t+T)} \right|, \tag{6}$$

where

$$x(t) = X\mathrm{e}^{-ct} \cos(\omega t + \varphi) + perturbation \tag{7}$$

denotes the free response of system (5) and T the pseudo-period of this response. It is successful when dealing with s.d.o.f. systems in association with noise pre-filtering, but soon becomes inaccurate when the system has m.d.o.f. Because of noise measurement or because response (7) consists of several modal features, system (5) is interesting to test the reliability of damping identification procedures. For numerical purposes, the perturbation is now standing for a reduced centered Gaussian noise of standard deviation $B \in [0\%, 50\%]$, and c denotes the damping ratio. Standard damping measurements and criteria may be found in references [2, 3]. Alternative methods such as multi-input–multi-output Volterra series [14] are available to analyze non-linear systems from the corresponding higher order frequency response functions (HFRFs). Since the scope of the present study only deals with time-domain procedures, we do not give exhaustive explanations on frequency-domain methods. In reference [13], Ruzzene *et al.* developed a continuous wavelet procedure based on a progressive Morlet wavelet analysis to extract modal parameters of m.d.o.f. systems: processing discrete approximates of the continuous wavelet transform (CWT) of a multi-modal response, they provide damping estimates from the instantaneous amplitude and phase of the signal using an approach similar to the Hilbert transform (HT). Band-pass frequency components are filtered by selecting the CWT dilation scale that matches the corresponding frequency mode, in the peculiar case of the Morlet analysis. In reference [1], Staszewski introduces a discrete wavelet reconstruction formula permitting one to recover the impulse response of m.d.o.f. systems, selecting narrow frequency bands of a damped signal near the natural frequency to be extracted. A logarithmic regression technique similar to the logarithmic decrement is then used to extract a damping ratio estimate from the impulse response envelope. Here we describe a new wavelet logarithmic decrement formula developed in the frame of both a general continuous wavelet analysis and a general multiresolution analysis [15]. This method simultaneously provides the modal decoupling and a damping ratio estimate.

3. CONTINUOUS WAVELET TRANSFORM—DAMPING ESTIMATE

In this section, a new wavelet-based logarithmic decrement formula is introduced to provide the damping of a specific mode in a multi-modal signal response depicted hereafter by

$$f(t) = \left[A\mathrm{e}^{-ct}\cos(\omega t + \varphi) + \sum_{i=1}^{N-1} A_i \mathrm{e}^{-c_i t}\cos(\omega_i t + \varphi_i) \right] \mathrm{H}_y(t) \tag{8}$$

or

$$f(t) = \left[A\mathrm{e}^{-ct}\cos(\omega t + \varphi) + \sum_{i=1}^{N-1} \alpha_i A \mathrm{e}^{-\delta_i ct}\cos(\omega + \varepsilon_i)t + \varphi_i) \right] \mathrm{H}_y(t) \tag{9}$$

or still

$$f(t) = f_\omega(t) + \sum_{i=1}^{N-1} f_{\omega_i}(t) \quad \text{with} \quad \left| \begin{array}{l} f_\omega(t) = A\mathrm{e}^{-ct}\cos(\omega t + \varphi)\,\mathrm{H}_y(t), \\ f_{\omega_i}(t) = A\mathrm{e}^{-c_i t}\cos(\omega_i t + \varphi_i)\,\mathrm{H}_y(t), \end{array} \right. \tag{10}$$

where A and $A_i = \alpha_i A$ denote the amplitudes, c and $c_i = \delta_i c$ denote the viscous dampings, ω and $\omega_i = \omega + \varepsilon_i$ denote the natural frequencies, φ and φ_i stand for the phases of the fundamental mode f_ω and the ith harmonic response f_{ω_i}, H_y denoting the Heaviside function. Here, we assume that the sequence of natural frequencies is sorted as

$$\omega < \omega_1 < \cdots < \omega_i < \cdots < \omega_{N-1}. \tag{11}$$

Considering the CWT of a signal f defined by

$$W_f^g(b, a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(t)\, \overline{g\left(\frac{t-b}{a}\right)}\, \mathrm{d}t \quad \forall (b, a) \in \mathbb{R} \times \mathbb{R}^{*+} \tag{12}$$

bound to an analyzing function $g \in L^2(\mathbb{R})$ checking some oscillation properties [16, pp. 909–910]

$$C_g = \int_{-\infty}^{+\infty} \frac{|\hat{g}(\omega)|^2}{|\omega|}\, \mathrm{d}\omega < +\infty, \tag{13}$$

one obtains integrating by part the wavelet transform of f:

$$\begin{aligned} W_f^g(b, a) = \sqrt{a} A \mathrm{e}^{-cb} \Bigg[& \int_{-\infty}^{+\infty} \mathrm{e}^{-cau} \cos(\omega(au + b) + \varphi)\ \mathrm{H}_y(au + b)\overline{g(u)}\, \mathrm{d}u \\ & - \mathrm{e}^{cb}\, \overline{g\left(-\frac{b}{a}\right)} \sum_{i=1}^{N-1} \frac{\alpha_i}{\varepsilon_i a} \sin(\varphi_i) - \sum_{i=1}^{N-1} \frac{\alpha_i \mathrm{e}^{(1-\delta_i)cb}}{\varepsilon_i a} \int_{-\infty}^{+\infty} \mathrm{e}^{-\delta_i cau}\, \mathrm{H}_y(au + b) \\ & \times \sin((\omega + \varepsilon_i)(au + b) + \varphi_i) [\overline{g'(u)} - \delta_i ca \overline{g(u)}]\, \mathrm{d}u \\ & - \sum_{i=1}^{N-1} \frac{\alpha_i \mathrm{e}^{(1-\delta_i)cb} a\omega}{\varepsilon_i a} \int_{-\infty}^{+\infty} \mathrm{e}^{-\delta_i cau}\, \mathrm{H}_y(au + b) \cos((\omega + \varepsilon_i)(au + b) + \varphi_i)\, \overline{g(u)}\, \mathrm{d}u \Bigg]. \end{aligned} \tag{14}$$

Choosing a wavelet $g \in L^2(\mathbb{R})$ whose support is completely included in $\mathbb{R}^+$ and so that the integral $\int_{\mathbb{R}} \mathrm{e}^{-cau} \cos(\omega au + \varphi)\overline{g(u)}\, \mathrm{d}u$ does not vanish (this condition being essentially reduced to consider a wavelet g whose first momentum is non-trivial), a damping estimation of c may be exhibited as for the logarithmic-decrement formula when focusing on the successive peaks $b_n = nT$ and $b_m = mT$ of signal f. Injecting the T-pseudo-periodicity of f in formula (14), it yields

$$\frac{W_f^g(nT,a)}{W_f^g(mT,a)} = \mathrm{e}^{(m-n)Tc}\left[\frac{1-\dfrac{\sum_{i=1}^{N-1}\alpha_a \mathrm{e}^{(1-\delta_i)cnT}(M_0^{|g'-\delta_i cag|}+a\omega M_0^{|g|})O\left(\dfrac{1}{\inf_{1\leqslant i<N}(\varepsilon_i a)}\right)}{\int_{-\infty}^{+\infty}\mathrm{e}^{-cau}\cos(\omega au+\varphi)\overline{g(u)}\,\mathrm{d}u}}{1-\dfrac{\sum_{i=1}^{N-1}\alpha_i \mathrm{e}^{(1-\delta_i)cmT}(M_0^{|g'-\delta_i cag|}+a\omega M_0^{|g|})O\left(\dfrac{1}{\inf_{1\leqslant i<N}(\varepsilon_i a)}\right)}{\int_{-\infty}^{+\infty}\mathrm{e}^{-cau}\cos(\omega au+\varphi)\overline{g(u)}\,\mathrm{d}u}}\right], \tag{15}$$

where $T = 2\pi/\omega$ is the signal fundamental pseudo-period and $M_n^g = \int_{-\infty}^{+\infty} x^n g(x)\,\mathrm{d}x$ is the nth momentum of g. From expression (15), a damping estimate of signal (9) is finally given by

$$c = \frac{1}{(m-n)T}\ln\left|\frac{W_f^g(nT,a)}{W_f^g(mT,a)}\right| + \sum_{i=1}^{N-1}\alpha_i\left|\frac{\mathrm{e}^{(1-\delta_i)mTc}-\mathrm{e}^{(1-\delta_i)nTc}}{(m-n)T}\right| O\left(\frac{1}{\inf_{1\leqslant i<N}(\varepsilon_i a)}\right) \tag{16}$$

as soon as the analyzing scale a accurately separates the natural frequencies, checking the forth-coming admissibility conditions

$$a\omega = O(1) \quad \text{and} \quad \inf_{1\leqslant i<N}(\varepsilon_i a) \gg 1. \tag{17}$$

Several notices may be pointed out:

- Formula (16) clearly looks like the logarithmic-decrement formula (6). Admissibility conditions $a\omega = O(1)$ and $\varepsilon_i a \gg 1$, $\forall i \in [1, N-1]$ are closely related to the intrinsic properties of the time-scale analysis of a signal, to the optical analogy (focus on b with magnification $1/a$) and to the frequential separation power. It is sufficient to adapt the analyzing scale so that higher frequency modes are filtered ($\varepsilon_i a \geqslant 1$) and so that the fundamental mode remains captured ($a\omega = O(1)$). Assuming that the analyzing wavelet g is well localized both in time and frequency domains, a relationship between the dilation parameter a and the corresponding frequency f which is focused by the wavelet transform may be obtained by $a = f_g/f$, f_g denoting the frequency that matches the maximum value of the Fourier transform of g.
- Physical arguments claim that the wavelet-based formula produces relevant damping estimations, stabilized by small relative amplitudes $\alpha_i = A_i/A$ and damping ratios $\delta_i = c_i/c$ so that either $\delta_i > 1$ or still $\delta_i \gg 1$ according to the higher frequency modes. Moreover, corrective errors are exponentially decreased by an increase of peak deviation $b_n - b_m = (n-m)T$. It is also noticeable that the phases φ and φ_i have no impact on the damping estimation.
- The procedure can be generalized straightforwardly to extract the damping c_i attached to the ith mode of frequency ω_i as soon as it is applied to a filtered component of signal f cancelling the fundamental and the first $(i-1)$ harmonic modes (low-pass filtering):

$$c_i \simeq \frac{1}{(m-n)T_i}\ln\left|\frac{W^g_{f-f_\omega-\sum_{j=1}^{i-1}f_{\omega_j}}(nT_i,a)}{W^g_{f-f_\omega-\sum_{j=1}^{i-1}f_{\omega_j}}(mT_i,a)}\right| \quad \forall 1\leqslant i<N, \tag{18}$$

where $T_i = 2\pi/\omega_i$ is the pseudo-period of the ith frequency mode.

3.1. CONCLUSION OF THE CONTINUOUS ANALYSIS

Having carefully chosen a scale of analysis a so that $\omega a = O(1)$ and $\inf_{1 \leqslant i < N}(\varepsilon_i a) \gg 1$ we have for $(n, m) \in \mathbb{N}^2$,

$$c \simeq \frac{1}{(m-n)T} \ln \left| \frac{W_f^g(nT, a)}{W_f^g(mT, a)} \right|, \quad m \neq n. \tag{19}$$

The wavelet-logarithmic formula just built permits one to identify the modal damping of a perturbed signal (9). Formula (19) is valid whether the perturbation is either a modal super-position with a discrete spectrum, or a purely stochastic perturbation or still a combination of two perturbations of these two types.

A Fourier expansion would permit us to classify perturbations into two classes:

- Perturbation with small amplitudes and a spectrum close to the eigenfrequency ω of the oscillator (9).
- Perturbation with mean amplitude, and a spectrum possessing only high frequencies according to ω.

In the first case, corrective terms provided by the perturbation with small amplitudes are small (order $O(1/\inf_i(\varepsilon_i a))$). In the second case, perturbations of large amplitudes do not modify the estimate of damping for easily filtered by the wavelet analysis. The time-frequency localization diagram of a wavelet ψ [17, pp. 27–28] presented in Figure 1 justifies this claim. Indeed, the admissibility condition (13) for ψ insures a good localization both in time (support$(\psi) \simeq [t_{min}, t_{max}]$) and in frequency (support$(\hat{\psi}) \simeq [\omega_{min}, \omega_{max}]$) of the expanded and translated pattern $\psi(\omega_0(t - t_0))$. Then any event (t_0, ω_0) of the signal is

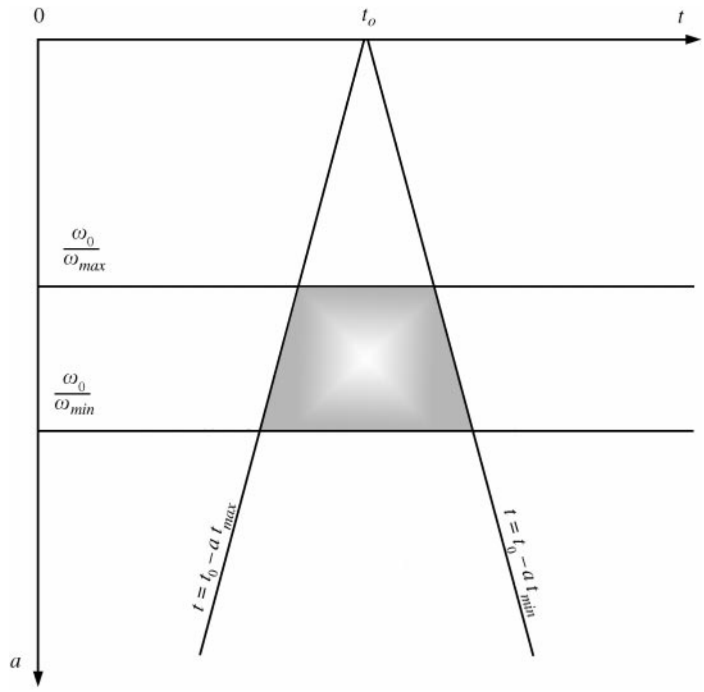

Figure 1. A wavelet time-frequency localization diagram.

localized by the wavelet analysis inside a trapezoidal hatched domain as depicted in Figure 1. Finally, a convenient choice of the scale analysis a can filter high signal frequencies, only keeping local informations of the time-frequency plane.

From a practical point of view, a wavelet filtering is required to successively cancel lower frequency modes and extract the viscous damping c_i of the ith harmonic response of signal (9). The frame of multiresolution analysis [18] of $L^2(\mathbb{R})$ and $L^2(\mathbb{R}/\mathbb{Z})$ (1-periodic signals) also provides a suitable way to build Hilbertian basis allowing to select specific frequency components of a signal. Therefore, a discrete counterpart of formula (19) is developed in this context.

4. MULTIRESOLUTION ANALYSIS AND DAMPING ESTIMATION

Wavelet theory is divided into two distinct parts so the Fourier analysis is: a continuous theory as depicted in Section 3 and a discrete formulation which develops concepts of wavelet series, Hilbertian bases of functional spaces [19, 20], and multiresolution analysis (MRA) [18]. At the same time, building adapted wavelets has become a great challenge [16, 21–23], the most famous ones probably being the compactly supported Daubechies wavelets and the Spline wavelets. The following study emphasizes a new version of the wavelet logarithmic decrement based on a periodic Spline MRA [17]. Though functional background and notations are recalled hereafter, the reader is invited to read references [15, 17, 18] for further details. In particular, MRA of $L^2(\mathbb{R})$ are separated from MRA of $L^2(\mathbb{R}/\mathbb{Z})$ the space of function periodized on [0, 1].

4.1. MULTIRESOLUTION ANALYSIS OF $L^2(\mathbb{R})$

Multiresolution analysis of a signal f consists in building successive approximations $(f_j)_{0 \leqslant j \leqslant J}$ of f including details $(f_{d_i})_{0 \leqslant i < j}$ up to a resolution 2^{-j}, the scale $j = 0$ and J being, respectively, the coarser and the finer scales. The reader may read the explanations in references [15, p. 915, 17, p. 38]. From a mathematical point of view, the finer approximation f_J of f may be indifferently expressed by its scaling series or its wavelet series representation

$$f \simeq f_J = \sum_{k \in \mathbb{Z}} \langle f, \varphi_{Jk} \rangle \varphi_{Jk} \tag{20}$$

$$\simeq f_0 + \sum_{j=0}^{J-1} f_{d_j} = \sum_{k \in \mathbb{Z}} \langle f, \varphi_{0k} \rangle \varphi_{0k} + \sum_{j=0}^{J-1} \sum_{k \in \mathbb{Z}} \langle f, \psi_{jk} \rangle \psi_{jk}, \tag{21}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $L^2(\mathbb{R})$, $(\varphi_{jk})_{k \in \mathbb{Z}}^{0 \leqslant j < J}$ and $(\psi_{jk})_{k \in \mathbb{Z}}^{0 \leqslant j < J}$ being two orthonormal bases involving dilated and translated patterns of a single pair (φ, ψ), respectively, the scaling function and the mother wavelet of the MRA:

$$\forall k \in \mathbb{Z} \quad \left| \begin{array}{l} \varphi_{jk}(x) = 2^{j/2} \varphi(2^j x - k), \\ \psi_{jk}(x) = 2^{j/2} \psi(2^j x - k). \end{array} \right. \tag{22}$$

The analogy with the continuous wavelet transform is noticeable, (a, b) being replaced by the pair $(2^{-j}, k.2^{-j})$. In practice, the construction of the other wavelet ψ is derived from the one of the scaling function φ which in turn is derived from the determination of admissible filtering functions [15, 16].

4.2. MULTIRESOLUTION ANALYSIS OF $L^2(\mathbb{R}/\mathbb{Z})$

A multiresolution analysis of 1-periodic signals of $L^2(\mathbb{R}/\mathbb{Z})$ leans on a periodization of an MRA of $L^2(\mathbb{R})$, by folding the approximation and details spaces on themselves with a Poisson summation [24]. By the end, an MRA of $L^2(\mathbb{R}/\mathbb{Z})$ may be associated with a pair of analyzing functions $(\tilde{\varphi}, \tilde{\psi})$ deduced from (φ, ψ) via the relationship

$$\tilde{\varphi}_{jk}(x) = \sum_{r\in\mathbb{Z}} \varphi_{jk}(x+r) \quad \text{and} \quad \tilde{\psi}_{jk}(x) = \sum_{r\in\mathbb{Z}} \psi_{jk}(x+r). \tag{23}$$

Associating the common inner product $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,\mathrm{d}x$ to $L^2(\mathbb{R}/\mathbb{Z})$, an approximation f_J of f may be recomposed from the finite-dimension scaling and wavelet series

$$f \simeq f_J = \sum_{k=0}^{2^J-1} \tilde{s}_{Jk}\tilde{\varphi}_{Jk} \tag{24}$$

$$\simeq f_0 + \sum_{j=0}^{J-1} f_{d_j} = \langle f, 1\rangle + \sum_{j=0}^{J-1}\sum_{k=0}^{2^J-1} \tilde{d}_{jk}\tilde{\psi}_{jk}, \tag{25}$$

where $\tilde{s}_{jk} = \langle f, \tilde{\varphi}_{jk}\rangle$ and $\tilde{d}_{jk} = \langle f, \tilde{\psi}_{jk}\rangle$ are, respectively, referred to as the scaling and the wavelet coefficients of signal f.

Practically speaking, the time-scale analysis of a periodic signal may be easily performed using fast trees algorithms [17, 18] allowing one to select or cancel details of peculiar channels of details of a signal $f = (f_k)_{0 \leqslant k < 2^J}$ sampled in the interval [0, 1].†

The decomposition mechanism detailed in the periodical case [16, pp. 934–938, 24, 25, pp. 55–59] and depicted in Figure 2 consists in successively convoluting the signal f, previously interpolated with a filter $\tilde{L}_J$, with a low-pass filter $\tilde{H}_j$ and a band-pass filter $\tilde{G}_j$ and then downsampling the obtained coefficients by a factor of two—convolution and downsampling being performed in the Fourier space—By the end, scaling coefficients $\tilde{s}_{jk}$ and wavelet coefficients $\tilde{d}_{jk}$ are recursively computed from the coefficients $\tilde{s}_{j+1,k}$ at scale $j+1$ with a complexity of order $O(N)$ [17]. Having stored the detail coefficients $\tilde{d}_{jk}$, the process is reversible and the signal may be recomposed with no loss of information, owing to similar convolution and upsampling operations. It should be pointed out that the existence of an interpolated filter $\tilde{L}_J$ is proved only in the peculiar case of periodic Spline wavelets [19].

4.3. A DISCRETE WAVELET-LOGARITHMIC DECREMENT FORMULA

A discrete wavelet-logarithmic decrement formula is straightforwardly derived from the continuous formula (19) in the case of an MRA of $L^2(\mathbb{R}/\mathbb{Z})$, using the scaling function $\tilde{\varphi}$ as the underlying analyzing function g. In this context, the dilation parameter $a = 2^{j_a}$ is set to both capture the fundamental frequency mode f_ω and to filter the higher frequency modes $(f_{\omega_i})_{1 \leqslant i < N}$ and eventually additional noisy perturbations in the pyramidal decomposition of signal f. Local extrema of scaling coefficients $\tilde{s}_{jk}$ bound to translation factors $b_k = k/2^{j_a}$ and $b_l = l/2^{j_a}$, $0 \leqslant k, l < 2^{j_a}$ are the counterparts of the terms $W_f^g(kT, a)$ and $W_f^g(lT, a)$ in the continuous formula (16). The fast tree algorithms exposed in section 4.2 have been implemented in C++ language using a periodic Spline wavelet analysis [25, p. 61].

†Whether one is concerned with a signal f sampled on a window $[0, T]$, the scaling and wavelet coefficients exposed in the algorithm are the ones of function g defined as g: $x \in [0, T] \to f(xT)$.

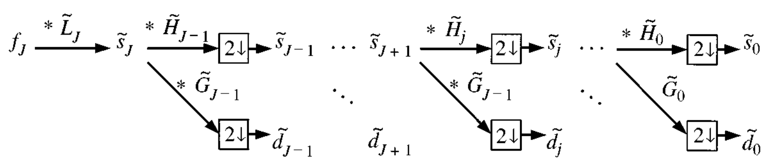

Figure 2. Pyramidal decomposition tree in periodic wavelets.

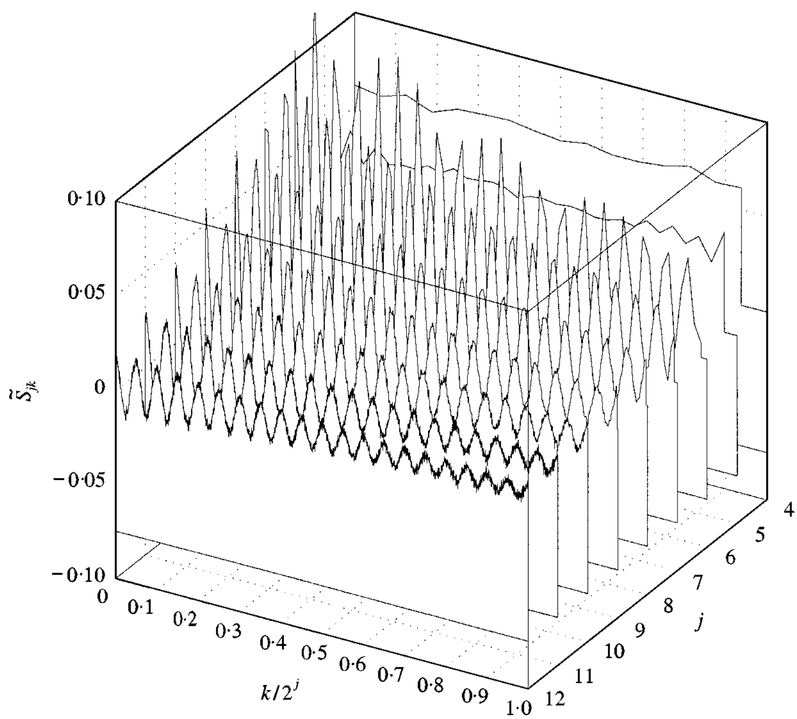

Figure 3. A signal decomposition ($c = 0{\cdot}4$, $\omega = 10\pi$ rad s^{-1}, $B = 0{\cdot}5$): scaling coefficients pyramid $\tilde{s}_{jk}$.

Assuming that the sampled signal f of the damping ratio c is living inside a given time window $[0, T]$, we also study the renormalized signal g of damping ratio c/T defined as $g : x = t/T \to f(xT)$ on $[0, 1]$. By extension, the discrete formula involving an analyzing scale j_a is written as

$$c \simeq \frac{2^{j_a}}{k-l} \ln \left| \frac{\tilde{s}_{j_a l}}{\tilde{s}_{j_a k}} \right| \tag{26}$$

In practice, only the scaling function $\tilde{\varphi}$ is considered as a potential applicant to analyze signal f for its first moment is non-trivial. Formula (26) may be viewed as a classical logarithmic decrement applied to the filtered component f_{j_a} of signal f.

Handling the validation of formula (26), a signal f of amplitude $X = 1{\cdot}0$, of damping $c = 0{\cdot}4$, of fundamental frequency $\omega = 10\pi$ rad s^{-1} and disrupted with a Gaussian noise of standard deviation $B = 0{\cdot}5$, has been decomposed. Figure 3 displays a distribution of the scaling coefficients in terms of the focal parameters $(k/2^{j_a})_{k \in [0, 2^{j_a}]}$ and the analyzing scale j_a. It is noticeable that the more the scale j_a becomes rough, the more noise is filtered, as it was

TABLE 1

Comparison of performance between a few identification procedures

		Logarithmic decrement (Wavelet pre-filtering)		Wavelet decrement (Standard analysis)		Wavelet decrement (Superabundant analysis)	
c (%)	b (%)	$\frac{\Delta c}{c}$ (%)	c	$\frac{\Delta c}{c}$ (%)	c	$\frac{\Delta c}{c}$ (%)	c
0·1	0·0	151·20	0·00251	151·20	0·00251	0·00	0·00100
	0·1	160·34	0·00260	160·95	0·00261	3·55	0·00103
	0·2	154·14	0·00254	154·42	0·00254	3·75	0·00104
	0·5	191·12	0·00291	189·69	0·00290	22·33	0·00122
1	0	0·00	0·01000	0·00	0·01000	0·00	0·01000
	1	6·00	0·01000	6·60	0·01000	2·45	0·00998
	2	11·32	0·01027	10·80	0·01020	4·04	0·01003
	5	72·94	0·01729	74·93	0·01749	19·46	0·01177
10	0	0·00	0·10000	0·00	0·10000	0·02	0·10002
	1	0·72	0·10018	0·72	0·10023	0·36	0·10002
	2	1·21	0·10023	1·07	0·10021	0·59	0·10001
	5	2·76	0·10020	2·65	0·10015	1·28	0·09968
	10	6·43	0·10357	5·78	0·10411	2·73	0·09984
	20	25·19	0·12418	22·82	0·12075	6·22	0·10147
	50	139·52	0·23952	135·19	0·23519	26·33	0·12492
100	0	0·15	1·00151	0·15	1·00151	0·02	0·99981
	1	0·76	1·00695	0·62	1·00528	0·12	0·99978
	2	1·90	1·01869	1·93	1·01903	0·23	1·00014
	5	8·31	1·08311	8·70	1·08700	0·69	0·99809
	10	22·93	1·22932	21·72	1·21721	1·43	1·00132
	20	35·06	1·34444	34·48	1·33180	2·70	1·00548
	50	19·45	1·08466	17·19	1·05384	10·60	1·09308

expected. One may yet distinguish a loss of definition in curves associated with coarse analyzing scales, with too few significant coefficients. Consequently, estimating the damping c with formula (26) does not provide results with better accuracy than with the logarithmic decrement associated with a pre-filtering as pointed out in Table 1. Improving the coefficient extrema localization, we introduce the concept of superabundant analysis developed by Perrier [25].

5. SUPERABUNDANT ANALYSIS AND DAMPING ESTIMATION

The standard decomposition drawback is to represent a signal with a small number of analyzing scales $j \in [0, J-1]$ and the number of coefficients getting more and more smaller as j becomes coarser (2^j coefficients per scale j). The reliability of formula (26) strongly depends on an efficient detection of successive coefficient extrema, it is common to use an "oblique" analysis which associates 2^{2J-1} "intermediate" scaling coefficients to 2^J sampling points: 2^J coefficients for each 2^{J-1} scales. Heuristically, the superabundant analysis brings a compromise between the continuous analysis (see Section 3) and a standard orthonormal MRA (see Section 4.1).

5.1. INTERMEDIATE SCALING FUNCTIONS

Let us consider an MRA of $L^2(\mathbb{R}/\mathbb{Z})$ generated by a particular periodization of an MRA of $L^2(\mathbb{R})$ bound to φ. Freezing the finest analyzing scale J, we define an intermediate scaling function by

$$\Phi_{\alpha k}(x) = \sqrt{\alpha} \sum_{r \in \mathbb{Z}} \varphi\left(\alpha\left(x + r - \frac{k}{2^J}\right)\right), \quad k \in [0, 2^J - 1], \qquad \alpha = \frac{2^{J-1}}{m}, \quad 1 \leqslant m \leqslant 2^{J-1}. \tag{27}$$

For specific intermediate scales $\alpha = 2^j$ and equating $E(k/2^{J-j}) = E(k/2^{J-j}) + k/2^{J-j} - E(k/2^{J-j})$, where $E(x)$ stands for the integer part of x, it yields

$$\Phi_{2^j k}(x) = \tilde{\varphi}_{jE(k/2^{J-j})}\left(x - \frac{k}{2^J} + 2^{-j}E\left(\frac{k}{2^{J-j}}\right)\right). \tag{28}$$

with $E(k/2^{J-j})$ ranging over the whole interval $[0, 2^J - 1]$, it follows that $(\Phi_{\alpha k})_{k \in [0,\, 2^J - 1]}$ also constitutes an oblique family of patterns spanning the approximation space up to a resolution 2^{-j}. Identically as in Section 4.1, we introduce intermediate scaling coefficients of a signal belonging to $L^2(\mathbb{R}/\mathbb{Z})$ by

$$O_{\alpha k}(f) = \int_0^1 f(x)\Phi_{\alpha k}(x)\,\mathrm{d}x. \tag{29}$$

These coefficients in superabundant quantity improve the signal definition, allowing to better localize the signal extrema which results in more relevant damping estimates than with the wavelet logarithmic decrement formula. Also, fast tree algorithms exist in the context of the superabundant multiresolution analysis and permit an effective computation of coefficients $O_{\alpha k}(f)$ in an $O(N)$ procedure thoroughly detailed in reference [25, p. 64].

5.2. A SUPERABUNDANT WAVELET LOGARITHMIC DECREMENT FORMULA

Extending the discrete wavelet formula (26) to the superabundant configuration, one obtains

$$c \simeq \frac{2^J}{k - l} \ln \left| \frac{O_{2^{j_\alpha} l}(f)}{O_{2^{j_\alpha} k}(f)} \right|, \tag{30}$$

where J is the finest scale equal to the interpolation scale and j_α is an admissible analysis scale according to the criteria (17).

Alternatively, the damping bound the ith modal response may be extracted using the following formula:

$$c_i \simeq \frac{2^J}{k - l} \ln \left| \frac{O_{2^{j_\alpha} l}(f - f_\omega - \sum_{j=1}^{j=i-1} f_{\omega_j})}{O_{2^{j_\alpha} k}(f - f_\omega - \sum_{j=1}^{j=i-1} f_{\omega_j})} \right| \quad \forall i \in [1, N-1] \tag{31}$$

with indices k and l corresponding to some extrema of the ith mode.

5.3. NUMERICAL EXPERIMENTS

Superabundant pyramidal algorithms bound to a periodical spline analysis have been implemented to validate formula (30). As a descriptive example, we analyzed a signal

$f(X = 1{\cdot}0, \quad c = 0{\cdot}4, \quad \omega = 10\,\pi\ \mathrm{rad\,s^{-1}}, \quad B = 0{\cdot}5)$ and displayed its superabundant decomposition pyramid in Figure 4. It is noticeable that the more the decomposition channel $\log_2(\alpha)$ decreases, the more the perturbation noise is filtered. Numerical validation of formula (30) requires to identify the position of location of intermediate coefficients extrema. Validity conditions (17) must be respected, i.e., fitting the decomposition channel $\alpha = 2^{j_a}$ to the channel of noise appearance, taking advantage of the filtering abilities of wavelets. Experimentally speaking, the difficulty is to fit the frontier between noise and "pure" signal components as witnessed in Figure 4.

Automatic recognition of local extrema being proceeded, a campaign of simulations has been developed in order to test the reliability and noise resistance of formula (30). Mean relative error distributions have been drawn for signals of type (7), of amplitude $X = 1{\cdot}0$ perturbed with a random Gaussian noise characterized by a null mean and a standard deviation $B \in [0, 50]$ (%) and for a given damping ratio c ranging between 0·1 and 100%. For each parameter set, a median relative error was estimated using a data base including 100 random processes. The relative error map displayed in Figure 5 reveals a good behavior of the superabundant wavelet decrement formula: the mean relative error is indeed very often below the threshold of 10% according to $c \in [10\%, 100\%]$, with a linear growth in terms of the noise level. Wavelet-based formula's performance highlights an excellent noise resistance of the so-called identification procedure, sensitively improving estimates for c ranging in large damping area ($c \in [20\%, 100\%]$) as well as medium damping area ($c \in [5\%, 20\%]$). As physical systems are often characterized by dampings of weak and even very weak amplitude, a similar study was conducted in the range of $c \in [0{\cdot}1\%, 5\%]$. Few damping estimates are gathered in Table 1 and attest that the wavelet analysis is not only a wavelet filtering of the damped signal. As for the estimation of very small dampings ($c \simeq 0{\cdot}1\%$), Table 1 shows that formula (30) greatly improves the forecasts compared to the

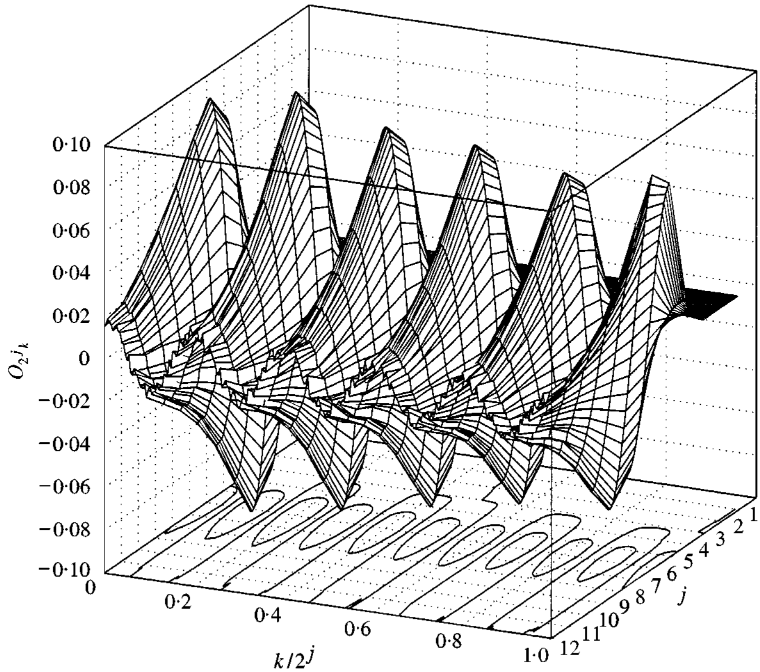

Figure 4. Superabundant decomposition of signal f with $c = 0{\cdot}4$, $\omega = 10\pi$ rad s^{-1}, $B = 0{\cdot}5$: intermediate scaling coefficients pyramid $O_{2^j k}$.

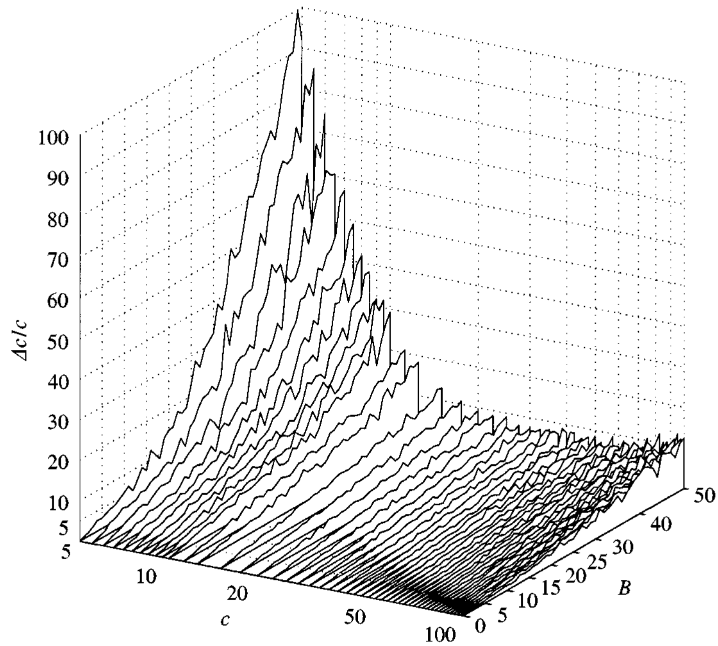

Figure 5. Wavelet-based logarithmic decrement (30): Relative error distribution $\Delta c/c$ (%) in terms of the damping ratio c (%) and the noise level B (%).

ones computed with its competitive methods whose estimations literally explode. Generally, speaking, the experimenter may consider the wavelet logarithmic decrement as bringing a sensitive improvement with forecasts' errors roughly less than 20% of the "ultimate" damping value, while at the same time its classical counterparts give unrealistic values easily reaching the level of 150 or 190%. Yet, these results must be handled with care if we speak in terms of signal-to-noise ratios that are utopian.

6. CONCLUSION

This study allowed us to build a new procedure to identify the damping ratio of dynamic systems leaning on a wavelet analysis of the time-response. From a signal processing point of view, this procedure may be assimilated as a logarithmic decrement transform applied to a filtered component of the original response. The corresponding formula which is based upon rigorous mathematical developments has been validated first analytically in the context of a general continuous wavelet transform and then numerically by comparison with exact reference solutions in the frame of a superabundant multiresolution analysis. Comparisons between exact and numerical estimates revealed an excellent noise resistance when the signal-to-noise ratio is fairly bad and demonstrated the relevance of the wavelet-based procedure which may be distinguished from standard counterparts by its faculty to analyze a signal at different resolution scales. Though physical arguments are required to set the optimizing analyzing scale which permits to separate modal (respectively purely random) contributions from the fundamental mode, the wavelet-logarithmic formula is mathematically very robust and a few improvements are still investigated to cope with ill-separated dynamic responses involving close frequency modes. Part II of this paper is now concerned with the practical application of the wavelet-based formula to estimate damping ratios of a m.d.o.f. system like a civil engineering building from *in situ* dynamic responses to shock or harmonic excitations.

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