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APPLICATION OF MULTIPOLE EXPANSIONS TO SOUND GENERATION FROM DUCTED UNSTEADY COMBUSTION PROCESSES

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Predicting and controlling the behavior of oscillatory combustion systems requires an understanding of the interactions between the periodic combustion and acoustic processes. Due to the complexity of these interactions, a number of past analyses of these problems have assumed a "concentrated", infinitely thin, combustion region. Although such treatments of the combustion process as a lumped element are attractive because of their simplicity, this paper shows that they may produce significant errors. It is further shown that such errors can be minimized by use of multipole expansion techniques similar to those used in acoustic radiation problems. Specifically, this paper develops a formalism for describing the combustion region as a series of lumped parameters whose magnitudes are proportional to ascending powers of the ratio L/λ, where L and λ are the length of the combustion region and the acoustic wavelength respectively. In the limit as L/λ goes to zero, only the first term of this expansion is significant and the "concentrated" combustion approximation is recovered. As L/λ increases, additional parameters (e.g., higher order terms) are needed to accurately describe the effect of the combustion process on the acoustic oscillations and must be included in the analysis. The paper closes with example calculations showing that accuracy in modelling combustion–acoustics interactions can be significantly increased by implementation of the developed technique.

1. INTRODUCTION

Predicting and controlling the behavior of systems undergoing oscillatory combustion requires an understanding of the interactions between the periodic combustion and acoustic processes. This generally requires determining the characteristics of the unsteady combustion process and its effect on the acoustic field by modelling or measurements. Prior efforts at modelling these interactions have either assumed that the combustion zone is infinitely thin and can be treated as a lumped heat source (e.g., see references [1–8]) or modelled the spatial dependence of the combustion process heat release throughout the combustion region (e.g., see reference [9]). In the latter case, measured data was sometimes used to describe the distributed combustion region [9].

The lumped parameter approach has been used extensively because it only requires modelling a single quantity that contains all relevant information about the combustion process. However, while it can be shown that such an approach is rigorously correct in the limit as the combustion zone width, L, approaches zero, it is unclear at what point (in terms of the ratio of the combustion zone thickness and the wavelength, i.e., L/λ) this approximation fails for finite-sized combustion regions. It will be shown in this paper that

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the errors in this approximation become significant even when L/λ is still quite small. On the other hand, while modelling the spatial dependence of the combustion zone appears to be the more desirable approach, in practice it is very difficult (if not impossible) to accurately model or measure the spatial distribution of the complex combustion process.

This paper develops a generalization of the lumped parameter approach that is applicable to a much wider class of unsteady combustion problems, while avoiding the difficulties associated with modelling a distributed combustion region. Clearly, the errors produced by the lumped parameter approach increase as L/λ increases because the coupling between the acoustic and combustion processes can no longer be effectively described by a single parameter. This suggests, however, that the accuracy of the lumped parameter approach could be increased by included additional parameters to describe the unsteady combustion process.

The rest of this paper describes the results of a study that developed such an approach. Its principal result is a formalism for "breaking" a compact combustion zone into fundamental "building blocks", analogous to the multiple expansions used in classical acoustics [10]. These "buildings blocks" are a series of lumped elements whose effect on the acoustic field are related to ascending powers of L/λ. In the limit as this ratio approaches zero, only the first term of this expansion is significant and the solution provided by the single lumped parameter approach is recovered. As this ratio increases, terms proportional to higher powers of L/λ become significant in describing the effect of the combustion process on the unsteady flow field and must be included in the analysis.

This decomposition has a number of applications. First, it allows for a clear interpretation of the dominant characteristics of the distributed combustion process insofar as it affects the acoustic field. Second, it shows what simplifications can be made to model distributed combustion process–acoustic interactions to a given order of accuracy. Finally, it suggests measurements that can be made to obtain information about the unsteady combustion system, such as the net acoustic energy flux out of the combustion zone.

2. BACKGROUND

Sound radiation in classical acoustics is often described in terms of fundamental sources, such as a pulsating sphere (i.e., a "monopole"), and from an oscillating sphere (i.e., a "dipole"). Such fundamental radiators serve as building blocks for understanding and classifying the directional and power transmission characteristics of more complex radiators. It is also well known [10] that if the characteristic dimension of an arbitrary body, L, is much smaller than the wavelength, its farfield radiation can be decomposed into that generated by a number of these fundamental sources. The goal of this investigation is to develop an analogous formulation for application to a ducted unsteady combustion process. The approach developed in this paper follows from the observation that, for sufficiently low-frequency oscillations, only the plane wave modes in a duct can propagate while the transverse modes are evanescent and decay rapidly. Thus, the acoustic field in the duct is nearly one dimensional except in a region in and near the flame (see Figure 1). A number of studies (e.g., references [1, 2, 5, 6]) have utilized this fact to derive conservation conditions relating the one-dimensional acoustic field variables across a multi-dimensional flame region. The expressions developed in these studies are equivalent to the Rankine–Hugoniot conditions [11], and are exact only for an infinitely thin flame. For example, it has been shown [2] that for an infinitely thin flame in a duct without mean flow of constant area, A, that these conditions take the form

$$p'_2 - p'_1 = 0, \qquad A(v'_2 - v'_1) = \frac{(\gamma - 1)Q'}{\gamma p}, \tag{1, 2}$$

where subscripts 1 and 2 denote the value of the variables on the up- and downstream sides of the flame region, respectively, and

$$Q' = \iiint_V q' \, \mathrm{d}V. \tag{3}$$

Outside of the combustion region, the propagation of acoustic waves can generally be modelled using one-dimensional acoustics, and equations (1, 2) provide the matching conditions that couple these one-dimensional oscillations across a complex combustion region (see Figure 1), with the aid of the lumped parameter Q'. While equations (1, 2) describe this coupling in terms of a single parameter, this paper seeks to increase the accuracy of this expression by including additional (higher order) parameters that will allow for a description of the characteristics of the unsteady combustion region to greater accuracy (i.e., a generalization of the Rankine–Hugoniot relations to unsteady domains of finite extent). To illustrate the solution approach, it is convenient to work with the following acoustic energy and momentum equations where it is assumed that the oscillations have an $\mathrm{e}^{-\mathrm{i}\omega t}$ time dependence and that mean flow affects are negligible (i.e., the Mach number of the mean flow is very small†):

$$-\mathrm{i}\omega p' + \gamma p \frac{\partial v'_\mathrm{i}}{\partial x_i} = (\gamma - 1)q', \qquad -\mathrm{i}\omega \rho v'_i + \frac{\partial p'}{\partial x_i} = 0. \tag{4, 5}$$

The subsequent derivation will also assume a constant mean density, but these results can be generalized to a region of arbitrarily varying density.

Integrating equations (4, 5) over the volume of the combustion region and applying the divergence theorem to the resulting integrals yields the following equations:

$$\oiint_S \gamma p v'_i n_i \, \mathrm{d}S = (\gamma - 1)\iiint_V q' \, \mathrm{d}V + \mathrm{i}\omega \iiint_V p' \, \mathrm{d}V, \qquad \oiint_S p' n_i \, \mathrm{d}S = \mathrm{i}\omega\rho \iiint_V v'_i \, \mathrm{d}V. \tag{6, 7}$$

In their present form, equations (6, 7) are valid for any region, but not very useful for our purposes. These equations are greatly simplified by evaluating the surface integrals at locations up and downstream of the flame where the acoustic field is one dimensional (see Figure 1). If it is assumed that the flame region is infinitely thin, the last terms in equations (6, 7) become zero and the resulting equations are equivalent to equations (1, 2). However, since the flame zone has a finite thickness L, these terms are not identically zero. Rather, they are $O(kL)$ smaller than the remaining terms in equations (6, 7) (because they are multiplied by the angular frequency $\omega = kc$ and are integrated over the volume $V \sim AL$).

Evaluating these $O(kL)$ volume integral terms in their present form is difficult (or impossible), however, because it requires knowledge of the acoustic field in the combustion

†As pointed out by a reviewer, these neglected mean flow effects have a significant influence upon convected entropy fluctuations (also excited by the combustion process), even in the limit of vanishing Mach number. These entropy disturbances affect the acoustic pressure and velocity matching conditions only in high Mach number flows (e.g., through terms like $\rho' u^2$ in the momentum equation), however, and are not further discussed in the ensuing analysis.

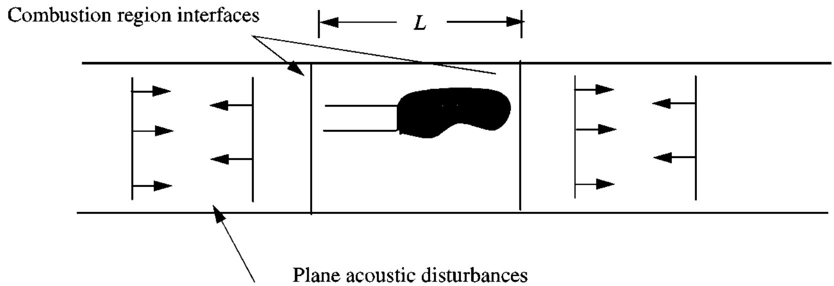

Figure 1. Schematic diagram of ducted combustion system interacting with acoustic disturbances.

region. Consequently, it would be preferable to recast these volume terms into surface integrals because, as previously discussed, it is much easier to determine the one-dimensional acoustic quantities on the up- and downstream surfaces of the combustion zone (see Figure 1).

The following paragraphs will show that these volume integral terms can be evaluated in terms of moments of acoustic quantities over the surface of the combustion region and "higher order" combustion process parameters. To obtain these expressions, take the spatial moments of equations (4, 5) (i.e., multiply by the spatial co-ordinate, **x**), integrate the resulting expressions over the volume of the combustion zone, and integrate by parts to obtain

$$-\mathrm{i}\omega \iiint_V p' x_j \,\mathrm{d}V + \oint\!\!\int_S \gamma p v_i' n_i x_j \,\mathrm{d}S - \iiint_V \gamma p v_j' \,\mathrm{d}V = (\gamma - 1) \iiint_V q' x_j \,\mathrm{d}V, \tag{8}$$

$$-\mathrm{i}\omega\rho \iiint_V v_i' x_j \,\mathrm{d}V + \oint\!\!\int_S p' x_j n_i \,\mathrm{d}S - \iiint_V \delta_{ij} p' \,\mathrm{d}V = 0. \tag{9}$$

Setting $i = j$ in equation (9) and summing over the indices $i = 1, \ldots, N$ yield

$$-\mathrm{i}\omega\rho \iiint_V v_i' x_i \,\mathrm{d}V + \oint\!\!\int_S p' x_i n_i \,\mathrm{d}S - \iiint_V N p' \,\mathrm{d}V = 0. \tag{10}$$

Examination of equations (8) and (10) shows that each contains one of the volume terms from equations (6, 7) in terms of surface integrals and a new volume term that is of $O(kL)^2$. Thus, equation (10) can be used to solve for the volume integral term of the acoustic pressure

$$\iiint_V p' \,\mathrm{d}V = \frac{1}{N}\left(-\mathrm{i}\omega\rho \iiint_V v_i' x_i \,\mathrm{d}V + \oint\!\!\int_S p' x_i n_i \,\mathrm{d}S\right). \tag{11}$$

This volume integral of the pressure can be substituted into equation (6) to obtain

$$\oint\!\!\int_S \gamma p v_i' n_i \,\mathrm{d}S = (\gamma - 1) \iiint_V q' \,\mathrm{d}V + \frac{\mathrm{i}\omega}{N}\left(-\mathrm{i}\omega\rho \iiint_V v_i' x_i \,\mathrm{d}V + \oint\!\!\int_S p' x_i n_i \,\mathrm{d}S\right). \tag{12}$$

If the $O(kL)^2$ volume integral term is neglected, equation (12) is an acoustic energy equation for the combustion region that is accurate to $O(kL)^2$, as opposed to the $O(kL)$ accuracy of equation (2). To produce a correspondingly accurate momentum equation, equation (8) is substituted into equation (7) to obtain

$$\oiint_S p' n_i \,\mathrm{d}S = \frac{\mathrm{i}\omega\rho}{\gamma p}\left(-\,\mathrm{i}\omega \iiint_V p' x_i \,\mathrm{d}V + \oiint_S \gamma p v_j' n_j x_i \,\mathrm{d}S - (\gamma - 1)\iiint_V q' x_i \,\mathrm{d}V\right). \tag{13}$$

To further increase the accuracy of the derived conservation equations (i.e., equations (12, 13)) from $O(kL)^2$ to $O(kL)^3$, it is necessary to evaluate the $O(kL)^2$ volume integrals in these equations (e.g., the term $\mathrm{i}\omega/N(-\mathrm{i}\omega\rho\iiint_V v_i' x_i \,\mathrm{d}V)$ in equation (12)). This is accomplished by taking the second moment of equations (4) and (5) and integrating by parts. The resulting equations will each contain one of the volume terms from equations (12, 13) in terms of surface integrals, and a new volume term that is of $O(kL)^3$. Substituting these expressions into equations (12, 13) produces the following expressions that are accurate to $O(kL)^3$:

$$\iint_S \left(\rho c v_i'\left(1 - \frac{(kx_j)(kx_j)}{2N}\right) - \frac{ikx_i}{N} p'\right) n_i \,\mathrm{d}S = \frac{(\gamma - 1)}{c}\iiint_V q'\left(1 - \frac{(kx_j)(kx_j)}{2N}\right)\mathrm{d}V + O((kL)^3), \tag{14}$$

$$\iint_S \left(-p'\left(1 - \frac{(kx_j)(kx_i)}{N+1}\right) + \mathrm{i}kx_j \rho c v_i'\right) n_i \,\mathrm{d}S = \frac{\mathrm{i}(\gamma - 1)}{c}\iiint_V q' (kx_j)\,\mathrm{d}V + O((kL)^3). \tag{15}$$

While equations (14, 15) are quite general and may be applied to a variety of problems, they will now be simplified to problems where the up and downstream surfaces have the same area, and where the acoustic fields are one dimensional over these surfaces. Defining the origin of the co-ordinate system to be at the midpoint between these two surfaces, equations (14, 15) become

$$A\left[(v_2' - v_1') - \mathrm{i}(kL)\frac{\kappa_1}{\rho c N}(p_2' - p_1') - (kL)^2\frac{\kappa_2}{2N}(v_2' - v_1')\right] = \frac{(\gamma - 1)}{\gamma p}\left(Q' - \frac{(kL)^2}{2N} Q_2'\right) + B_1, \tag{16}$$

$$A\left[\frac{1}{\rho c}(p_2' - p_1') - \mathrm{i}(kL)\kappa_1(v_2' - v_1') - (kL)^2\frac{\kappa_2}{\rho c(N+1)}(p_2' - p_1')\right] = -\,\mathrm{i}\frac{(\gamma - 1)}{\gamma p}(kL)Q_1' + B_2, \tag{17}$$

where the geometric parameters κ_1 and κ_2 are given by the following surface integrals over the regions of integration:

$$\kappa_1 = \frac{1}{A}\iint_S \left(\frac{x_j n_j}{L}\right)\mathrm{d}S, \qquad \kappa_2 = \frac{1}{A}\iint_S \left(\frac{x_j}{L}\right)\left(\frac{x_j}{L}\right)\mathrm{d}S, \tag{18}$$

and the lumped combustion process parameters are given by

$$Q' = \iiint_V q' \,\mathrm{d}V, \qquad Q_1' = \iiint_V q'\left(\frac{x}{L}\right)\mathrm{d}V, \qquad Q_2' = \iiint_V q'\left(\frac{x_j}{L}\right)\left(\frac{x_j}{L}\right)\mathrm{d}V. \tag{19}$$

The boundary terms in equations (16, 17), B_1 and B_2, represent surface integrals from equations (14, 15) evaluated over any additional surfaces within the combustion volume, such as those enclosing flameholding devices or the combustor walls (see Figure 1).

Equations (16, 17) are the principal result of this study. They relate the values of the one-dimensional acoustic fields up and downstream of the combustion region to a series of lumped parameters whose influence on the solution diminishes by powers of kL. For $kL \ll 1$, these equations show, in agreement with the discussion in the Introduction, that all the information necessary to describe the coupling between the unsteady combustion process and the acoustic field is contained in the quantity Q'. However, as kL increases, an accurate description of this coupling requires a second parameter, Q'_1, with influence on the solution of $O(kL)$. In general, a description of this coupling accurate to $O(kL)^n$ requires specifying n lumped combustion process parameters. The examples in the following section show $O(1)$, $O(kL)$ and $O(kL)^2$ calculations that illustrate this point. However, before proceeding to this section, two comments on the solution procedure are in order.

First, although this solution procedure could, in principle, be carried out to an arbitrary order to obtain any specified degree of accuracy, there is a practical limit to its accuracy and utility. That is, applying it to problems where $kL - O(1)$, or where a very high degree of accuracy is desired, may require specification of a large number of parameters, making the complexity of the developed procedure equivalent to that associated with modelling a distributed combustion region. Also, although the fact that these lumped parameters are integrated quantities suggests that their value should be relatively insensitive to errors in the description of the combustion process heat addition, q' (since the local details are averaged out), the higher order terms will be increasingly more sensitive to errors. That is, lumped parameters containing terms of the form $(x)^n q'$ will be more sensitive to errors than terms like $(x)^{n-1} q'$. Thus, in the solution of practical problems, a point will be reached where the accuracy of the solution will not increase by including higher order corrections. For example, the accuracy of the solution may be increased by adding the $O(kL)$ correction, but may be unaffected by including higher order terms. An example in the following section will illustrate this point.

Second, it should be noted that although detailed information about the combustion process is not required in this procedure, the solution does not provide any information about the values of the acoustic variables within the combustion region. That is, information is only obtained about the acoustic quantities on the surface of the region. Thus, a slightly different procedure than the one outlined here must be applied to problems where the combustion process is assumed to be a function of acoustic quantities in the combustion region, as opposed to being known (e.g., $q' = Rp' + Sv'$ as opposed to $q' = f(x)$, where $f(x)$ is known). To apply the multipole procedure to this problem, the functional expression for q' can be inserted into equation (4) and the procedure that resulted in the derivation of equations (14, 15) should be applied to equations (4, 5). This will produce a set of relations analogous to equations (14, 15), but without an explicit source term.

3. EXAMPLE CALCULATIONS

In this section, an example problem is solved in order to illustrate the application and results of the developed approach. Specifically, this section will consider a variation of a problem discussed in reference [2] that applied the first order matching conditions in equations (1, 2) across an infinitely thin combustion region. By allowing the combustion region to have finite width, the advantages of the developed approach will be illustrated by comparing the errors introduced by the various orders of approximation.

For simplicity, this section considers a one-dimensional problem (i.e., $N = 1$) in a duct of length L_{comb} with a combustion region of length L (see Figure 1). The plane waves propagating in the duct outside of the combustion region are described by the following expressions:

$$p'_n = e^{-i\omega t}(A_n e^{ikx} + B_n e^{-ikx}), \qquad v'_n = \frac{1}{\rho c} e^{-i\omega t}(A_n e^{ikx} - B_n e^{-ikx}), \tag{20, 21}$$

where the subscript n denotes the region of interest. The unknown amplitudes, A_n and B_n of the propagating waves are determined by boundary conditions at both ends of the duct and matching conditions across the flame (equations (16, 17)). The boundary conditions used in these calculations are

$$\text{at } x = 0; \quad p' = p_0 e^{-i\omega t}, \qquad \text{at } x = L_{comb}; \quad p' = 0. \tag{22}$$

Finally, it is assumed that the spatial distribution of the unsteady heat release within the combustion zone is given by $q' = \sin(\pi x/L)$. The exact solution for this problem can be determined by solving equations (4, 5).

Figure 2 compares the spatial dependence of the exact acoustic pressure with solutions obtained by the mulipole expansions (i.e., using equations (16, 17) and (20–22)) of different $O(kL)$ accuracy when $L/\lambda = 0{\cdot}02$ (i.e., $kL = 0{\cdot}04\pi$) and $L_{comb}/\lambda = 0{\cdot}77$. In this example, the combustion region thickness is small relative to a wavelength and Figure 2 shows that the acoustic field is well described by the $O(1)$ expansion, although the accuracy increases when the $O(kL)$ or $O(kL)^2$ terms are included. To illustrate the errors introduced when L/λ increases. Figure 3 compares the exact and approximate solutions when $L/\lambda = 0{\cdot}08$ and $L_{comb}/\lambda = 0{\cdot}77$. The figure shows that significant errors result from the $O(1)$ expansion (e.g. errors of up to eighty percent), and that these errors significantly decrease as the higher order terms are included in the matching conditions (e.g., errors of up to 20 and 4 per cent with the $O(kL)$ and $O(kL)^2$ expansions respectively). Since the size of combustion regions in practical combustors are likely to be closer to (or larger than) that employed in the last example, this result shows that including higher order terms in calculations may be necessary to obtain reasonable results.

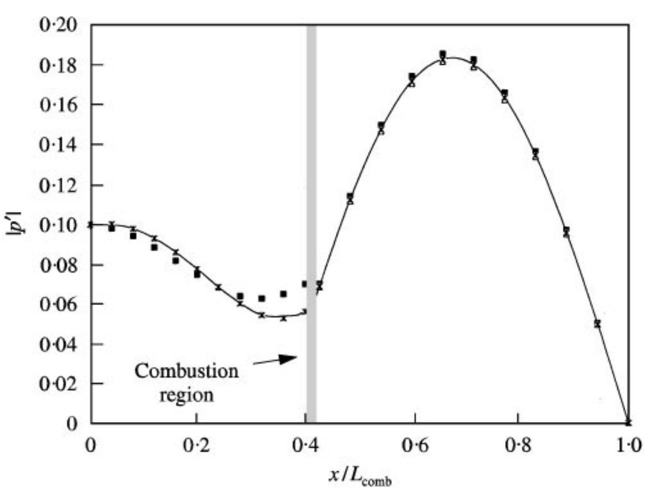

Figure 2. Plot of the exact and predicted mode shape using multipole expansions for $L/\lambda = 0{\cdot}02$ and $L_{comb}/\lambda = 0{\cdot}77$ (——, exact; ■, $O(1)$; △, $O(kL)$, $*$, $O(kL)^2$).

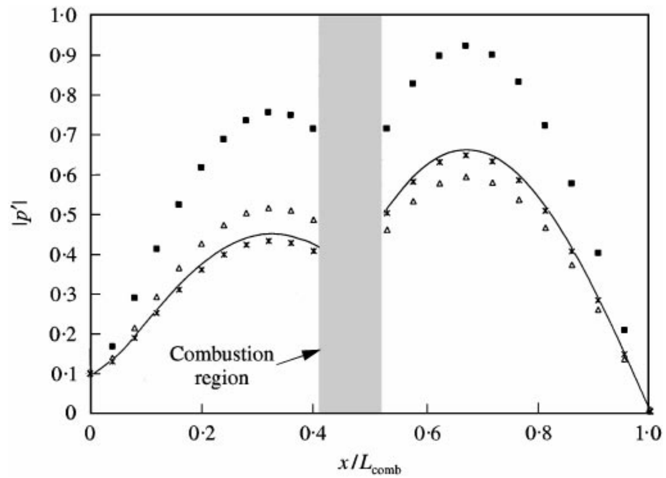

Figure 3. Plot of the exact and predicted mode shape using multipole expansions for $L/\lambda = 0{\cdot}08$ and $L_{comb}/\lambda = 0{\cdot}77$ (——, exact; ■, $O(1)$; △, $O(kL)$, ∗, $O(kL)^2$).

Figures 2 and 3 were obtained assuming "perfect knowledge" of q'. Figure 4 compares the solutions obtained for the same conditions as in Figure 3, but with a 30 per cent, normally distributed, random error inserted in q'. The figure shows that the addition of this error scarcely effects the $O(1)$ calculation, but has larger effects on the higher order corrections. This example emphasizes the increased sensitivity of the higher order corrections to errors that was discussed in the previous section. It should be pointed out that the relatively minor effect this error has on the predicted field emphasizes a point that is the basis for this work: only global information about the combustion process is required to describe low-frequency interactions between the combustion process and the acoustic field.

Having compared the exact modes shapes with those predicted by the multipole expansion procedure, the rest of this section considers the errors introduced by the procedure in calculating the energy added to the acoustic field by the combustion process. The intensity flux through the boundaries of the combustion zone is given by [10]

$$I = A(p'_2 v'_2 - p'_1 v'_1). \tag{23}$$

Figure 5 shows the absolute value of the error ($|(I_{exact} - I_{predicted})/I_{exact}|$) introduced by different orders of the multiple expansion solution. The figure shows that the error tends to zero as the width of the combustion zone approaches zero, but increases rapidly with increasing thickness. It is surprising to note that the error predicted by the standard $O(1)$ expansion reaches a value of almost 100 per cent when the combustion zone is still only one-tenth of a wavelength in size. However, accuracy is substantially increased by including higher order terms.

4. CONCLUSIONS

This paper has developed an approach analogous to the multiple expansions used in classical acoustics that improves current capabilities for modelling combustion–acoustic interaction in duct systems. Example calculations demonstrated substantial improvements

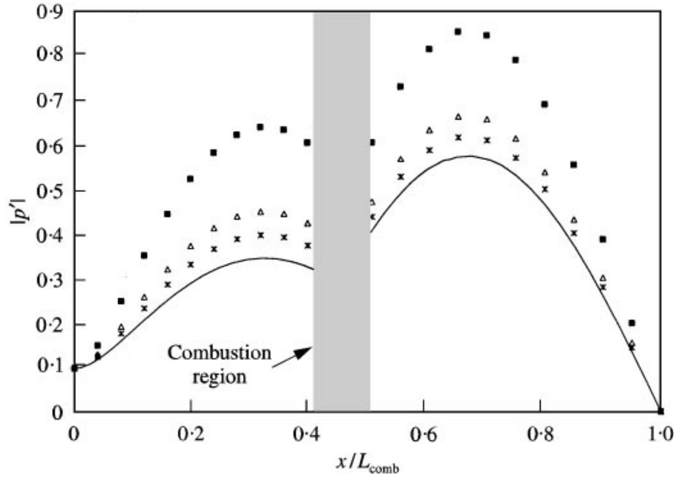

Figure 4. Plot of the exact and predicted mode shape using multipole expansions for $L/\lambda = 0{\cdot}08$ and $L_{comb}/\lambda = 0{\cdot}77$ with the addition of a 30 per cent, normally distributed, random error to q' (——, exact; ■, $O(1)$; △, $O(kL)$, $*$, $O(kL)^2$).

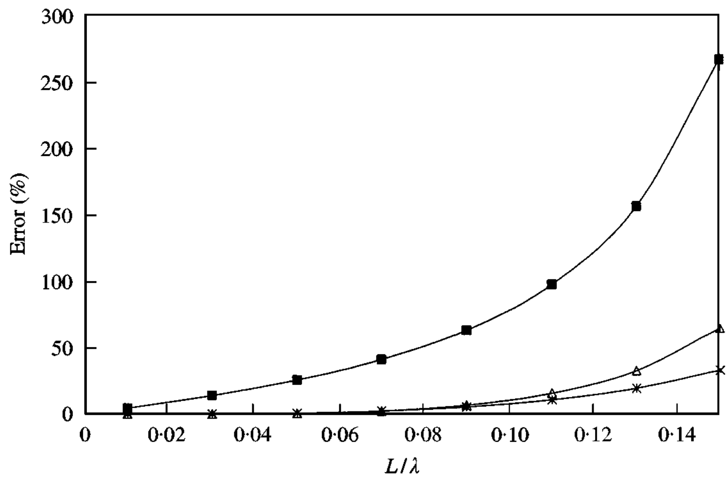

Figure 5. Plot of the error $(|I_{exact} - I_{predicted})/I_{exact}|)$ between the exact and predicted energy flux out of the combustion zone for three different orders of approximation (■, $O(1)$; △, $O(kL)$, $*$, $O(kL)^2$).

(e.g., reduction in errors by a factor of 20) in accuracy in modelling these interactions without significant increases in model complexity. While this paper has primarily emphasized the use of this procedure in improving models of unsteady combustion systems, in closing, several other applications should be pointed out.

One promising application of this technique is to aid the interpretation of experimental or numerical results. Even as increasing levels of information are becoming available about the complex processes occurring in unsteady combustion systems through application of sophisticated experimental techniques and numerical computations, it is often difficult to interpret this data, and more importantly, to determine which features in the mass of information are and are not important. Since decomposing the combustion region into lumped parameters offers a clear interpretation of the dominant characteristics of the distributed combustion process insofar as it affects the acoustic field, the multipole procedure is a convenient tool for such interpretation.

Second, the multipole approach discussed here can be applied or generalized to a variety of situations where volumetric information (that may be difficult to obtain) can be inferred from surface measurements. For example, the procedure discussed in equations (4–17) showed how volume integrals of the acoustic pressure or velocity could be approximated as moments of acoustic quantities over the surface of the volume.

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APPENDIX: NOMENCLATURE

A	area
k	wave number; $2\pi/\lambda$
L	length of the combustion region
n	unit normal vector
N	number of spatial dimensions
p	pressure
q	volumetric rate of heat release
Q	total rate of heat release
x	position vector, $xe_x + ye_y + ze_z$
v	velocity
V	volume

Greek

γ	ratio of specific heats
λ	acoustic wavelength
ρ	fluid density
ω	angular frequency