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MODAL ANALYSIS OF MECHANICAL SYSTEMS WITH IMPACT NON-LINEARITIES: LIMITATIONS TO A MODAL SUPERPOSITION

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This paper presents an attempt to generalize the modal superposition formula to mechanical systems with impact type non-linearities following the procedure introduced for smooth non-linearities. The study is restricted to simple one- and two-degree-of-freedom systems with a unilateral constraint on one of the degrees of freedom, for which the response can be analytically determined. Generalized frequencies, modes and masses are built in the procedure. The results obtained for various sets of parameters point out some limitations to the validity of a general modal superposition formula.

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1. INTRODUCTION

Analysis of the response of structures is convenient if a linear model can fully describe the structure. Within this framework, it is useful to introduce in either the finite or infinite dimension (Hilbertian case) the notion of eigenmodes of the structure. They are either normal modes (defined by adding conservative conditions to the model) or complex modes (taking into account viscous damping for example) [1, 2]. The linear theory of differential systems provides the response of the structure to an external elementary sinusoïdal excitation in an interesting form: the full response is simply the superposition of the responses of each mode to the excitation. Such a formula is well known; this is the superposition formula which is the basis of modal synthesis [2, 3]. The notion of modal synthesis can be extended to the case of sub-structures by using linear operator theory [3, 4].

In the non-linear case, the notion of non-linear modes had been considered first. In the case of mathematically smooth non-linearities and for a finite number of degrees of freedom (d.o.f.) with particular polynomial non-linearities, Rosenberg and others first introduced natural modes [5] and then non-linear normal modes [6–9], and investigated their stability. Since then, many methods have been used to introduce modes (natural, non-linear, non-linear normal, minimal normal, non-linear similar normal, etc.) in the case of non-linear structures; methods derived from the works of Rosenberg [10–13]; a stroboscopic method [14]; methods based on averaging and modal truncation [15–17]; direct or geometrical methods for conservative systems [18–20] or in the Hamiltonian frame [21–25]; Padé approximation [26, 27]; multi-spectral, Volterra series and HFRFs [28, 29]; integral transforms [30]; Lie series [31]; methods based on normal forms in the Hamiltonian [32–34] or general frame [35–40]; or methods using centre manifold theory [41–47] or amplitude equations [48]. Jézéquel and Lamarque extended the modal

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superposition starting from non-linear modes built via normal forms, for systems with a few d.o.f. and smooth non-linearities [49–51], even for the case of complex modes [52]; this method is valid for sufficiently small non-linear oscillations. The non-linear modes and generalized masses obtained depend on the amplitudes of the normal co-ordinates. They are built up in order to agree at best with a resonance equation. The modes do not always verify the reciprocity condition which exists in the linear case.

The question of non-linear modes is obviously related to the search for periodic solutions to non-linear dynamical systems and the study of their stability [52, 53]. It therefore involves a huge amount of literature dealing with numerous analytical methods, perturbation methods, and methods for bifurcation analysis for example references [32, 54–66].

In the field of stochastic behaviour, the question of non-linear modes has been examined already [67] and tools are available [68]. In the case of non-smooth non-linearities, only particular cases have been investigated; that is, normal modes for piecewise linear systems [69, 70]. Sometimes, in order to deal with localized or weak non-linear non-smooth phenomena using linear methods, one can introduce a modified dissipation or stiffness matrix.

From the point of view of dynamics, vibro-impact systems have been thoroughly studied in the literature: global behaviours and periodic solutions have been investigated in the single-d.o.f. case (both analytically and numerically in references [71–74] or by means of a change of variables in reference [75]) and in the two-d.o.f. case (double impact oscillator in references [76, 77] or impact damper in references [78, 79]). Singularities in the dynamics of such systems have been pointed out in references in [80–82], and some authors have examined the effect of dry friction on mechanical systems with impacts in [83–86]. General results for vibro-impact systems can also be found in references [87, 88]. Moreover, a modal approach has been introduced in reference [89] to deal with direct and inverse problems in discrete systems with impacts, based on the theory of non-linear normal modes (see reference [27]).

Nevertheless, to our knowledge, no attempt to build a modal superposition similar to the linear case exists in the case of hard non-smooth non-linearities such as friction or impact. The main aim of this work is to seek some answers to the question about possibility of building a modal superposition and a modal synthesis in the case of structures exhibiting a non-linearity of impact type. The problem is considered for the case of simple systems with one and two d.o.f.

In section 2, a single d.o.f. system is considered. Using a piecewise exact integration (2.1), the periodic responses under sinusoïdal excitation are studied (2.2). The building of a modal superposition in equation (2.3) is then tested by introducing successively a generalized eigenfrequency from free vibrations, a generalized mode and a generalized mass associated with forced oscillations.

In section 3, two-d.o.f. systems are considered. First the case of weak coupling and impact of a mass against a rigid stop is dealt with (3.1). Then the case of strong coupling is examined, again with impacts against a external rigid stop (3.2). The results obtained are applied to the case of direct impacts between two rigid solids (3.3). Finally, in section 4 conclusions are drawn on the relevance of the modal superposition formula obtained.

2. SINGLE-DEGREE-OF-FREEDOM SYSTEM

The system studied consists of a single-d.o.f. damped harmonic oscillator with a unilateral constraint, for which an impact law is defined (see Figure 1). The impact process

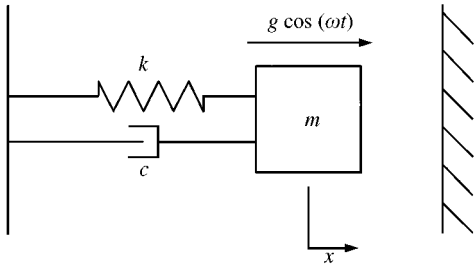

Figure 1. Single-degree-of-freedom system with $k = \omega_1^2 m$, $c = am$, $g = fm$.

is considered to be instantaneous and the behaviour of the system at the time of impact is described using the coefficient of restitution $e \in [0,1]$, characterizing the energy loss during impact. The equations governing the dynamics of the system are then

$$\begin{aligned} &\ddot{x} + a\dot{x} + \omega_1^2 x = f\cos(\omega t), \\ &x(t) \leqslant x_{max}, \\ &\dot{x}(t^+) = -e\dot{x}(t^-) \quad \text{if } x(t) = x_{max}. \end{aligned} \tag{1}$$

If the velocity immediately before the impact at $\bar{t}$ is zero, several cases can occur: if at $\bar{t}$ the acceleration is negative, then the system is still described by equation (1) after the impact and the trajectory is tangent to the stop at $\bar{t}$. If, on the other hand, the acceleration is positive, then the system remains in contact with the stop for a non-zero time interval. It can be shown that sticking never occurs if $f < \omega_1^2 x_{max}$. In the following, it is assumed that the system's parameters satisfy this condition.

2.1. ANALYTICAL SOLUTION

The system with sinusoïdal forcing can be written:

$$\begin{aligned} &\ddot{x} + a\dot{x} + \omega_1^2 x = f\cos(\omega t), \\ &\dot{x}(t^+) = -e\dot{x}(t^-) \quad \text{if } x(t) = x_{max}, \\ &x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0. \end{aligned} \tag{2}$$

Since this system is linear between two consecutive impacts, it is possible to determine a piecewise analytical form of the solution on $\mathbb{R}^+$. Setting $\tilde{\omega}_1 = \sqrt{\omega_1^2 - a^2/4}$ and $\eta = a/2\tilde{\omega}_1$. $\forall k \in \mathbb{N}^*$, the solution on $[t_{k-1}, t_k]$ can be written in the form

$$\begin{aligned} &x(t) = \mathrm{e}^{-at/2}[A_k\cos(\tilde{\omega}_1 t) + B_k\sin(\tilde{\omega}_1 t)] + f_1\cos(\omega t) + f_2\sin(\omega t), \\ &x(t_k^+) = x(t_k^-) = x_{max}, \\ &\dot{x}(t_k^+) = -e\dot{x}(t_k^-), \end{aligned} \tag{3}$$

where $t_0 = 0$ and

$$f_1 = f\frac{\omega_1^2 - \omega^2}{(\omega_1^2 - \omega^2)^2 + a^2\omega^2},$$

$$f_2 = f\frac{a\omega}{(\omega_1^2 - \omega^2)^2 + a^2\omega^2}.$$

From equations (3), the following recursive relation gives the values of the constants A_k and B_k:

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} x_0 - f_1 \\ \dfrac{\dot{x}_0 - f_2\omega}{\tilde{\omega}_1} + \eta(x_0 - f_1) \end{pmatrix},$$

$$\begin{pmatrix} A_{k+1} \\ B_{k+1} \end{pmatrix} = \begin{pmatrix} A_k \\ B_k \end{pmatrix} + (1+e)\,u(A_k, B_k, t_k)\begin{pmatrix} -\sin(\tilde{\omega}_1 t_k) \\ \cos(\tilde{\omega}_1 t_k) \end{pmatrix}, \tag{4}$$

$$\begin{aligned} u(A_k, B_k, t_k) = {} & [\sin(\tilde{\omega}_1 t_k) + \eta\cos(\tilde{\omega}_1 t_k)]\,A_k \\ & + [-\cos(\tilde{\omega}_1 t_k) + \eta\sin(\tilde{\omega}_1 t_k)]\,B_k \\ & + \frac{\omega}{\tilde{\omega}_1}\,\mathrm{e}^{at_k/2}\,[f_1\sin(\omega t_k) - f_2\cos(\omega t_k)]. \end{aligned}$$

2.2. PERIODIC SOLUTIONS

Owing to the analytical form of the solution given by equations (3) and (4) it is possible to seek analytically a periodic solution, similar to that achieved in reference [72] or [71] for example. A solution of period nT with k impacts per cycle is here called (n, k)-periodic, where $T = 2\pi/\omega$ is the period of the external excitation.

2.2.1. $(n, 0)$-*periodic solutions*

The simplest case consists of looking for nT-periodic solutions which never impact against the stop. Such a case implies $n = 1$ and the initial conditions leading to a (1,0)-periodic solution are

$$x_0 = f_1,$$

$$\dot{x}_0 = f_2\,\omega.$$

Setting:

$$\begin{aligned} \omega_+ &= \sqrt{\tilde{\omega}_1^2 - \frac{a^2}{4} + \sqrt{\frac{f^2}{x_{max}^2} - a^2\tilde{\omega}_1^2}}, \\ \omega_- &= \sqrt{\tilde{\omega}_1^2 - \frac{a^2}{4} - \sqrt{\frac{f^2}{x_{max}^2} - a^2\tilde{\omega}_1^2}} \end{aligned} \tag{5}$$

and as it is assumed that $f < \omega_1^2 x_{max}$, it can be shown that $(n, 0)$-periodic solutions exist if and only if $\omega \leqslant \omega_-$ or $\omega \geqslant \omega_+$.

The stability of these periodic solutions can be determined using a Poincaré map defined by a constant phase plane $Z = T$ in the co-ordinates $(X, Y, Z) = (x, \dot{x}, t \bmod T)$. As pointed out in reference [80], such a mapping is not everywhere continuous nor differentiable. Therefore, for each periodic solution corresponding to a fixed point of the Poincaré map where it is continuously differentiable, the stability can be investigated. It can be shown that $(1, 0)$-periodic solutions are always stable.

2.2.2. $(n, 1)$-*periodic solutions*

Similar to section 2.2.1, nT-periodic solutions with one impact per cycle can be sought by using equations (3) and (4). In this case, the impact time can be analytically determined, and the initial conditions leading to $(n, 1)$-periodic solutions are then given by

$$x_0 = A_1(t_1) + f_1,$$

$$\dot{x}_0 = \tilde{\omega}_1 [B_1(t_1) - \eta A_1(t_1)] + \omega f_2,$$

where $A_1(t_1)$ and $B_1(t_1)$ depend analytically on the system's parameters. An example of $(1, 1)$-periodic solution is shown in Figure 2.

As in section 2.2.1, the Poincaré map can be used (when it is defined and of class C^1 locally) in order to determine the type of the periodic solutions obtained (see in Figure 3). The Jacobian matrix can be calculated analytically from the analytical form of the Poincaré map by taking into account the influence of the partial derivatives of the impact time t_1 with respect to the initial conditions.

It can be shown that such a stability study is valid only if $\omega \neq \omega_+$ and $\omega \neq \omega_-$ for in that case the Poincaré map is not differentiable at the fixed point considered.

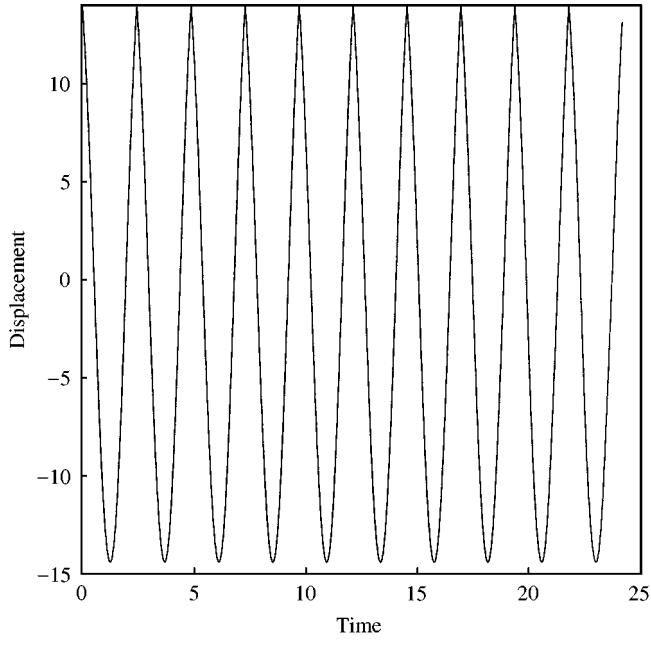

Figure 2. $(1, 1)$-periodic solution to the system (2) for $\omega = 2{\cdot}6$, $x_0 = 13{\cdot}32968$ and $\dot{x}_0 = 19{\cdot}36619$.

2.2.3. $(n, 2)$-*periodic solutions*

The method for seeking nT-periodic solutions with two impacts per cycle is identical to the case with one impact per cycle, with a new unknown due to the second impact time t_2. By characterizing the periodicity of the solution and by using the analytical form (3), (4) a system of two non-linear equations with two unknowns (t_1, t_2) can be obtained which can for example be solved using Newton's method. Once the value of t_1 and t_2 are known, the initial conditions of the system are given by

$$x_0 = A_1(t_1, t_2) + f_1,$$

$$\dot{x}_0 = \tilde{\omega}_1(B_1(t_1, t_2) - \eta A_1(t_1, t_2)) + \omega f_2.$$

Figures 4–6 show three examples of $(n, 2)$-periodic solutions that can be obtained analytically.

As in the case of $(n, 1)$-periodic solutions, the stability of the $(n, 2)$-periodic solutions can be studied by using the Poincaré map: when $\omega \notin \{\omega_+, \omega_-\}$, it is possible to calculate analytically the Jacobian matrix of the Poincaré map, and its eigenvalues determine the stability or instability of the periodic solution.

2.3. MODAL SUPERPOSITION

2.3.1. *Free oscillations*

The free oscillations of the system are described by the following equation:

$$\ddot{x} + a\dot{x} + \omega_1^2 x = 0,$$

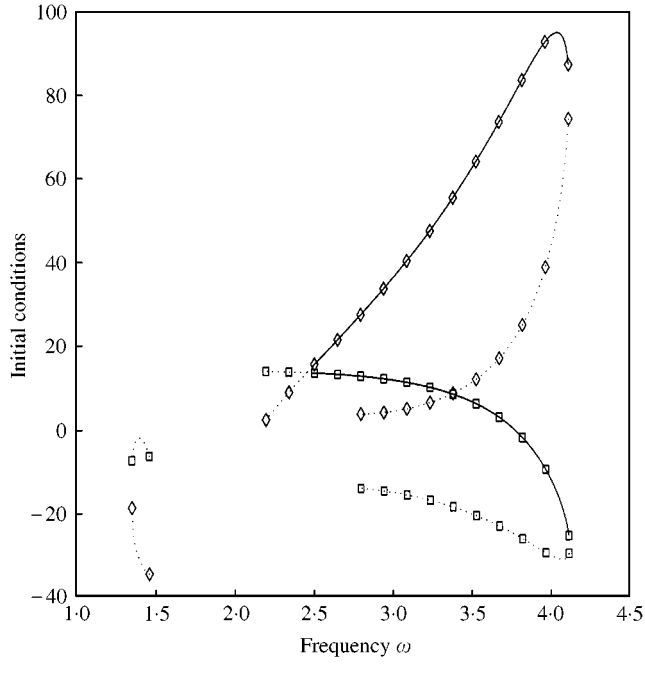

Figure 3. Existence of (1, 1) periodic solutions. Stable (square: x_0, diamond: $\dot{x}_0$) and unstable (dotted curve) solutions.

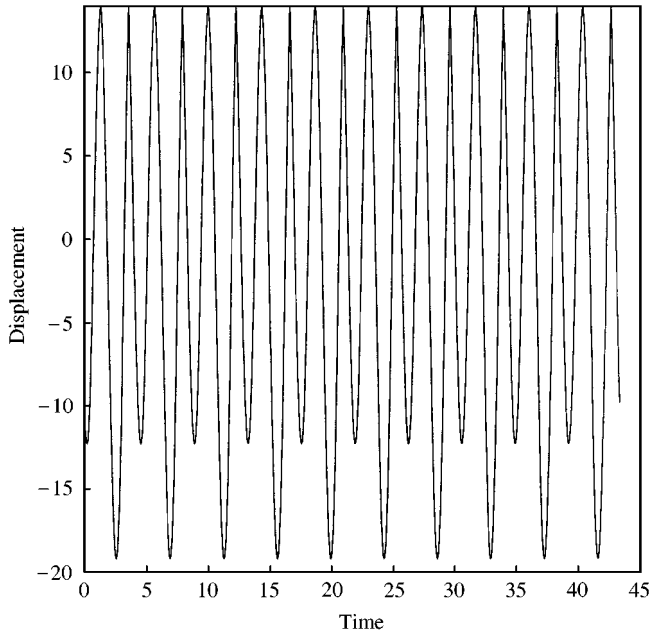

Figure 4. (1, 2)-periodic solution for $\omega = 1{\cdot}45$, $x_0 = -9{\cdot}89493$ and $\dot{x}_0 = -20{\cdot}71102$.

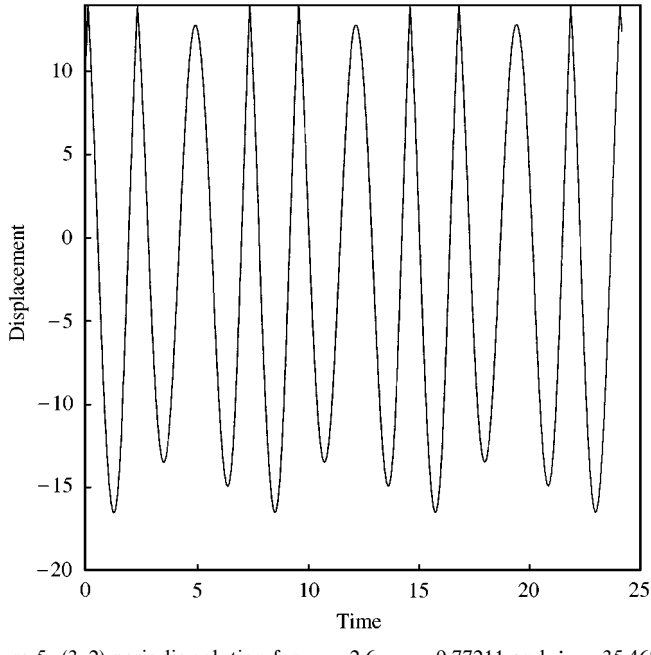

Figure 5. (3, 2)-periodic solution for $\omega = 2{\cdot}6$, $x_0 = 9{\cdot}77211$ and $\dot{x}_0 = 35{\cdot}46891$.

$$\dot{x}(t^+) = -e\dot{x}(t^-) \quad \text{if } x(t) = x_{max}, \tag{6}$$
$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0.$$

The solution of this equation can also be piecewise analytically written: the expression is identical to equation (3), the constants of integration being given by equation (4), with $f_1 = f_2 = 0$.

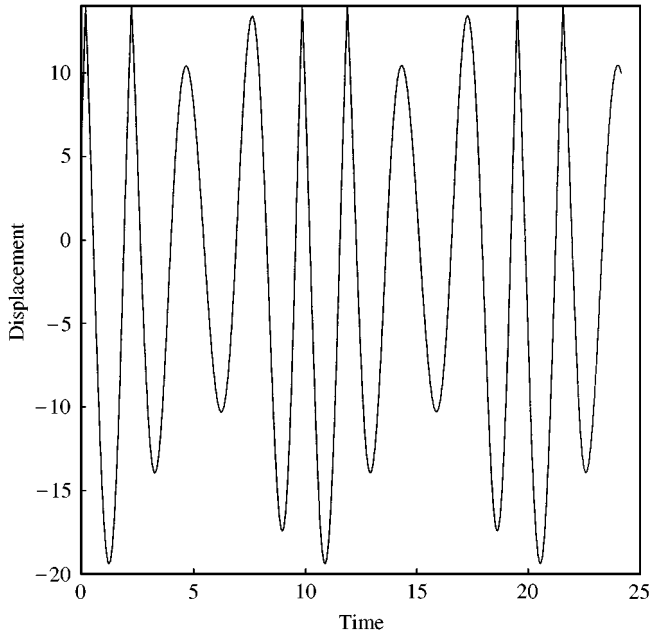

Figure 6. (4, 2)-periodic solution for $\omega = 2{\cdot}6$, $x_0 = 4{\cdot}76445$ and $\dot{x}_0 = 47{\cdot}91445$.

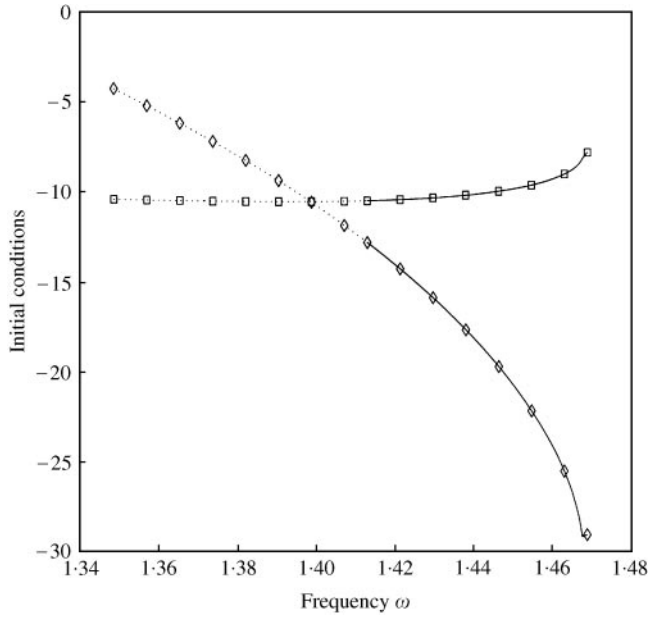

Figure 7. Existence of (1, 2) periodic solutions. Stable (square: x_0, diamond: $\dot{x}_0$) and unstable (dotted curve) solutions.

If the equilibrium is a position that the system can physically reach, namely if $x_{max} > 0$, and if $a \neq 0$, it can be shown that the system (6) has a finite number K of impacts. This result is interesting from the modal point of view; it means that the free oscillations of the system are governed by the frequency ω_1, except for a bounded time span. Hence, it will be considered later on that the natural frequency of the system is ω_1.

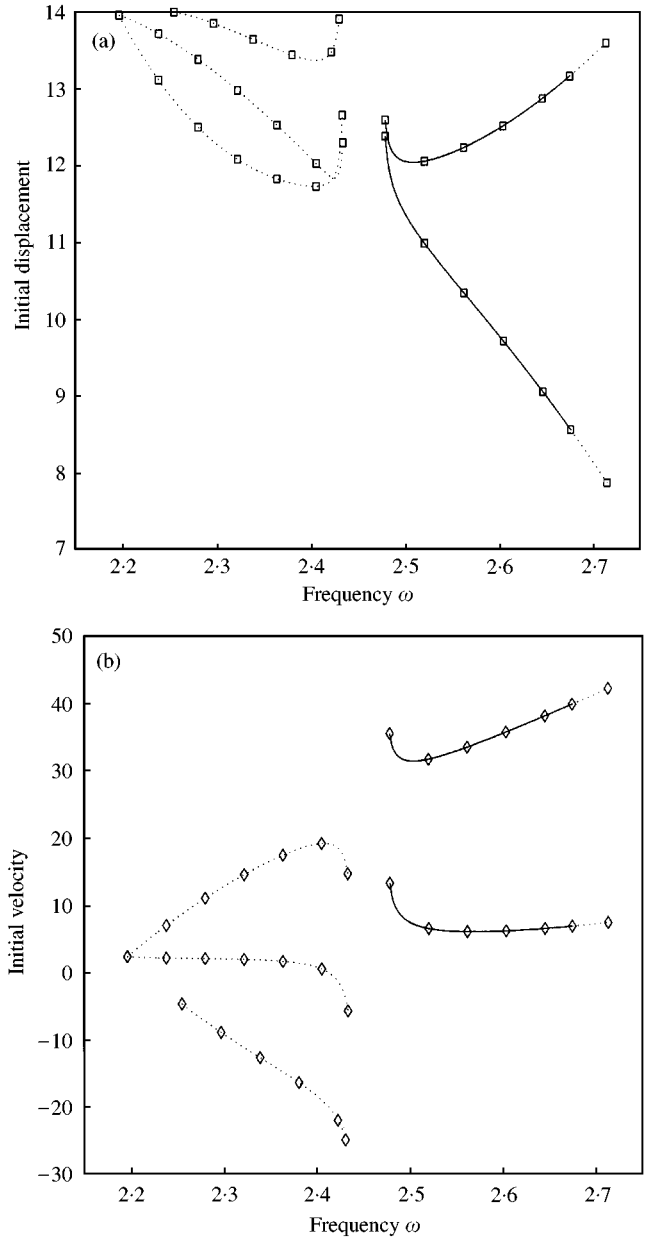

Figure 8. Existence of (3, 2)-periodic solutions: stable (solid curve) and unstable solutions (dotted curve): (a) initial displacements; (b) initial velocities.

Remark. When $a = 0$, the number of impacts is infinite and $\lim_{k \to +\infty} t_{k+1} - t_k = 2\pi/\tilde{\omega}_1$. If $x_{max} = 0$, there also is an infinite number of impacts and $t_{k+1} - t_k = \pi/\tilde{\omega}_1$, $\forall k$. Finally, if $x_{max} < 0$ then $\lim_{k \to +\infty} t_{k+1} - t_k = 0$.

2.3.2. *Generalized mass and modal superposition*

In this section, it is intended to establish a modal superposition formula for the irregular non-linear model previously introduced. The case $x_{max} > 0$ is considered for which the

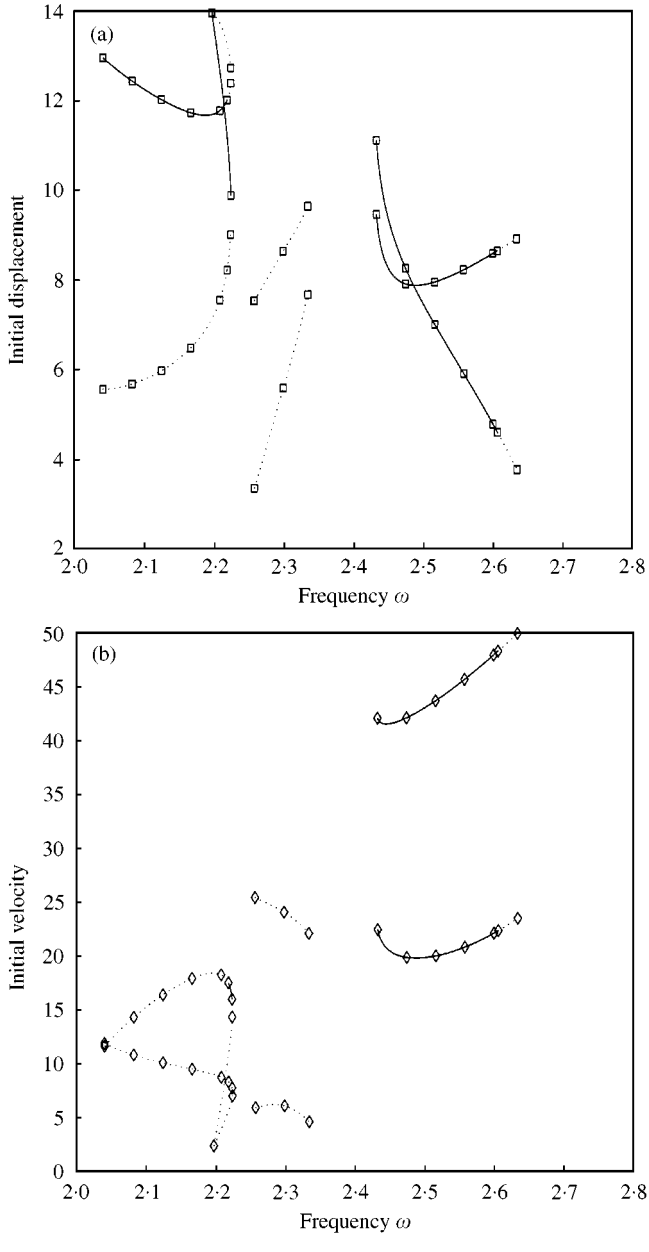

Figure 9. Existence of (4, 2)-periodic solutions: stable (solid curve) and unstable solutions (dotted curve): (a) initial displacements; (b) initial velocities.

natural frequency is ω_1, and an attempt is made to obtain a modal superposition formula similar to the linear case, connecting the nth harmonic amplitude of the response to the amplitude of the forcing.

If a (n, k)-periodic solution is considered, i.e. a solution with period nT and k impacts per cycle. Due to the nT-periodicity of the response, the Fourier coefficients can be defined; for

all $j \in \mathbb{Z}$ we have

$$c_j(\omega) = \frac{1}{nT}\int_0^{nT} x(t)\exp\left(-\mathrm{i}\frac{j}{n}\omega t\right)\mathrm{d}t$$
$$= \frac{1}{nT}\left[\int_0^{t_1} x(t)\exp\left(-\mathrm{i}\frac{j}{n}\omega t\right)\mathrm{d}t + \sum_{m=1}^{k-1}\int_{t_m}^{t_{m+1}} x(t)\exp\left(-\mathrm{i}\frac{j}{n}\omega t\right)\mathrm{d}t\right.$$
$$\left. + \int_{t_k}^{nT} x(t)\exp\left(-\mathrm{i}\frac{j}{n}\omega t\right)\mathrm{d}t\right].$$

$x(t)$ is known to be piecewise via equation (3). These $k+1$ integrals can then be calculated analytically. If $k \neq 0$ define

$$H_{n,k}^j(\omega) = \frac{2\tilde{\omega}_1}{nT}\{B_1 + A_{k+1}\mathrm{e}^{-anT/2}\sin(n\tilde{\omega}_1 T) - B_{k+1}\mathrm{e}^{-anT/2}\cos(n\tilde{\omega}_1 T)$$
$$+ (\eta + \mathrm{i}\gamma_j)[A_1 - A_{k+1}\mathrm{e}^{-anT/2}\cos(n\tilde{\omega}_1 T) - B_{k+1}\mathrm{e}^{-anT/2}\sin(n\tilde{\omega}_1 T)] \tag{7}$$
$$+ (1+e)\sum_{m=1}^{k}\mathrm{e}^{-at_m/2}\exp\left(-\mathrm{i}\frac{j}{n}\omega t_m\right)u(A_m, B_m, t_m)\Big\},$$

where $\gamma_j = j\omega/n\tilde{\omega}_1$, and if $k = 0$, $H_{n,0}^j(\omega) = 0$. Then

$$c_j(\omega) = \frac{f_1 - \mathrm{i}f_2}{2}\delta_j^n + \frac{f_1 - \mathrm{i}f_2}{2}\delta_j^{-n} + \frac{H_{n,k}^j(\omega)}{2}. \tag{8}$$

Let

$$\Delta I_1(\omega) = \omega_1^2 - \omega^2 + a\mathrm{i}\omega. \tag{9}$$

The nth Fourier coefficient is then given by

$$c_n(\omega) = \frac{f_1 - \mathrm{i}f_2}{2} + \frac{H_{n,k}^n(\omega)}{2\Delta I_1(\omega)} = \frac{f + H_{n,k}^n(\omega)}{2\Delta I_1(\omega)}. \tag{10}$$

Let the modal mass $m_{n,k}$ be:

$$m_{n,k} = \frac{1}{1 + H_{n,k}^n(\omega)/f}. \tag{11}$$

Because the free damped steady state oscillations are linear oscillations, the generalized mode is here represented by the scalar 1. The nth harmonic amplitude can then be written in the form

$$\mathscr{A}_n(\omega) = \sqrt{2(|c_n|^2 + |c_{-n}|^2)} = 2|c_n| = \left|\frac{f}{m_{n,k}\Delta I_1(\omega)}\right|. \tag{12}$$

This is a modal superposition formula connecting the nth harmonic amplitude of the forced response to the amplitude of the forcing via the free response. This formula is similar to the formula that obtained in the linear case, but the mass (which should be equal to 1) is replaced by a modal mass (11).

Due to this definition, the mass is a complex: to give it a more physical meaning, it is necessary to consider its module. Thus, the unitary mass system with impact is modelled and, subjected to a sinusoïdal forcing of frequency ω, like a linear system without impact of

mass $|m_{n,k}|$, subjected to the same forcing. This modelling holds in terms of spectral amplitude for a (n, k)-periodic response: the spectral amplitude is the same one for both systems.

Remark.

— In the case of a nT-periodic response without any impacts, it was seen that only the case $n = 1$ was possible. Thus, $m_{1,0} = 1$ which is coherent since in this case the classical modal superposition formula applies and gives $\mathscr{A}_1(\omega) = |f/\Delta I_1(\omega)|$.

— There is no longer a unique modal superposition formula as in the linear case, but an infinity *a priori*; it depends on the period and number of impacts per cycle of the periodic solution.

— Preceding calculations give access to the whole Fourier series of a (n, k)-periodic response.

— An analytical expression of the module of the modal mass is obtained if an only if $k = 1$, i.e., when the periodic solution has only one impact by period. Setting

$$X_1 = \frac{2(1+e)\omega}{\eta f T} \frac{-1 + 2e^{anT/2}\cos(n\tilde{\omega}_1 T) - e^{anT}}{e + e^{anT/2}\{(1-e)\cos(n\tilde{\omega}_1 T) + (1+e)\eta\sin(n\tilde{\omega}_1 T)\} - e^{anT}}$$
$$\times [f_1 \sin(\omega t_1) - f_2 \cos(\omega t_1)]$$

for the modal mass:

$$m_{n,1} = \frac{1}{1 + X_1 e^{-i\omega t_1}} \tag{13}$$

Since t_1 is analytically known in the case of a $(n, 1)$-periodic response, the modal mass $|m_{n,1}|$ can be expressed analytically as a function of the parameters of the system.

2.3.3. *Examples of modal superposition*

It is now possible to show some applications of the modal superposition formula previously established, using the analytical search for periodic solutions carried out in section 2.2. First consider the following set of parameters for the system: $\omega_1 = 2{\cdot}5$, $a = 0{\cdot}05$, $x_{max} = 14$, $e = 0{\cdot}9$ and $f = 20$. Note that for these values of parameters $\omega_1^2 x_{max} > f$ and $e > 0$, so that sticking never occurs.

Figures 10 outlines an important difference compared to the linear case: the peak of amplitude corresponding to resonance does no longer exist around ω_1. Consequently, for ω close to the natural frequency of the system, the modal mass is the largest because there is no resonance between the external forcing and the system. The maximum amplitude occurs at a frequency higher than the natural frequency, but the maximum reached is much weaker than in the linear case; there is not a true peak of amplitude in the usual sense.

In addition, the modal superposition formula previously established enables the amplitude of nth harmonic of a solution, whose period is nT to be computed. In the linear case (without impact), this is sufficient to know the whole spectral response of the system: only $n = 1$ occurs and the Fourier coefficients are all zero except the first one. In this case, it is no longer sufficient, for the occurrence of impacts leads to periodic solutions with many harmonics, and the nth harmonic amplitude can be a poor approximation to the spectral amplitude of the response. For example, Figure 12 shows that the second-harmonic amplitude for a T-periodic solution can be the largest one. In the same way, the Fourier coefficient c_0 can become large, can be seen in Figure 10. Table 1 summarizes the difference

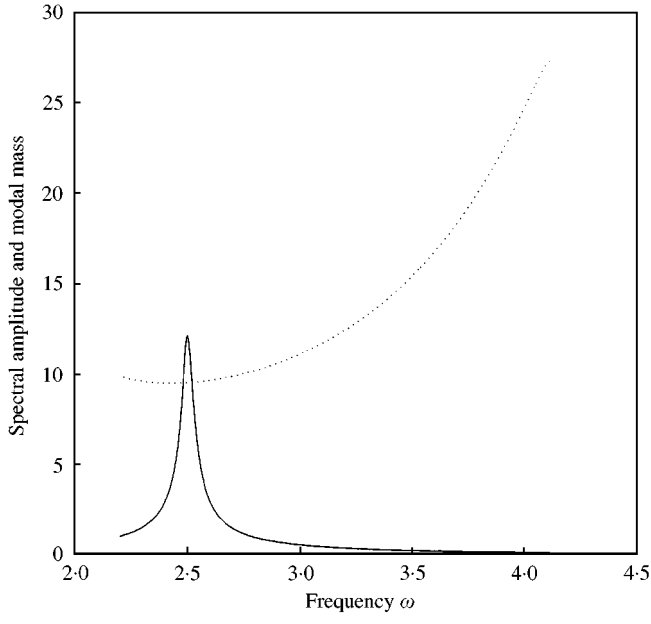

Figure 10. First-harmonic amplitude (dotted curve) and modal mass (solid curve) of the (1, 1)-periodic solutions.

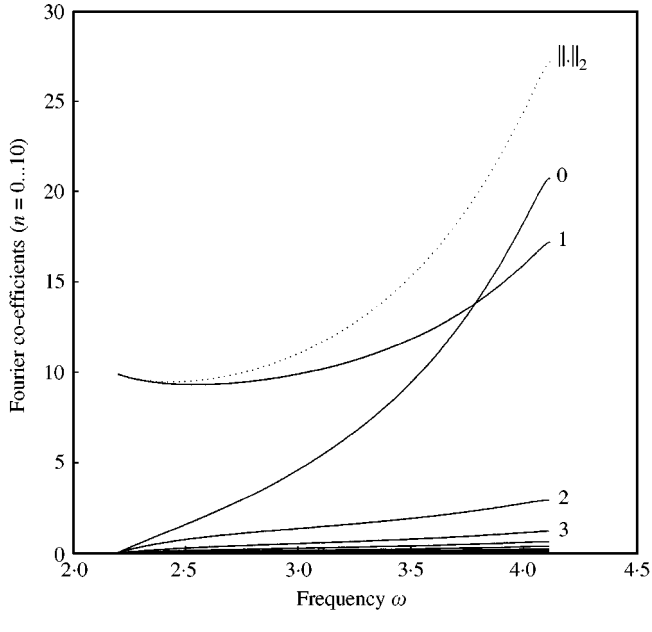

Figure 11. Fourier coefficients of the (1, 1)-periodic responses.

between the spectral amplitude and the nth harmonic amplitude for various periodic solutions.

For this example, a modal superposition formula can be built and is worthwhile as long as ω remains close to the "primary resonance". Nevertheless, this frequency area does not always correspond to the largest amplitudes.

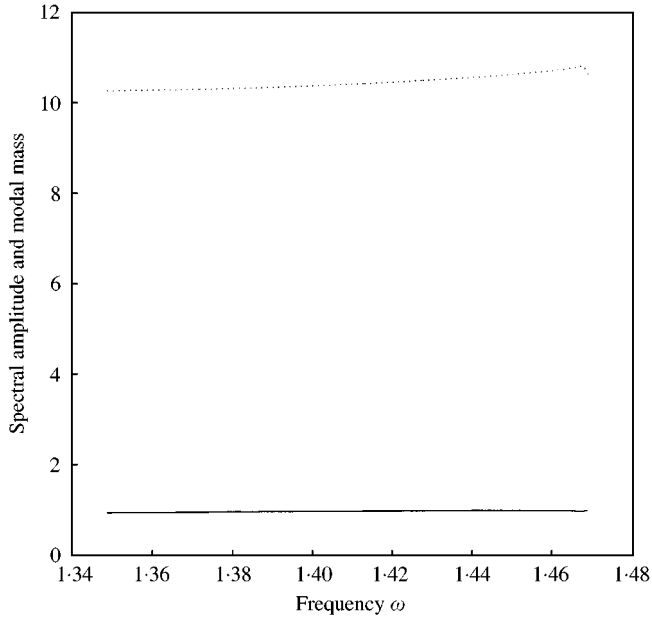

Figure 12. First-harmonic amplitude (dotted curve) and modal mass (solid curve) of the (1, 2)-periodic solutions.

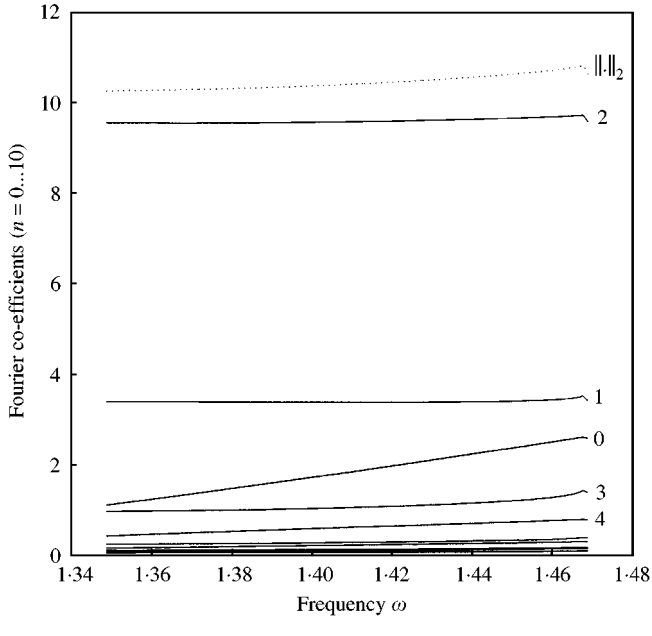

Figure 13. Fourier coefficients of the (1, 2)-periodic responses.

An example showing a spectral behaviour different from the preceding one can now be considered. In this case $\omega_1 = 1$, $a = 0{\cdot}02$, $x_{max} = 1$, $e = 0{\cdot}9$ and $f = 20$ (sticking to the stop may then occur, but only responses without sticking will be considered).

In Figure 18 two important characteristics can be seen. First of all, and contrary to the previous example, the response exhibits a true peak of spectral amplitude similar to the

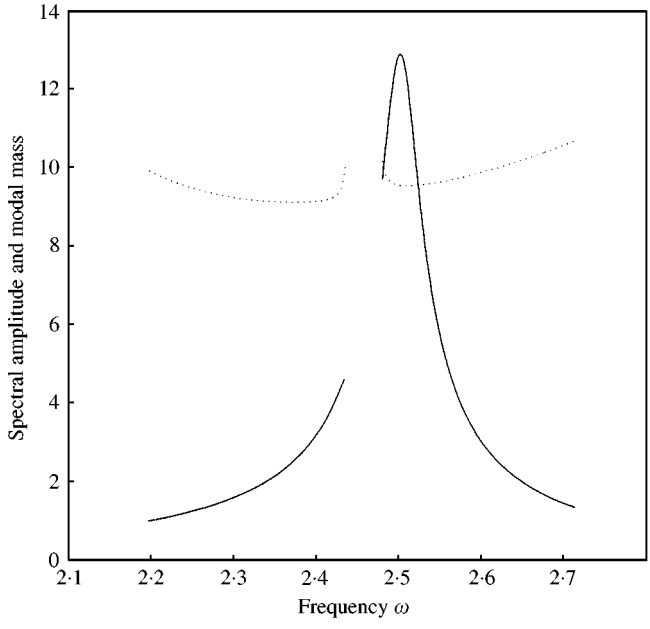

Figure 14. Third-harmonic amplitude (dotted curve) and modal mass (solid curve) of the (3, 2)-periodic solutions.

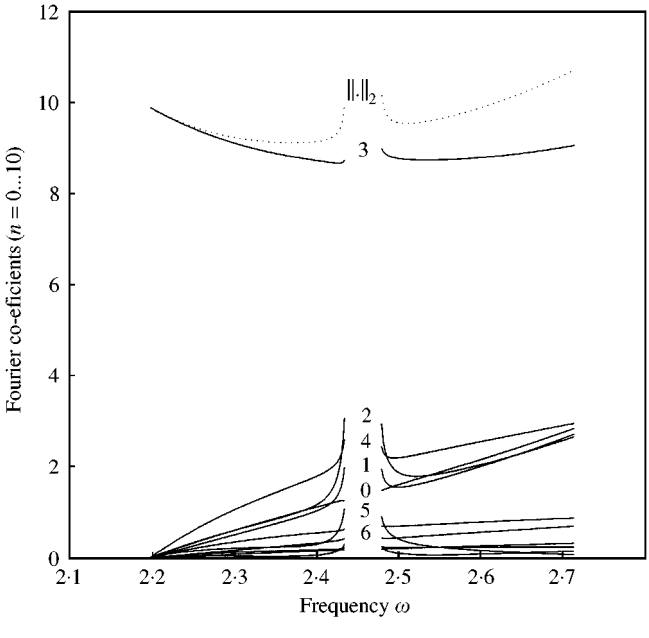

Figure 15. Fourier coefficients of the (3, 2)-periodic responses.

resonance peaks observed in linear systems. However, the peak does not occur at the natural frequency ω_1, but at $\omega \approx 2\omega_1$. Furthermore, the approximation to the whole amplitude by the amplitude of first harmonic is very rough: resonance occurs through the Fourier coefficient c_0 of the solution, and the remainder of the harmonics are negligible in the neighbourhood of the peak.

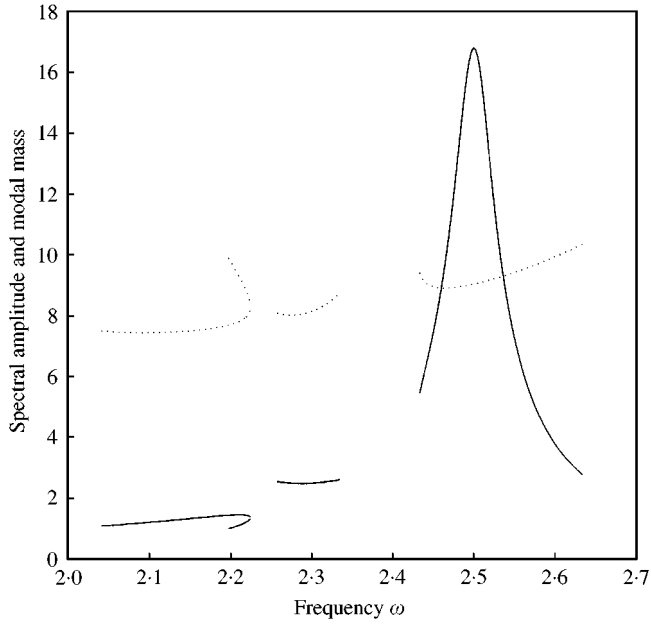

Figure 16. Fourth-harmonic amplitude (dotted curve) and modal mass (solid curve) of the (4, 2)-periodic solutions.

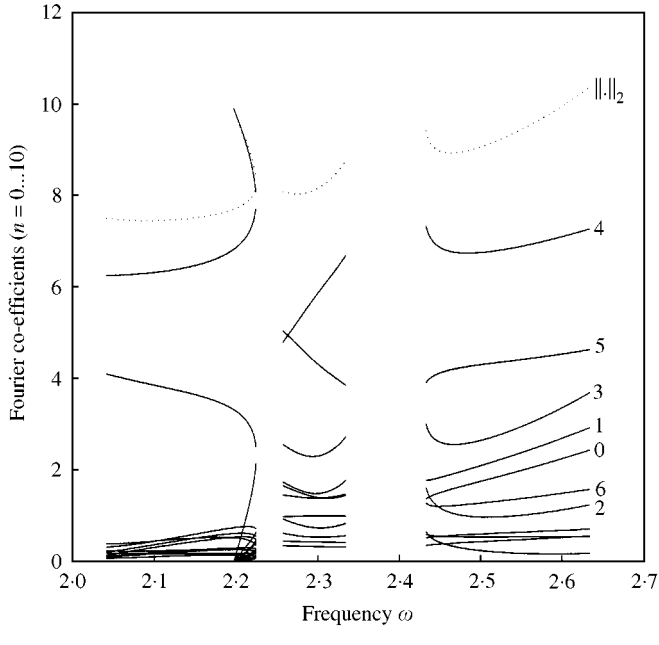

Figure 17. Fourier coefficients of the (4, 2)-periodic responses.

The last two examples have shown that the building of a modal superposition formula similar to the linear case comes up against two major difficulties. On one hand, the multiplicity of types of periodic solutions prevents a unique formula which holds for any case: it is necessary to know *a priori* the period of the response and the number of impacts per cycle. On the other hand, taking into account only one harmonic in this superposition

TABLE 1

Difference between Fourier amplitude and nth harmonic amplitude

Periodic solution	Difference (%) stable or unstable solution	Difference (%) stable solution
(1, 1)	36·65	36·65
(1, 2)	67·89	67·89
(3, 2)	15·56	13·96
(4, 2)	40·89	28·96

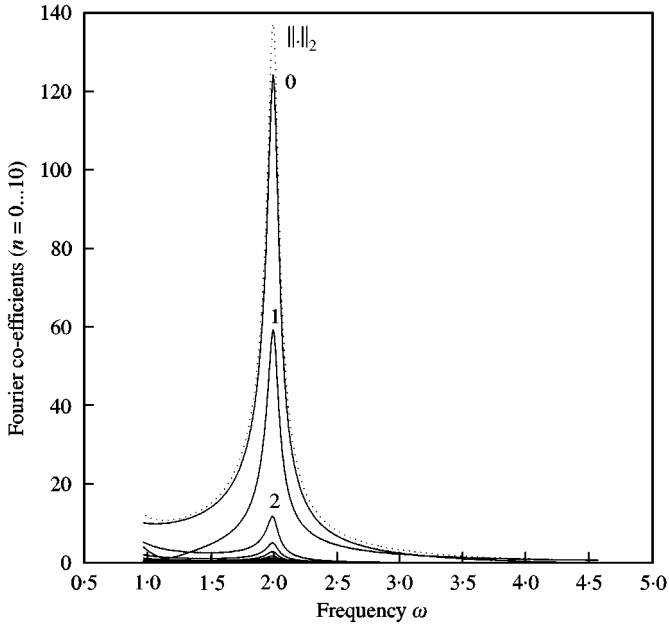

Figure 18. Fourier coefficients of the (1, 1)-periodic responses.

may not be sufficient any more; it would be necessary to include at least the four first harmonics in the formula in order to get a closer approximation in the examples shown.

3. TWO-DEGREE-OF-FREEDOM SYSTEMS

This section deals with a system with two d.o.f., one of them being constrained by a stop:

$$\ddot{x} + a\dot{x} + \omega_1^2 x + k_1 y = f_1 \cos(\omega t),$$

$$x \in \mathscr{C}^0(\mathbb{R}^+,]-\infty, x_{max}]),$$

$$x(t) = x_{max} \Rightarrow \dot{x}(t^+) = -e\dot{x}(t^-),$$

$$\ddot{y} + a\dot{y} + \omega_2^2 y + k_2 x = f_2 \cos(\omega t),$$

$$y \in \mathscr{C}^1(\mathbb{R}^+, \mathbb{R}).$$

This system can be written in the form

$$\ddot{\mathbf{X}} + \mathbf{A}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{F}\cos(\omega t) + \text{impact}, \tag{14}$$

with

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} \omega_1^2 & k_1 \\ k_2 & \omega_2^2 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

and the term " + impact" represents the constraint induced on x by the occurrence of impacts.

The model just introduced is a general mathematical model. It will be studied in three stages: first of all, consider the case $k_1 = 0$ (which will be referred to as "weak coupling") for which the study is very close to the single-degree-of-freedom case in term of periodic solutions: it will give an initial outline of modal superposition with two d.o.f.

The general case where $k_1 \neq 0$ (referred to as "strong coupling") will then be studied, for which the modal superposition is similar to the case $k_1 = 0$, except that the search for periodic solutions becomes more complex. Finally it can be seen how the modal superposition can be written in the case of two rigid bodies colliding.

Note, initially, that the first case is a mathematical model which cannot easily be expressed in mechanical terms. Indeed, the action–reaction principle prevents x from acting on y without y acting on x. Therefore, it is necessary to consider carefully the physical conclusions that could be drawn from this model.

3.1. WEAK COUPLING

System (14), the special case where $k_1 = 0$, which is close to the single-d.o.f. case previously studied can be dealt with.

3.1.1. *Analytical solution of the system*

3.1.1.1. *Decoupling equations.* The system (14) can be decoupled in order to obtain explicit solutions for x and y. If

$$v_2 = \frac{k_2}{\omega_1^2 - \omega_2^2} \quad \text{and} \quad P = \begin{pmatrix} 1 & 0 \\ v_2 & 1 \end{pmatrix},$$

and pre-multiply the system 14 without impacts by P^{-1}, a system in the new co-ordinates $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = P^{-1}X$ is obtained:

$$\begin{aligned} \ddot{x}_1 + a\dot{x}_1 + \omega_1^2 x_1 &= f^1 \cos(\omega t), \\ \ddot{x}_2 + a\dot{x}_2 + \omega_2^2 x_2 &= f^2 \cos(\omega t), \end{aligned} \tag{15}$$

where

$$\begin{aligned} f^1 &= f_1, \\ f^2 &= -v_2 f_1 + f_2. \end{aligned}$$

Written in that form, the system is easy to solve and as long as there is no impact:

$$x_1(t) = \mathrm{e}^{-at/2}[A^1\cos(\tilde{\omega}_1 t) + B^1\sin(\tilde{\omega}_1 t)] + f_1^1\cos(\omega t) + f_2^1\sin(\omega t),$$
$$x_2(t) = \mathrm{e}^{-at/2}[A^2\cos(\tilde{\omega}_2 t) + B^2\sin(\tilde{\omega}_2 t)] + f_1^2\cos(\omega t) + f_2^2\sin(\omega t), \tag{16}$$

where $\tilde{\omega_1} = \sqrt{\omega_1^2 - a^2/4}$, $\tilde{\omega}_2 = \sqrt{\omega_2^2 - a^2/4}$ and

$$f_1^1 = f^1\frac{\omega_1^2 - \omega^2}{(\omega_1^2 - \omega^2)^2 + a^2\omega^2},$$

$$f_2^1 = f^1\frac{a\omega}{(\omega_1^2 - \omega^2)^2 + a^2\omega^2},$$

$$f_1^2 = f^2\frac{\omega_2^2 - \omega^2}{(\omega_2^2 - \omega^2)^2 + a^2\omega^2},$$

$$f_2^2 = f^2\frac{a\omega}{(\omega_2^2 - \omega^2)^2 + a^2\omega^2}.$$

Moreover, if $\eta_1 = a/2\tilde{\omega}_1$ and $\eta_2 = a/2\tilde{\omega}_2$, the solution is given in the initial co-ordinate system by

$$x = x_1,$$
$$y = v_2 x_1 + x_2. \tag{17}$$

3.1.1.2. *Gluing at impact times.* An impact occurs when $x(t) = x_{max}$, which is equivalent to $x_1(t) = x_{max}$. If it is assumed that there is an impact at t_k, on $[t_{k-1}, t_k]$, the solution (16) is given by

$$x_1(t) = \mathrm{e}^{-at/2}[A_k^1\cos(\tilde{\omega}_1 t) + B_k^1\sin(\tilde{\omega}_1 t)] + f_1^1\cos(\omega t) + f_2^1\sin(\omega t),$$
$$x_2(t) = \mathrm{e}^{-at/2}[A_k^2\cos(\tilde{\omega}_2 t) + B_k^2\sin(\tilde{\omega}_2 t)] + f_1^2\cos(\omega t) + f_2^2\sin(\omega t).$$

From this equation it can be inferred that t_k is solution of the equation

$$f(t_k) = \mathrm{e}^{-at_k/2}[A_k^1\cos(\tilde{\omega}_1 t_k) + B_k^1\sin(\tilde{\omega}_1 t_k)] + f_1^1\cos(\omega t_k) + f_2^1\sin(\omega t_k) - x_{max} = 0.$$

Moreover, by assumptions on x and y, at impact time

$$x(t_k^+) = x(t_k^-),$$
$$\dot{x}(t_k^+) = -e\dot{x}(t_k^-),$$
$$y(t_k^+) = y(t_k^-),$$
$$\dot{y}(t_k^+) = \dot{y}(t_k^-),$$

which yields for the new variables:

$$x_1(t_k^+) = x_1(t_k^-),$$
$$\dot{x}_1(t_k^+) = -e\dot{x}_1(t_k^-),$$
$$x_2(t_k^+) = x_2(t_k^-),$$
$$\dot{x}_2(t_k^+) = \dot{x}_2(t_k^-) + (1+e)v_2\dot{x}_1(t_k^-).$$

Thus, obtain the following relationships provide the constants of integration

$$
\begin{aligned}
A^1_{k+1} &= A^1_k - (1+e)_2 \sin(\tilde{\omega}_1 t_k)\, u(A^1_k, B^1_k, t_k),\\
B^1_{k+1} &= B^1_k + (1+e)\cos(\tilde{\omega}_1 t_k)\, u(A^1_k, B^1_k, t_k),\\
A^2_{k+1} &= A^2_k + (1+e) v_2 \frac{\tilde{\omega}_1}{\tilde{\omega}_2} \sin(\tilde{\omega}_2 t_k)\, u(A^1_k, B^1_k, t_k),\\
B^2_{k+1} &= B^2_k - (1+e) v_2 \frac{\tilde{\omega}_1}{\tilde{\omega}_2} \cos(\tilde{\omega}_2 t_k)\, u(A^1_k, B^1_k, t_k),
\end{aligned}
\tag{18}
$$

where u is given by

$$
\begin{aligned}
u(A,B,t) = {} & A\,[\sin(\tilde{\omega}_1 t) + \eta_1 \cos(\tilde{\omega}_1 t)] + B[-\cos(\tilde{\omega}_1 t) + \eta_1 \sin(\tilde{\omega}_1 t)]\\
& + \mathrm{e}^{at/2} \frac{\omega}{\tilde{\omega}_1} [f^1_1 \sin(\omega t) - f^1_2 \cos(\omega t)].
\end{aligned}
$$

3.1.2. *Search for periodic solutions*

The search for periodic solutions (x, y) is equivalent to the search for periodic solutions (x_1, x_2). According to equation (17), $x_1 = x$, therefore the search for periodic solutions for x_1 is similar to the one carried out in the case of a single-d.o.f. system. Thus, it is possible to determine x^0_1 and $\dot{x}^0_1$ so that x_1 is (n, k)-periodic where $k \in \{0, 1, 2\}$. As regards x_2, the method is identical; only the recursion that gives A^2_2 and B^2_2 being different. Thus 2×2 linear systems are obtained in $(A^2_1(t_1), B^2_1(t_1))$. When the determinant is non-zero, x^0_2 and $\dot{x}^0_2$ are obtained so that x_2 is (n, k)-periodic where $k \in \{0, 1, 2\}$.

The initial conditions for the original system co-ordinates (x, y) are given by equation (17):

$$
\begin{aligned}
x_0 &= x^0_1,\\
y_0 &= v_2 x^0_1 + x^0_2,\\
\dot{x}_0 &= \dot{x}^0_1,\\
\dot{y}_0 &= v_2 \dot{x}^0_1 + \dot{x}^0_2.
\end{aligned}
$$

3.1.3. *Modal superposition*

3.1.3.1. *Free oscillations of the system.* It was seen in Section 2.3.1 that, in the case of a single-d.o.f. system without external forcing, the number of impacts is finite and the steady state response is periodic with frequency ω_1. Hence, in the case of weak coupling, the free response also exhibits a finite number of impacts, and the steady state response for x_1 is periodic with frequency ω_1. As for x_2, since there are no more impacts once the steady state response is reached, the equation of its movement is given by equation (15):

$$
\ddot{x}_2 + a\dot{x}_2 + \omega_2^2 x_2 = 0.
$$

This is the equation of a classical damped oscillator without external forcing, whose steady state response has frequency ω_2. In order to establish a modal superposition formula, it is necessary to start from the natural frequencies ω_1 and ω_2. Moreover, in that case again, the generalized modes correspond to linear modes.

3.1.3.2. *Generalized masses and modal superposition.* Consider a (n, k)-periodic solution. The Fourier coefficients of the functions x_1 and x_2 can be calculated. These are quite similar to the coefficients found for the single-d.o.f. system. Setting

$$H^j_{n,k}(\omega) = \frac{2\tilde{\omega}_1}{nT} \frac{1}{\omega_1^2 - \dfrac{j^2\omega^2}{n^2} + \mathrm{i}a\dfrac{j\omega}{n}}$$

$$\Big\{B_1^1 + A_{k+1}^1 \mathrm{e}^{-anT/2} \sin(n\tilde{\omega}_1 T) - B_{k+1}^1 \mathrm{e}^{-anT/2} \cos(n\tilde{\omega}_1 T)$$

$$+ (\eta_1 + \mathrm{i}\gamma_j^1)[A_1^1 - A_{k+1}^1 \mathrm{e}^{-anT/2} \cos(n\tilde{\omega}_1 T) - B_{k+1}^1 \mathrm{e}^{-anT/2} \sin(n\tilde{\omega}_1 T)]$$

$$+ (1 + e) \sum_{m=1}^{k} \mathrm{e}^{-at_m/2} \exp\left(-\mathrm{i}\frac{j}{n}\omega t_m\right) u(A_m^1, B_m^1, t_m)\Big\}$$

$$G^j_{n,k}(\omega) = \frac{2\tilde{\omega}_2}{nT} \frac{1}{\omega_2^2 - \dfrac{j^2\omega^2}{n^2} + \mathrm{i}a\dfrac{j\omega}{n}}$$

$$\times \Big\{B_1^2 + A_{k+1}^2 \mathrm{e}^{-anT/2} \sin(n\tilde{\omega}_2 T) - B_{k+1}^2 \mathrm{e}^{-anT/2} \cos(n\tilde{\omega}_2 T)$$

$$+ (\eta_2 + \mathrm{i}\gamma_j^2)\,[A_1^2 - A_{k+1}^2 \mathrm{e}^{-anT/2} \cos(n\tilde{\omega}_2 T) - B_{k+1}^2 \mathrm{e}^{-anT/2} \sin(n\tilde{\omega}_2 T)]$$

$$- (1 + e) v_2 \frac{\tilde{\omega}_1}{\tilde{\omega}_2} \sum_{m=1}^{k} \mathrm{e}^{-at_m/2} \exp\left(-\mathrm{i}\frac{j}{n}\omega t_m\right) u(A_m^1, B_m^1, t_m)\Big\}$$

with $\gamma_j^1 = j\omega/n\tilde{\omega}_1$ and $\gamma_j^2 = j\omega/n\tilde{\omega}_2$, the Fourier coefficients are given by

$$c_j^1(\omega) = \frac{f_1^1 - \mathrm{i}f_2^1}{2}\,\delta_j^n + \frac{f_1^1 + \mathrm{i}f_2^1}{2}\,\delta_j^{-n} + \frac{H^j_{n,k}(\omega)}{2},$$

$$c_j^2(\omega) = \frac{f_1^2 - \mathrm{i}f_2^2}{2}\,\delta_j^n + \frac{f_1^2 + \mathrm{i}f_2^2}{2}\,\delta_j^{-n} + \frac{G^j_{n,k}(\omega)}{2}.$$

The Fourier coefficient corresponding to the nth harmonic are inferred by using the coordinate transformation (17)

$$c_n^x(\omega) = c_n^1(\omega) = \frac{f^1}{2m_1^{n,k}\Delta I_1},$$

$$c_n^y(\omega) = v_2 c_n^1(\omega) + c_n^2(\omega) = v_2 \frac{f^1}{2m_1^{n,k}\Delta I_1} + \frac{f^2}{2m_2^{n,k}\Delta I_2}, \tag{19}$$

where, $\Delta I_1 = \omega_1^2 - \omega^2 + ai\omega$, $\Delta I_2 = \omega_2^2 - \omega^2 + ai\omega$ and

$$m_1^{n,k} = \frac{1}{1 + H^n_{n,k}(\omega)/f^1},$$

$$m_2^{n,k} = \frac{1}{1 + G^n_{n,k}(\omega)/f^2}. \tag{20}$$

The nth Fourier coefficients are then in the original basis:

$$c_n^x(\omega) = \frac{f_1}{2m_1^{n,k}\Delta I_1},$$
$$c_n^y(\omega) = v_2\frac{f_1}{2m_1^{n,k}\Delta I_1} + \frac{-v_2 f_1 + f_2}{2m_2^{n,k}\Delta I_2}.$$

The contribution of the nth harmonic to the Fourier spectrum is given (as seen in equation (12)) by

$$\mathscr{A}_n^X(\omega) = |X_n(\omega)| = \left|\frac{f_1}{m_1^{n,k}\Delta I_1}\right|,$$
$$\mathscr{A}_n^Y(\omega) = |Y_n(\omega)| = \left|v_2\frac{f_1}{m_1^{n,k}\Delta I_1} + \frac{-v_2 f_1 + f_2}{m_2^{n,k}\Delta I_2}\right|, \tag{21}$$

which can also be written as

$$\begin{pmatrix} X_n(\omega) \\ Y_n(\omega) \end{pmatrix} = \frac{\begin{pmatrix} 1 & 0 \\ v_2 & 0 \end{pmatrix}}{m_1^{n,k}\Delta I_1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \frac{\begin{pmatrix} 0 & 0 \\ -v_2 & 1 \end{pmatrix}}{m_2^{n,k}\Delta I_2}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{22}$$

If

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 \\ v_2 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{pmatrix} 0 & 0 \\ -v_2 & 1 \end{pmatrix},$$

we can express $\mathbf{A}_1$ in the form $\mathbf{T}_1^t\mathbf{T}_1'$, and $\mathbf{A}_2$ in the form $\mathbf{T}_2^t\mathbf{T}_2'$ where $\mathbf{T}_1, \mathbf{T}_1', \mathbf{T}_2$ and $\mathbf{T}_2'$ denote the generalized modes of the system.

If $\mathbf{T}_1 = (a_1, b_1)$ and $\mathbf{T}_1' = (a_1', b_1')$: then

$$\mathbf{T}_1^t\mathbf{T}_1' = \begin{pmatrix} a_1a_1' & a_1b_1' \\ b_1a_1' & b_1b_1' \end{pmatrix},$$

three equations with four unknowns are obtained

$$b_1' = 0,$$
$$a_1a_1' = 1,$$
$$b_1a_1' = v_2.$$

It is thus possible to arbitrarily set one of the unknowns; for example, as in the case of linear modes, if $a_1' = 1$, then

$$a_1 = 1,$$
$$b_1 = v_2,$$

hence

$$\mathbf{T}_1 = (1, v_2),$$
$$\mathbf{T}_1' = (1, 0). \tag{23}$$

The same type of calculations and assumptions for A_2 lead to

$$\mathbf{T}_2 = (0, 1),$$

$$\mathbf{T}'_2 = (-v_2, 1). \tag{24}$$

Using equations (23) and (24), relation (22) can then be written in the form

$$\begin{pmatrix} X_n(\omega) \\ Y_n(\omega) \end{pmatrix} = \frac{\mathbf{T}_1^t \mathbf{T}'_1}{m_1^{n,k} \Delta I_1(\omega)} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \frac{\mathbf{T}_2^t \mathbf{T}'_2}{m_2^{n,k} \Delta I_2(\omega)} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{25}$$

This has just established a modal superposition formula holding for a two-d.o.f. system with weak coupling. This formula links the nth harmonic amplitude to the free response and the forcing by means of a modal mass. The vectors $\mathbf{T}_1, \mathbf{T}'_1, \mathbf{T}_2$ and $\mathbf{T}'_2$ represent the modes (left and right respectively). Note that $\mathbf{T}_1$ and $\mathbf{T}_2$ are merely the eigenvectors associated with ω_1^2 and ω_2^2.

Finally, as in the case of the single-d.o.f. system, there is no unique modal superposition formula which holds in all cases, but according to the period of the response and the number of impacts per cycle, it is necessary to choose the right formula.

Remark. The modal superposition formula seems to exhibit a reciprocity breaking compared to the linear case: T_1 and T'_1 are different whatever choice is made when setting one of the unknowns. Indeed, the second component of T'_1 is always zero whereas that of T_1 is always non-zero. This is in fact due to the choice of the model, for which $k_1 = 0$ and $k_2 \neq 0$.

3.1.3.3. *Examples of modal superposition.* For the first example, the following parameters are chosen: $\omega_1 = 2{\cdot}5$, $\omega_2 = 3{\cdot}8$, $a = 0{\cdot}05$, $e = 0{\cdot}9$, $x_{max} = 14$, $f_1 = 20$, $f_2 = 18$ and $k_2 = 1$.

The second example is derived from the single-d.o.f. case and the chosen parameters are the following: $\omega_1 = 1$, $\omega_2 = 3{\cdot}8$, $a = 0{\cdot}02$, $e = 0{\cdot}9$, $x_{max} = 1$, $f_1 = 20$, $f_2 = 18$ and $k_2 = 1$. The spectral amplitude for x is the same one as that obtained in the single-d.o.f. case. The continuous d.o.f. y exhibits a resonance for $\omega \simeq \omega_2$ similar to the linear case, but several additional secondary resonances occur for $\omega \simeq \omega_2/2$ and $\omega \simeq \omega_2/3$. Furthermore, for the frequency associated with a spectral amplitude peak for x, a small peak for y is found, which can become large if k_2 is sufficiently large. Lastly, if ω_1 is close to $\omega_2/2$ and k_2 is rather large, then the main peak of resonance can occur in the neighbourhood of $\omega_2/2$.

3.2. STRONG COUPLING

In this section, the system (14) with $k_1 \neq 0$ is studied.

3.2.1. *Analytical solution of the system*

3.2.1.1. *Decoupling equations.* System (14) will be decoupled in order to be able to write the solutions x and y explicitly. The matrix K has the following characteristic polynomial:

$$P_K(\lambda) = \lambda^2 - (\omega_1^2 + \omega_2^2)\lambda + \omega_1^2\omega_2^2 - k_1 k_2$$

with discriminant $\Delta = (\omega_1^2 - \omega_2^2)^2 + 4k_1k_2$.

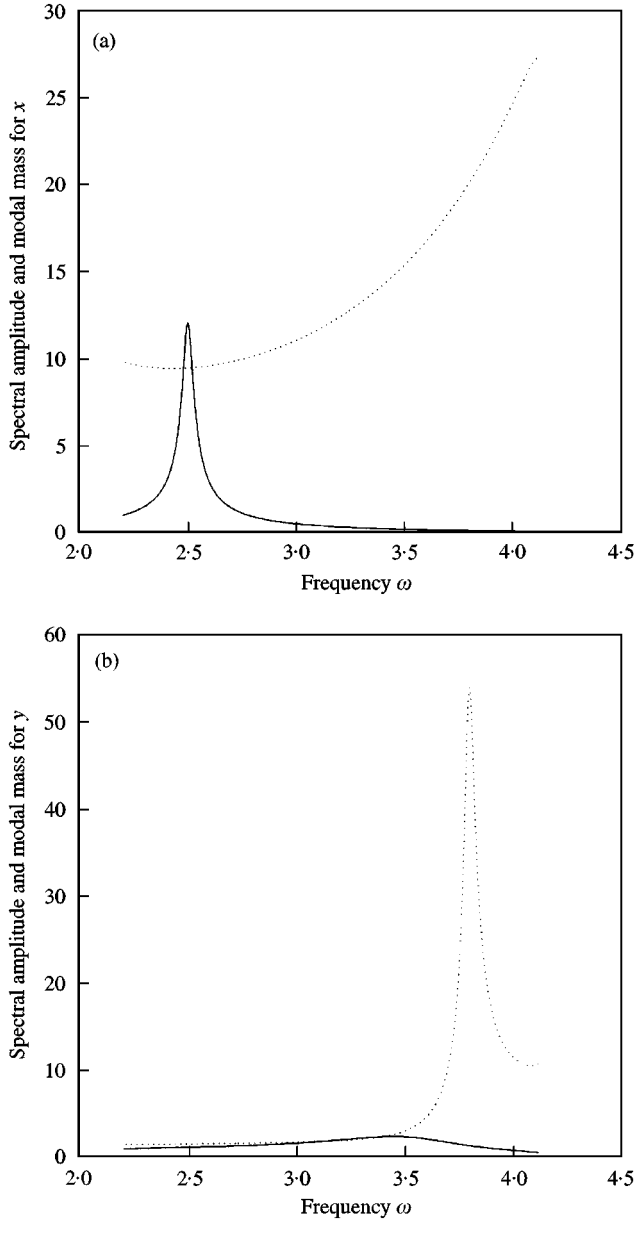

Figure 19. First-harmonic amplitude (dotted curve) and modal mass (solid curve) of the (1, 1)-periodic responses: (a) First degree of freedom x; (b) second degree of freedom y.

The system parameters will be chosen so that $\Delta > 0$. Then the matrix K admits two distinct real eigenvalues:

$$\lambda_1 = \frac{\omega_1^2 + \omega_2^2 - \sqrt{\Delta}}{2},$$
$$\lambda_2 = \frac{\omega_1^2 + \omega_2^2 + \sqrt{\Delta}}{2}.$$

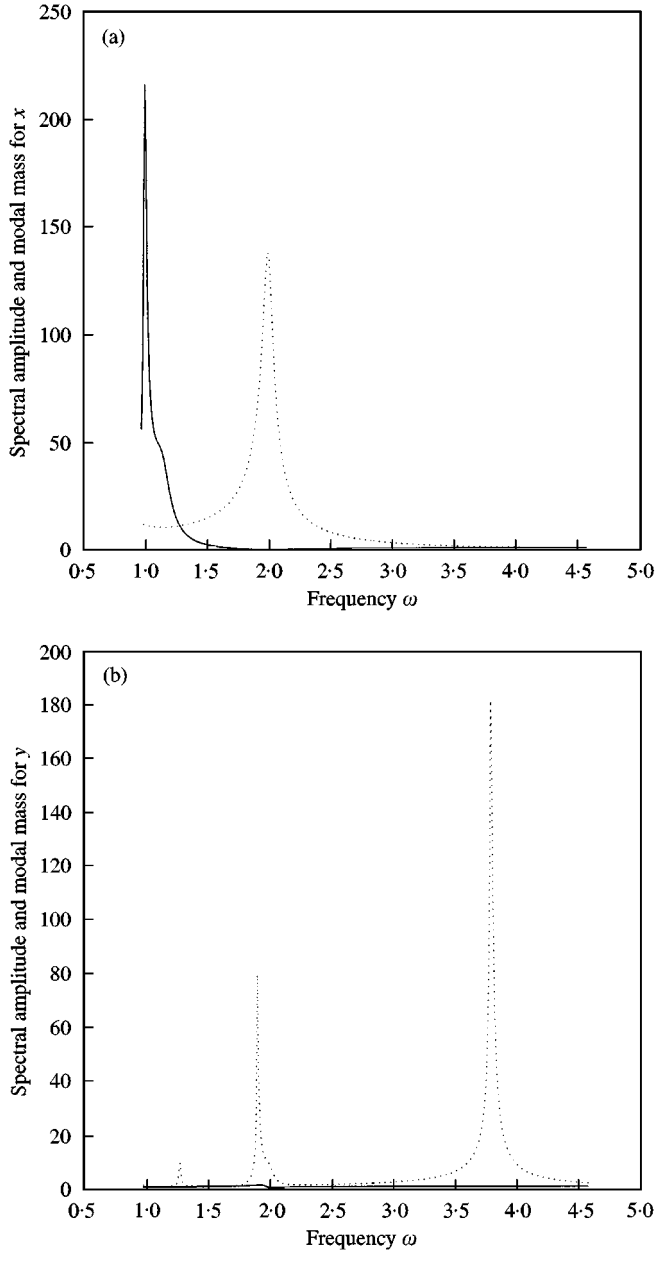

Figure 20. First-harmonic amplitude (dotted curve) and modal mass (solid curve) of the (1, 1)-periodic responses: (a) First degree of freedom x; (b) second degree of freedom y.

Assume that $\omega_1 \neq \omega_2$; if $v_1 = k_1/(\omega_1^2 - \omega_2^2)$ and $v_2 = k_2/(\omega_1^2 - \omega_2^2)$, then two eigenvectors associated with λ_1 and λ_2 are respectively $\binom{1}{v_2}$ and $\binom{v_1}{-1}$, which define the matrix

$$\mathbf{P} = \begin{pmatrix} 1 & -v_1 \\ v_2 & 1 \end{pmatrix}.$$

The assumption $\Delta > 0$ implies that its determinant is nonzero, and $\mathbf{P}$ is invertible with

$$\mathbf{P}^{-1} = \frac{1}{1 + v_1 v_2}\begin{pmatrix} 1 & v_1 \\ -v_2 & 1 \end{pmatrix}.$$

Then, if $\mathbf{X} = \mathbf{PY}$, the system (14) is multiplied by $\mathbf{P}^{-1}$ in order to obtain

$$\ddot{\mathbf{Y}} + a\dot{\mathbf{Y}} + \mathbf{P}^{-1}\mathbf{KPY} = \mathbf{P}^{-1}\mathbf{F}\cos(\omega t) + \text{impact}$$

Let

$$f^1 = \frac{f_1 + v_1 f_2}{1 + v_1 v_2},$$
$$f^2 = \frac{f_2 - v_2 f_1}{1 + v_1 v_2}.$$

The new system obtained using the new basis is given by

$$\begin{aligned} \ddot{x}_1 + a\dot{x}_1 + \lambda_1 x_1 &= f^1 \cos(\omega t), \\ \ddot{x}_2 + a\dot{x}_2 + \lambda_2 x_2 &= f^2 \cos(\omega t) \end{aligned} \tag{26}$$

as long as there is no impact. Thus these equations can be solved easily;

$$\begin{aligned} x_1(t) &= \mathrm{e}^{-at/2}[A^1 \cos(\tilde{\omega}_1 t) + B^1 \sin(\tilde{\omega}_1 t)] + f_1^1 \cos(\omega t) + f_2^1 \sin(\omega t), \\ x_2(t) &= \mathrm{e}^{-at/2}[A^2 \cos(\tilde{\omega}_2 t) + B^2 \sin(\tilde{\omega}_2 t)] + f_1^2 \cos(\omega t) + f_2^2 \sin(\omega t) \end{aligned} \tag{27}$$

with $\tilde{\omega}_1 = \sqrt{\lambda_1 - a^2/4}$, $\tilde{\omega}_2 = \sqrt{\lambda_2 - a^2/4}$ and

$$f_1^1 = f^1 \frac{\lambda_1 - \omega^2}{(\lambda_1 - \omega^2)^2 + a^2\omega^2},$$

$$f_2^1 = f^1 \frac{a\omega}{(\lambda_1 - \omega^2)^2 + a^2\omega^2},$$

$$f_1^2 = f^2 \frac{\lambda_2 - \omega^2}{(\lambda_2 - \omega^2)^2 + a^2\omega^2},$$

$$f_2^2 = f^2 \frac{a\omega}{(\lambda_2 - \omega^2)^2 + a^2\omega^2}.$$

The solution in the original basis is finally given by

$$\begin{aligned} x &= x_1 - v_1 x_2, \\ y &= v_2 x_1 + x_2. \end{aligned} \tag{28}$$

3.2.1.2. *Gluing solutions at impact times.* An impact occurs when $x(t) = x_{max}$, namely when $x_1(t) - v_1 x_2(t) = x_{max}$. Assume that there is an impact at t_k. For $t \in [t_{k-1}, t_k]$, the decoupled solution is given by

$$x_1(t) = \mathrm{e}^{-at/2}[A_k^1 \cos(\tilde{\omega}_1 t) + B_k^1 \sin(\tilde{\omega}_1 t)] + f_1^1 \cos(\omega t) + f_2^1 \sin(\omega t),$$

$$x_2(t) = \mathrm{e}^{-at/2}[A_k^2 \cos(\tilde{\omega}_2 t) + B_k^2 \sin(\tilde{\omega}_2 t)] + f_1^2 \cos(\omega t) + f_2^2 \sin(\omega t).$$

From these equations it can be inferred that the equation verified by t_k is

$$f(t_k) = \mathrm{e}^{-at_k/2}\{A_k^1\cos(\tilde{\omega}_1 t_k) + B_k^1\sin(\tilde{\omega}_1 t_k) - v_1[A_k^2\cos(\tilde{\omega}_2 t_k) + B_k^2\sin(\tilde{\omega}_2 t_k)]\}$$
$$+ (f_1^1 - v_1 f_1^2)\cos(\omega t_k) + (f_2^1 - v_1 f_2^2)\sin(\omega t_k) - x_{max} = 0.$$

Yet by assumptions on x and y, at the impact time:

$$x(t_k^+) = x(t_k^-),$$
$$\dot{x}(t_k^+) = -e\dot{x}(t_k^-),$$
$$y(t_k^+) = y(t_k^-),$$
$$\dot{y}(t_k^+) = \dot{y}(t_k^-)$$

which yields, according to (28),

$$x_1(t_k^+) = x_1(t_k^-),$$
$$\dot{x}_1(t_k^+) = \left(1 - \frac{1+e}{1+v_1v_2}\right)\dot{x}_1(t_k^-) + \frac{(1+e)v_1}{1+v_1v_2}\dot{x}_2(t_k^-),$$
$$x_2(t_k^+) = x_2(t_k^-),$$
$$\dot{x}_2(t_k^+) = \frac{(1+e)v_2}{1+v_1v_2}\dot{x}_1(t_k^-) + \left(\frac{1+e}{1+v_1v_2} - e\right)\dot{x}_2(t_k^-).$$

For $i \in \{1, 2\}$,

$$u_i(k) = A_k^i\sin(\tilde{\omega}_i t_k) + \eta_i\cos(\tilde{\omega}_i t_k)] + B_k^i[-\cos(\tilde{\omega}_i t_k) + \eta_i\sin(\tilde{\omega}_i t_k)]$$
$$+ \frac{\omega}{\tilde{\omega}_i}\mathrm{e}^{at_k/2}[f_1^i\sin(\omega t_k) - f_2^i\cos(\omega t_k)],$$

then:

$$\begin{aligned}
A_{k+1}^1 &= A_k^1 - \frac{1+e}{1+v_1v_2}\sin(\tilde{\omega}_1 t_k)\left[u_1(k) - v_1\frac{\tilde{\omega}_2}{\tilde{\omega}_1}u_2(k)\right],\\
B_{k+1}^1 &= B_k^1 + \frac{1+e}{1+v_1v_2}\cos(\tilde{\omega}_1 t_k)\left[u_1(k) - v_1\frac{\tilde{\omega}_2}{\tilde{\omega}_1}u_2(k)\right],\\
A_{k+1}^2 &= A_k^2 + \frac{(1+e)v_2}{1+v_1v_2}\sin(\tilde{\omega}_2 t_k)\left[\frac{\tilde{\omega}_1}{\tilde{\omega}_2}u_1(k) - v_1u_2(k)\right],\\
B_{k+1}^2 &= B_k^2 - \frac{(1+e)v_2}{1+v_1v_2}\cos(\tilde{\omega}_2 t_k)\left[\frac{\tilde{\omega}_1}{\tilde{\omega}_2}u_1(k) - v_1u_2(k)\right].
\end{aligned} \tag{29}$$

3.2.2. *Search for periodic solutions*

The search for periodic solutions (x, y) is equivalent to the search for periodic solutions (x_1, x_2). Only $(n, 0)$ and $(n, 1)$-periodic solutions will be sought. The following calculations are similar to those found in reference [76], with the addition of damping.

3.2.2.1. $(n, 0)$-*periodic solutions.* As for the single d.o.f system, it is easy to prove that the system admits a $(n,0)$-periodic solution if, and only if, x_1 and x_2 are given by

$$x_1(t) = f_1^1 \cos(\omega t) + f_2^1 \sin(\omega t),$$

$$x_2(t) = f_1^2 \cos(\omega t) + f_2^2 \sin(\omega t).$$

According to equation (28):

$$x(t) = (f_1^1 - v_1 f_1^2)\cos(\omega t) + (f_2^1 - v_1 f_2^2)\sin(\omega t).$$

It is thus possible to get results similar to those obtained for a single-d.o.f. system in section 2.2.1. This gives a condition that a $(n, 0)$-periodic solution exists; it requires $(f_1^1 - v_1 f_1^2)^2 + (f_2^1 - v_1 f_2^2)^2 \leqslant x_{max}^2$. From this inequality, a fourth degree polynomial in ω^2 can be obtained, allowing the values of ω for which the system admits a $(n, 0)$-periodic response to be determined. The remark made in Section 2.2.1 still holds; the existence of $(n, 0)$-periodic solutions requires $x_{max} > 0$.

3.2.2.2. $(n, 1)$-*periodic solutions.* nT-periodicity requires

$$x_1(nT) = x_1^0,$$

$$\dot{x}_1(nT) = \dot{x}_1^0,$$

$$x_2(nT) = x_2^0,$$

$$\dot{x}_2(nT) = \dot{x}_2^0$$

or using linear combination, for a solution with one impact per cycle:

$$\begin{aligned}
A_2^1 \cos(n\tilde{\omega}_1 T) + B_2^1 \sin(n\tilde{\omega}_1 T) - \mathrm{e}^{anT/2} A_1^1 &= 0,\\
A_2^2 \cos(n\tilde{\omega}_2 T) + B_2^2 \sin(n\tilde{\omega}_2 T) - \mathrm{e}^{anT/2} A_1^2 &= 0,\\
A_2^1 \sin(n\tilde{\omega}_1 T) - B_2^1 \cos(n\tilde{\omega}_1 T) + \mathrm{e}^{anT/2} B_1^1 &= 0,\\
A_2^2 \sin(n\tilde{\omega}_2 T) + B_2^2 \cos(n\tilde{\omega}_2 T) - \mathrm{e}^{anT/2} B_1^2 &= 0.
\end{aligned} \tag{30}$$

Using equation (29), a 4×4 linear system in $(A_1^1, B_1^1, A_1^2, B_1^2)$ is obtained. The resolution of this system leads to $A_1^1(t_1)$, $B_1^1(t_1)$, $A_1^2(t_1)$ and $B_1^2(t_1)$.

It remains to determine t_1 via the equation

$$\begin{aligned}
f(t_1) = \mathrm{e}^{-at_1/2}\{&A_1^1(t_1)\cos(\tilde{\omega}_1 t_1) + B_1^1(t_1)\sin(\tilde{\omega}_1 t_1)\\
&- v_1[A_1^2(t_1)\cos(\tilde{\omega}_2 t_1) + B_1^2(t_1)\sin(\tilde{\omega}_2 t_1)]\}\\
&+ (f_1^1 - v_1 f_1^2)\cos(\omega t_1) + (f_2^1 - v_1 f_2^2)\sin(\omega t_1) - x_{max} = 0.
\end{aligned}$$

Initial conditions of the system leading to a $(n, 1)$-periodic solution are then given by

$$x_0 = A_1^1(t_1) + f_1^1 - v_1(A_1^2(t_1) + f_1^2),$$

$$y_0 = v_2(A_1^1(t_1) + f_1^1) + A_1^2(t_1) + f_1^2,$$

$$\dot{x}_0 = \tilde{\omega}_1 [B_1^1(t_1) - \eta_1 A_1^1(t_1)] + f_2^1 \omega - v_1 \tilde{\omega}_2 [B_1^2(t_1) - \eta_2 A_1^2(t_1)] - v_1 f_2^2 \omega,$$

$$\dot{y}_0 = v_2 \tilde{\omega}_1 [B_1^1(t_1) - \eta_1 A_1^1(t_1)] + v_2 f_2^1 \omega + \tilde{\omega}_2 [B_1^2(t_1) - \eta_2 A_1^2(t_1)] + f_2^2 \omega.$$

3.2.3. *Modal superpositon*

3.2.3.1. *Free oscillation of the system.* It can be shown that the system (14) without external forcing has a finite number of impacts. Thus the steady state of the system (26) with $f^1 = f^2 = 0$ is periodic with frequency λ_1 for x_1 and λ_2 for x_2: these two frequencies will be used as natural frequencies for the forced system.

3.2.3.2. *Generalized masses and modal superposition.* It is now possible to consider a (n, k)-periodic solution and to calculate the Fourier coefficients of x_1 and x_2. The calculation of $c_j^1(\omega)$ and $c_j^2(\omega)$ is similar to the one carried out in the weak coupling case: only the recursive relation (29) changes. In this case

$$H_{n,k}^j(\omega) = \frac{2\tilde{\omega}_1}{nT} \frac{1}{\omega_1^2 - \dfrac{j^2\omega^2}{n^2} + \mathrm{i}a\dfrac{j\omega}{n}}$$

$$\times \Bigg\{ B_1^1 + A_{k+1}^1 \mathrm{e}^{-anT/2} \sin(n\tilde{\omega}_1 T) - B_{k+1}^1 \mathrm{e}^{-anT/2} \cos(n\tilde{\omega}_1 T)$$

$$+ (\eta_1 + \mathrm{i}\gamma_j^1) [A_1^1 - A_{k+1}^1 \mathrm{e}^{-anT/2} \cos(n\tilde{\omega}_1 T) - B_{k+1}^1 \mathrm{e}^{-anT/2} \sin(n\tilde{\omega}_1 T)]$$

$$+ \frac{1+e}{1+v_1 v_2} \sum_{m=1}^{k} \mathrm{e}^{-at_m/2} \exp\left(-\mathrm{i}\frac{j}{n}\omega t_m \right) \left[u_1(m) - v_1 \frac{\tilde{\omega}_2}{\tilde{\omega}_1} u_2(m) \right] \Bigg\}$$

$$G_{n,k}^j(\omega) = \frac{2\tilde{\omega}_2}{nT} \frac{1}{\omega_2^2 - \dfrac{j^2\omega^2}{n^2} + \mathrm{i}a\dfrac{j\omega}{n}}$$

$$\Bigg\{ B_1^2 + A_{k+1}^2 \mathrm{e}^{-anT/2} \sin(n\tilde{\omega}_2 T) - B_{k+1}^2 \mathrm{e}^{-anT/2} \cos(n\tilde{\omega}_2 T)$$

$$+ (\eta_2 + \mathrm{i}\gamma_j^2) [A_1^2 - A_{k+1}^2 \mathrm{e}^{-anT/2} \cos(n\tilde{\omega}_2 T) - B_{k+1}^2 \mathrm{e}^{-anT/2} \sin(n\tilde{\omega}_2 T)]$$

$$- \frac{(1+e)v_2}{1+v_1 v_2} \sum_{m=1}^{k} \mathrm{e}^{-at_m/2} \exp\left(-\mathrm{i}\frac{j}{n}\omega t_m \right) \left[\frac{\tilde{\omega}_1}{\tilde{\omega}_2} u_1(m) - v_1 u_2(m) \right] \Bigg\}$$

with $\gamma_j^1 = j\omega/n\tilde{\omega}_1$ and $\gamma_j^2 = j\omega/n\tilde{\omega}_2$. Then

$$c_j^1(\omega) = \frac{f_1^1 - \mathrm{i}f_2^1}{2} \delta_j^n + \frac{f_1^1 + \mathrm{i}f_2^1}{2} \delta_j^{-n} + \frac{H_{n,k}^j(\omega)}{2},$$

$$c_j^2(\omega) = \frac{f_1^2 - \mathrm{i}f_2^2}{2} \delta_j^n + \frac{f_1^2 + \mathrm{i}f_2^2}{2} \delta_j^{-n} + \frac{G_{n,k}^j(\omega)}{2}.$$

The nth Fourier coefficients of the system by the basis change (28) are then

$$c_n^x(\omega) = c_n^1(\omega) - v_1 c_n^2(\omega) = \frac{f^1}{2m_1^{n,k}\Delta I_1} - v_1 \frac{f^2}{2m_2^{n,k}\Delta I_2},$$
$$c_n^y(\omega) = v_2 c_n^1(\omega) + c_n^2(\omega) = v_2 \frac{f^1}{2m_1^{n,k}\Delta I_1} + \frac{f^2}{2m_2^{n,k}\Delta I_2}, \tag{31}$$

where $\Delta I_1 = \lambda_1 - \omega^2 + ai\omega$, $\Delta I_2 = \lambda_2 - \omega^2 + ai\omega$ and

$$m_1^{n,k} = \frac{1}{1 + H_{n,k}^n(\omega)/f^1},$$
$$m_2^{n,k} = \frac{1}{1 + G_{n,k}^n(\omega)/f^2}. \tag{32}$$

In the original basis,

$$c_n^x(\omega) = \frac{1}{1 + v_1 v_2}\left[\frac{f_1 + v_1 f_2}{2m_1^{n,k}\Delta I_1} - v_1 \frac{f_2 - v_2 f_1}{2m_2^{n,k}\Delta I_2}\right],$$
$$c_n^y(\omega) = \frac{1}{1 + v_1 v_2}\left[v_2 \frac{f_1 + v_1 f_2}{2m_1^{n,k}\Delta I_1} + \frac{f_2 - v_2 f_1}{2m_2^{n,k}\Delta I_2}\right].$$

The contribution of the nth harmonic is (as for the weak coupling in equation (21)) given by

$$\begin{pmatrix} X_n(\omega) \\ Y_n(\omega) \end{pmatrix} = \frac{1}{1 + v_1 v_2}\left\{\frac{\begin{pmatrix} 1 & v_1 \\ v_2 & v_1 v_2 \end{pmatrix}}{m_1^{n,k}\Delta I_1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \frac{\begin{pmatrix} v_1 v_2 & -v_1 \\ -v_2 & 1 \end{pmatrix}}{m_2^{n,k}\Delta I_2}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right\}. \tag{33}$$

Let

$$\mathbf{A}_1 = \frac{1}{1 + v_1 v_2}\begin{pmatrix} 1 & v_1 \\ v_2 & v_1 v_2 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_2 = \frac{1}{1 + v_1 v_2}\begin{pmatrix} v_1 v_2 & -v_1 \\ -v_2 & 1 \end{pmatrix},$$

we can express $\mathbf{A}_1$ in the form $\mathbf{T}_1^t \mathbf{T}_1'$ and $\mathbf{A}_2$ in the form $\mathbf{T}_2^t \mathbf{T}_2'$.

Let $\mathbf{T}_1 = (a_1, b_1)$ and $\mathbf{T}_1' = (a_1', b_1')$; then

$$\mathbf{T}_1^t \mathbf{T}_1' = \begin{pmatrix} a_1 a_1' & a_1 b_1' \\ b_1 a_1' & b_1 b_1' \end{pmatrix}.$$

Three equations with four unknowns are obtained

$$a_1 a_1' = \frac{1}{1 + v_1 v_2},$$

$$a_1 b_1' = \frac{v_1}{1 + v_1 v_2},$$

$$b_1 b_1' = \frac{v_1 v_2}{1 + v_1 v_2},$$

$$b_1 a_1' = \frac{v_2}{1 + v_1 v_2}.$$

As in the case of weak coupling, it is possible to fix one of the unknows; for example, in order to respect a reciprocity condition as long as possible, let $a_1 = 1/(\sqrt{1 + v_1 v_2})$. Then $1 + v_1 v_2 > 0$, for the parameters of the system verify $(\omega_1^2 - \omega_2^2)^2 + 4k_1 k_2 > 0$. Then

$$a_1' = \frac{1}{\sqrt{1 + v_1 v_2}},$$

$$b_1' = \frac{v_1}{\sqrt{1 + v_1 v_2}},$$

$$b_1 = \frac{v_2}{\sqrt{1 + v_1 v_2}}.$$

and

$$\begin{aligned} \mathbf{T}_1 &= \frac{1}{\sqrt{1 + v_1 v_2}}(1, v_2), \\ \mathbf{T}_1' &= \frac{1}{\sqrt{1 + v_1 v_2}}(1, v_1). \end{aligned} \tag{34}$$

In the same way, for A_2:

$$\begin{aligned} \mathbf{T}_2 &= \frac{1}{\sqrt{1 + v_1 v_2}}(-v_1, 1), \\ \mathbf{T}_2' &= \frac{1}{\sqrt{1 + v_1 v_2}}(-v_2, 1). \end{aligned} \tag{35}$$

Using equations (34) and (35), the relation (33) then becomes

$$\begin{pmatrix} X_n(\omega) \\ Y_n(\omega) \end{pmatrix} = \frac{T_1^t T_1'}{m_1^{n,k} \Delta I_1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \frac{T_2^t T_2'}{m_2^{n,k} \Delta I_2} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{36}$$

Thus a modal superposition formula for the two-d.o.f. system with strong coupling has been established.

Remark.

— T_1 and T_2 thus defined are in fact eigenvectors associated with λ_1 and λ_2.
— T_1' and T_2 are orthogonal, just as T_2' and T_1.
— There is no reciprocity when $k_1 \neq k_2$, but when the matrix K is symmetrical, reciprocity occurs since then $T_1 = T_1'$ and $T_2 = T_2'$.

3.3. TWO RIGID BODIES COLLIDING

In this section, a mechanical system consisting of two oscillating rigid bodies which can collide during their movement is studied.

It will be assumed that the equilibrium position x_1^0 and x_2^0 of the two solids are such that $x_2^0 - x_1^0 = x_{max} > 0$ (system without preload). The equations of the system in relative displacements are then

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + \kappa_1 x_1 = g_1\cos(\omega t),$$

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + \kappa_2 x_2 = g_2\cos(\omega t),$$

$$x_2 - x_1 \geqslant x_{max}.$$

Let $a_1 = c_1/m_1$, $\lambda_1 = \kappa_1/m_1$, $f^1 = g_1/m_1$ and similarly $a_2 = c_2/m_2$, $\lambda_2 = \kappa_2/m_2$, $f^2 = g_2/m_2$, and assume hereafter that $a_1 = a_2 = a$. The system becomes

$$\begin{aligned} &\ddot{x}_1 + a\dot{x}_1 + \lambda_1 x_1 = f^1\cos(\omega t),\\ &\dot{x}_2 + a\dot{x}_2 + \lambda_2 x_2 = f^2\cos(\omega t),\\ &x_2 - x_1 \geqslant x_{max}. \end{aligned} \tag{37}$$

When $x_2(t) - x_1(t) = x_{max}$, an impact occurs at t and the restitution law leads the relative velocity between the two solids to be multiplied by a factor $-e$ at the impact time:

$$\dot{x}_2(t^+) - \dot{x}_1(t^+) = -e(\dot{x}_2(t^-) - \dot{x}_1(t^-)).$$

Moreover, the conservation of the momentum provides the second equation to be able to determine post-impact velocities:

$$m_2\dot{x}_2(t^+) + m_1\dot{x}_1(t^+) = m_2\dot{x}_2(t^-) + m_1\dot{x}_1(t^-).$$

A simple change of co-ordinates can be applied in order to express the system (14) in the form previously studied; let

$$\begin{aligned} x &= x_1 - x_2,\\ y &= \frac{m_1}{m_2}x_1 + x_2. \end{aligned} \tag{38}$$

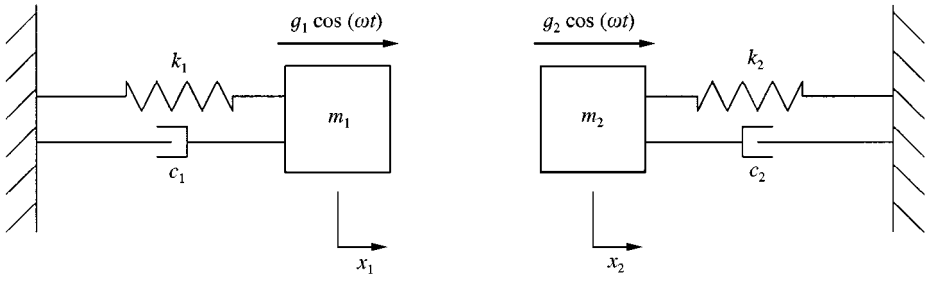

Figure 21. Mechanical model of two rigid oscillating bodies colliding.

For this new system of variables, the equations become

$$\begin{aligned} &\ddot{x} + a\dot{x} + \omega_1^2 x + k_1 y = f_1 \cos(\omega t) + \text{“impact”},\\ &\ddot{y} + a\dot{y} + \omega_2^2 x + k_2 x = f_2 \cos(\omega t),\\ &x_2 - x_1 \geqslant 0, \end{aligned} \tag{39}$$

where $\omega_1^2 = (m_1\lambda_2 + m_2\lambda_1)/(m_1 + m_2)$, $\omega_2^2 = (m_1\lambda_1 + m_2\lambda_2)/(m_1 + m_2)$, $k_1 = (m_2/(m_1 + m_2))(\lambda_1 - \lambda_2)$, $k_2 = (m_1/(m_1 + m_2))(\lambda_1 - \lambda_2)$, $f_1 = f^1 - f^2$ and $f_2 = (m_1/m_2)f^1 + f^2$.

Thus the system is of the type (14); the results of the preceding study are then applicable. First of all note that k_1 and k_2 are proportional, so that there is always the case of the strong coupling studied in section 3. As regards the solutions of the system between two impacts, there is no work to do in order to decouple the system, since the initial system is already written in decoupled form. Thus the eigenvalues of the system are always real, and are exactly λ_1 and λ_2.

In this case $v_1 = 1$ and $v_2 = m_1/m_2$. According to equations (34) and (35), the modes of the system are given by

$$\begin{aligned} T_1 &= \alpha_{1,2}\left(1, \frac{m_1}{m_2}\right),\\ T_1' &= \alpha_{1,2}(1, 1),\\ T_2 &= \alpha_{1,2}(-1, 1),\\ T_2' &= \alpha_{1,2}\left(-\frac{m_1}{m_2}, 1\right), \end{aligned}$$

where $\alpha_{1,2} = 1/\sqrt{1 + m_1/m_2}$.

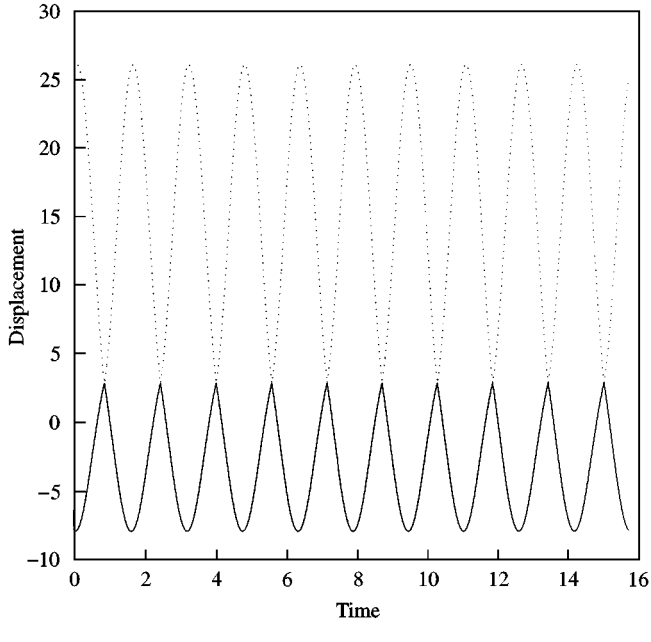

Figure 22. (1, 1)-periodic solution for $\omega = 4$, $x_0^1 = -7{\cdot}90482$, $\dot{x}_0^1 = -2{\cdot}07935$, $x_0^2 = 11{\cdot}20441$, and $\dot{x}_0^2 = 15{\cdot}14127$: absolute displacement.

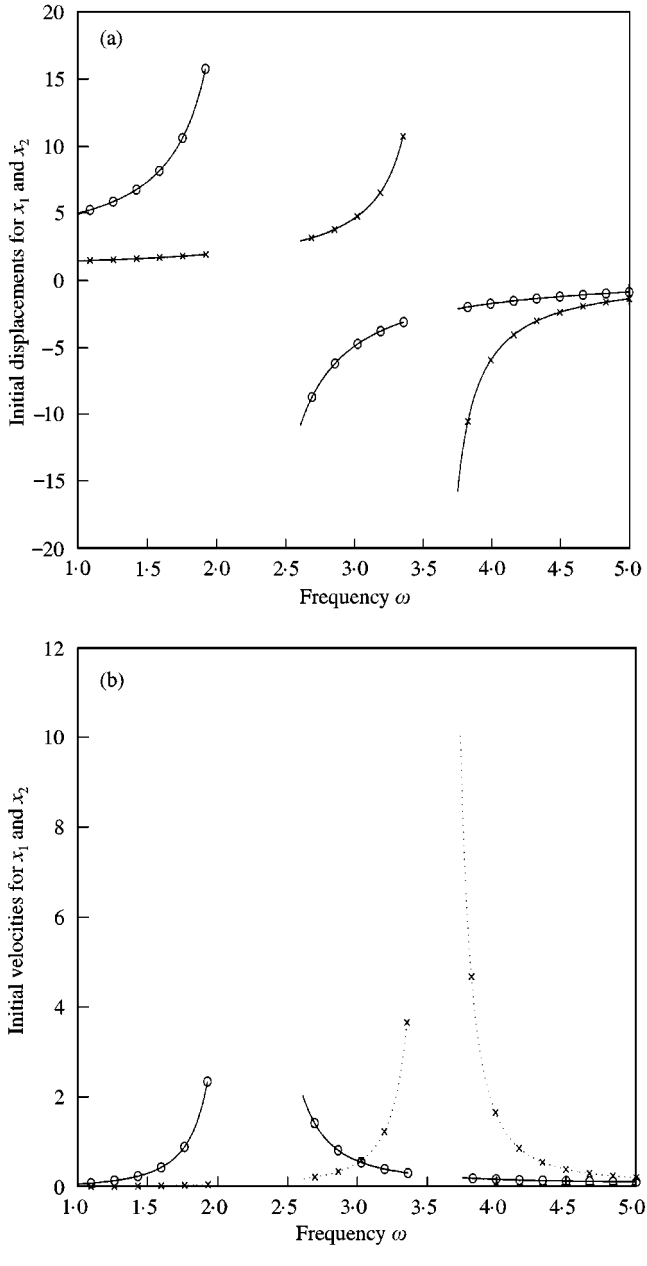

Figure 23. Existence of (1, 0)-periodic solutions: (a) initial displacements for x_1 (circle) and x_2 (x-mark); (b) initial velocities for x_1 (circle) and x_2 (x-mark).

Using these modes, the modal superposition formula is as in equation (36), the expression of the modal masses being given by equation (32). It is interesting to note that in general there is no reciprocity, except if $m_1 = m_2$.

Finally, the periodic responses of the system can be dealt with. First of all, according to section 3.2.2, there are $(n, 0)$-periodic solutions because it was assumed that $x_{max} > 0$. In the same way, calculations of section 3.2.2 can search for $(n, 1)$-periodic solutions. Diagrams of existence of periodic solutions versus the frequency of the external excitation are obtained.

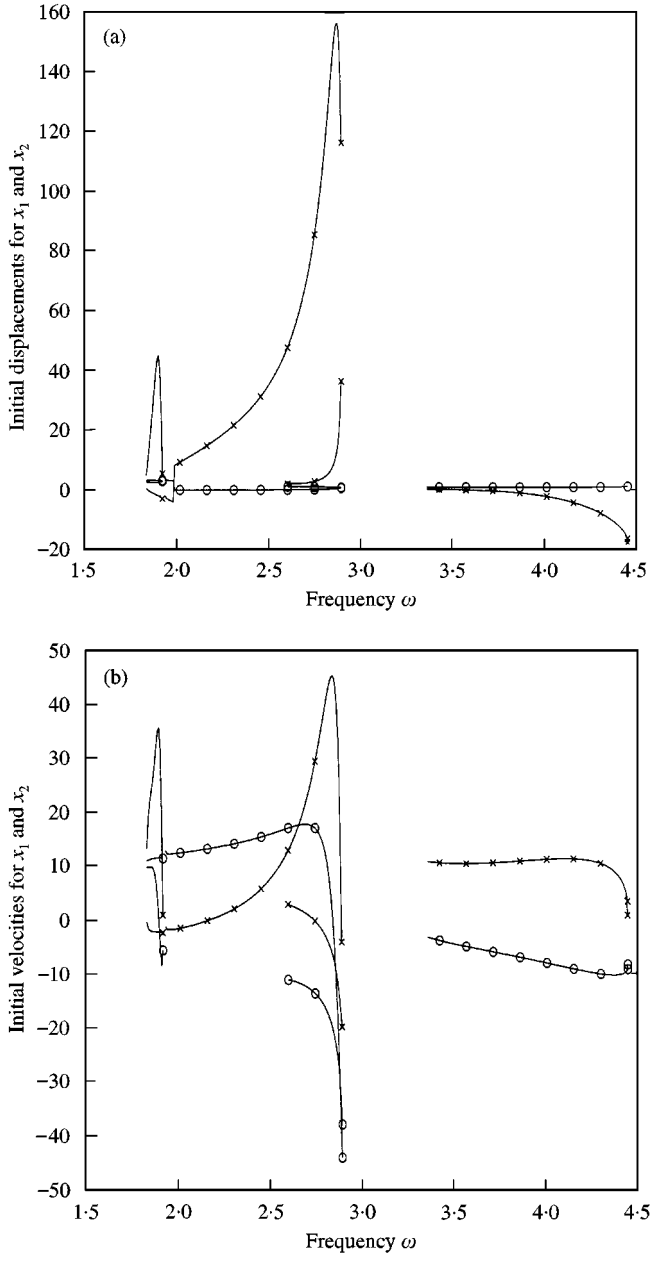

Figure 24. Existence of (1, 1)-periodic solutions: (a) initial displacements for x_1 (circle) and x_2 (x-mark); (b) initial velocities for x_1 (circle) and x_2 (x-mark).

From these periodic solutions, the modal superposition formula can then be tested, plotting the difference between the spectral amplitude of the system's response and the nth harmonic amplitude given by equation (36).

For the first example presented, let $m_1 = 1$, $m_2 = 0{\cdot}7$, $\lambda_1 = 5$, $\lambda_2 = 13$, $a = 0{\cdot}05$, $e = 0{\cdot}9$, $f^1 = 20$, $f^2 = 18$, and $x_{max} = 14$.

As it was the case for the single-d.o.f. system, note in Figure 25 that the usual resonance of linear systems no longer occurs. Indeed, when the frequency of the external excitation is

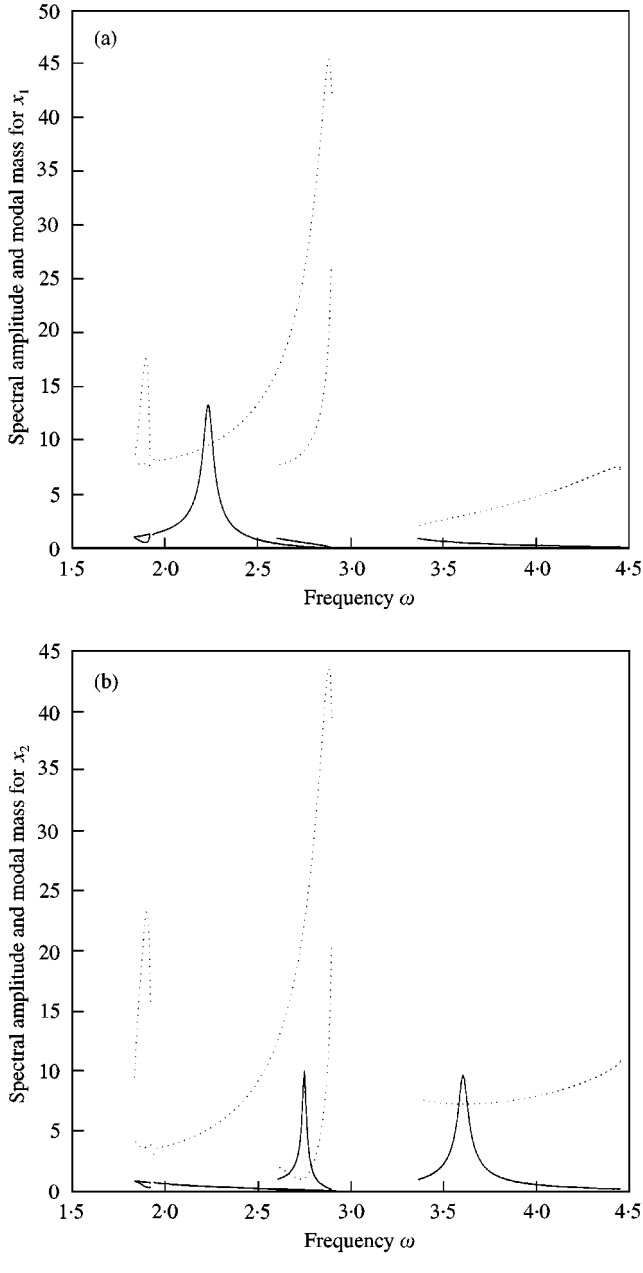

Figure 25. First-harmonic amplitude (dotted curve) and modal mass (solid curve) of the (1, 1)-periodic solutions: (a) first degree of freedom x_1; (b) second degree of freedom x_2.

equal to the natural frequency of the system, the modal mass corresponding to the excited mode goes through a local maximum, so that the associated spectral amplitude does not present a particular peak.

Nevertheless, spectral amplitude peaks appear for values of ω far away from the natural frequencies. The main peak is located in the neighbourhood of ω_2 (which is one of the natural frequencies of the system after the change of variables), but it is difficult to show a resonance in ω_2, for the peak is appreciably shifted from this frequency. From the modal

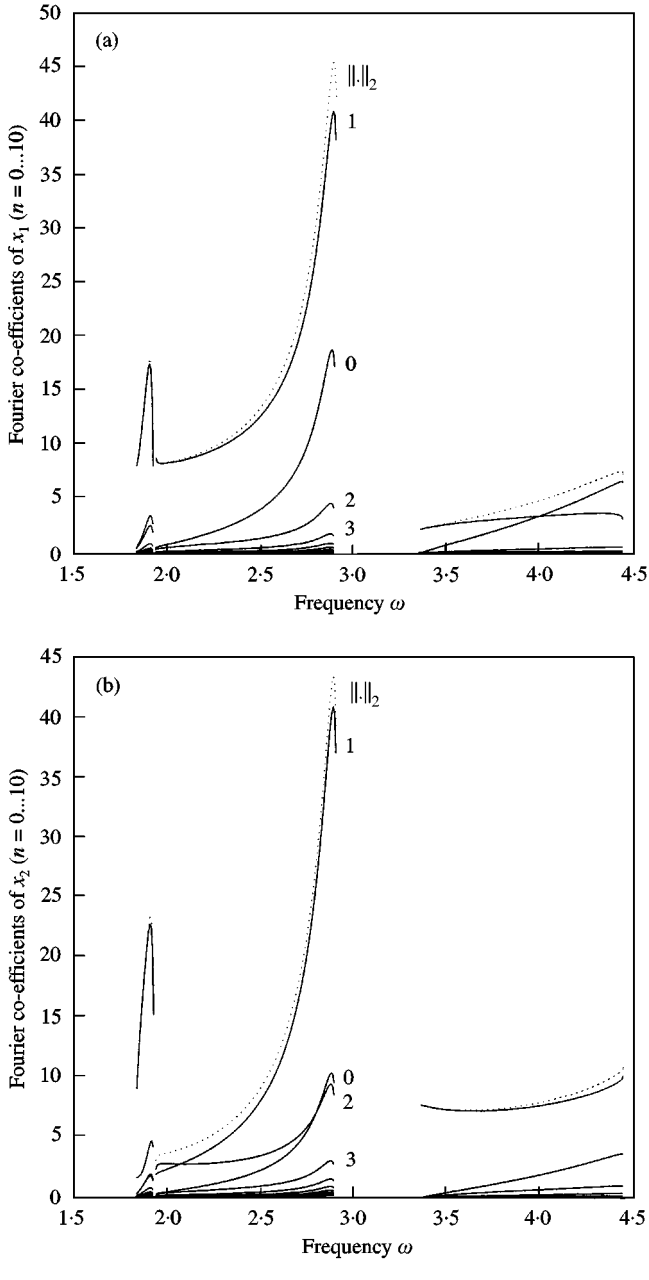

Figure 26. Fourier coefficients of the (1, 1)-periodic responses: (a) first degree of freedom x_1; (b) second degree of freedom x_2.

point of view, the occurrence of such peaks does not correspond to a classical resonance (where the generalized natural frequency of the system is close to the frequency of external excitation); resonance can be interpreted as the locus of frequency where the generalized modal mass is minimum with respect to ω.

As regards the closeness of the approximation to the spectral amplitude by the nth harmonic amplitude, note, as in the single-d.o.f. case, that the first harmonic is not always the leading term in the amplitude of a (1, 1)-periodic response (see Figure 26). Thus, the

TABLE 2

Difference between the whole spectral amplitude and the amplitude of nth harmonic

Degree of freedom	Difference (in %) whole range	Difference (in %) neighbourhood of the peak
x_1	77·31	12·82
x_2	82·75	4·89

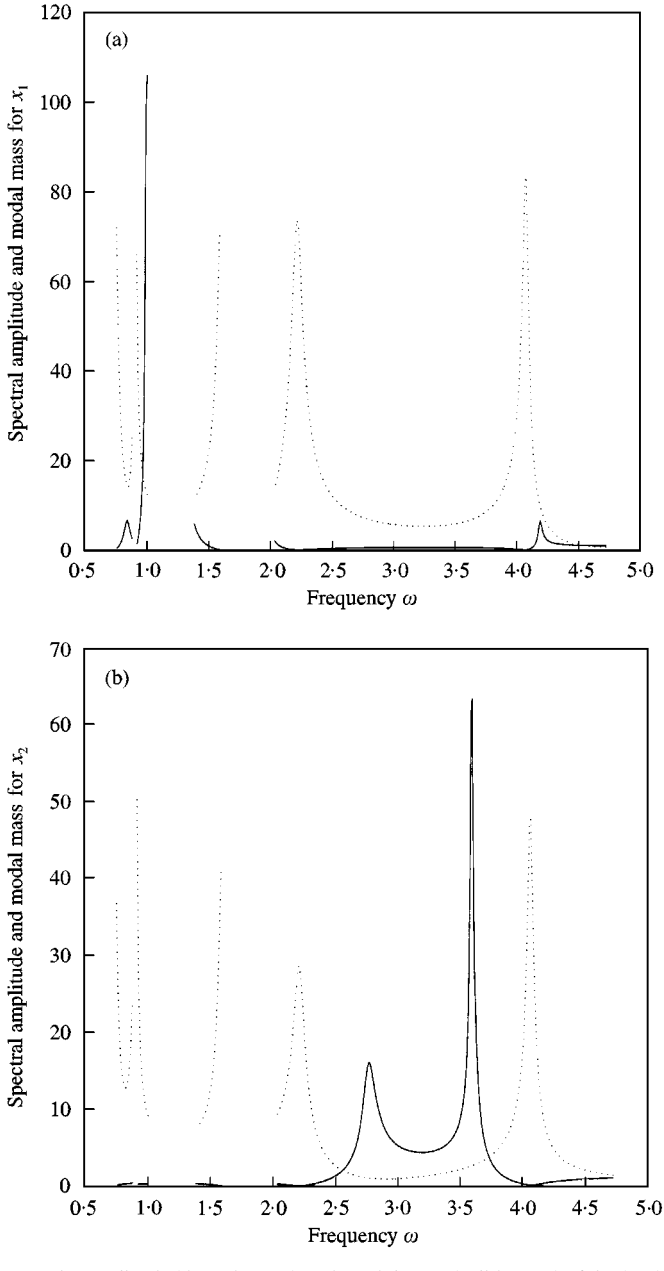

Figure 27. First-harmonic amplitude (dotted curve) and modal mass (solid curve) of the (1, 1)-periodic solutions: (a) first degree of freedom x_1; (b) second degree of freedom x_2.

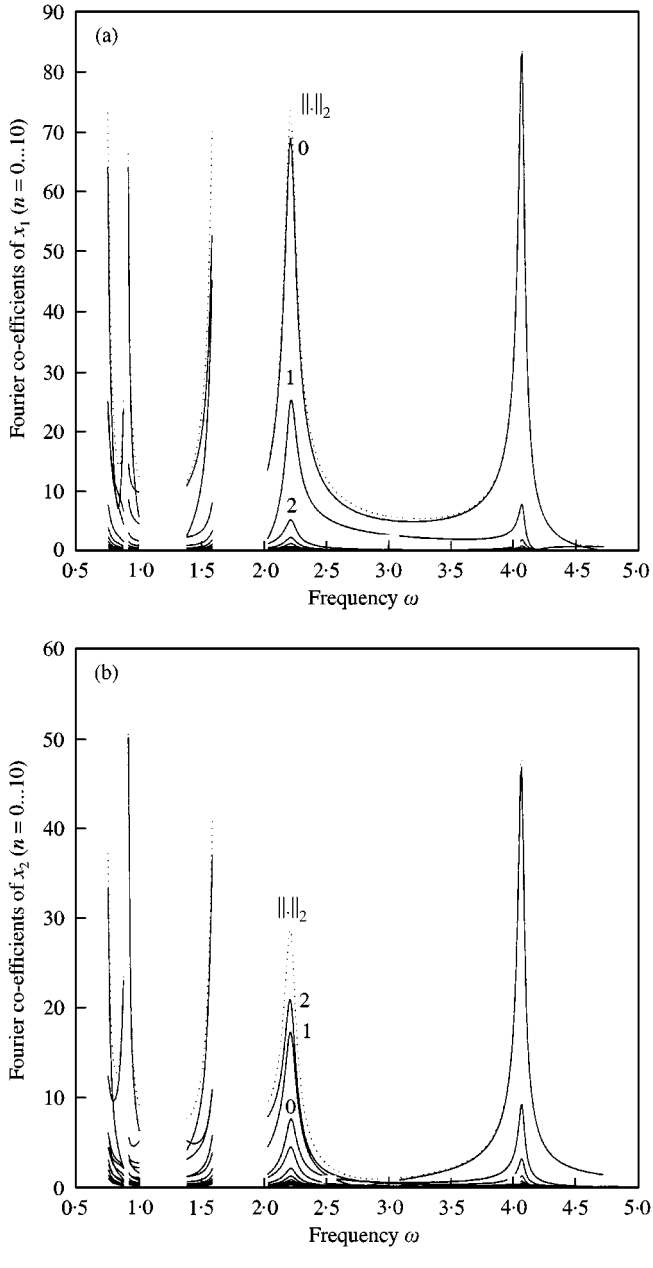

Figure 28. Fourier coefficients of the (1, 1)-periodic responses: (a) first degree of freedom x_1; (b) second degree of freedom x_2.

constant coefficient c_0 overrides all the others for some values of ω. Yet if ω is considered to be close to the peaks in the spectral response, the first harmonic gives a close approximation to the total amplitude (see Table 2).

For the last example, a set of parameters close to the second example in the single-d.o.f. system is chosen: $m_1 = 1$, $m_2 = 0{\cdot}7$, $\lambda_1 = 1$, $\lambda_2 = 13$, $a = 0{\cdot}02$, $e = 0{\cdot}9$, $f^1 = 20$, $f^2 = 18$, and $x_{max} = 1$. Looking at Figures 27 and 28 first note that amplitude peaks appear at the same frequencies for x_1 and x_2, which is due to energy transmission between the two bodies

through impacts. Moreover, the first harmonic amplitude can once again be far lower than the spectral amplitude; most of amplitude peaks in x_1 come from $\mathscr{A}_0$, and for x_2 the second harmonic is often overriding.

Only the (1, 1)-periodic responses of the system have been studied. Nevertheless, it must be kept in mind that many other types of periodic solutions are possible, as was seen for the single-d.o.f. system, depending on the time period and the number of impacts per cycle. Theoretically, nothing can prevent a modal superposition formula for any (n, k)-periodic response from being written, but in practice such a response is hard to find by analytical means when $k > 1$.

4. CONCLUSION

The feasibility of building a modal superposition formula for systems with irregular non-linearities of impact type has been investigated, imitating the procedure used in the smooth non-linear case [49, 51]. The formula has been built for simple single- and two-d.o.f. systems with unilateral constraint and restitution law. The generalized modes and frequencies obtained turn out to be identical to the linear case, the non-linearity of the system being concentrated into the modal masses.

The examples considered show that the formula is valid in the case of a primary resonance for which the spectral amplitude is given by the Fourier coefficient corresponding to the periodicity of the forced solution obtained. Nevertheless, these examples have above all illustrated the limitations to such a formulation. Firstly, the multiplicity of periodic solutions, with different periodicity or number of impacts per cycle, compels several potential formulas to be built, and it is not possible to know *a prori* which one has to be used. Furthermore, main amplitude peaks may appear away from any *a priori* clearly identifiable resonance, which may be overidden by some unusual harmonics and consequently cause the modal superposition formula to fail.

Therefore, it is not possible, in a general case, to build a modal superposition formula using only the usual sequence definition of generalized frequencies, definition of generalized modes and then definition of generalized modal masses; the non-linearities of impact type produce a limitation on the formulation of a general formula following the usual procedure.

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