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# OPTIMUM PARAMETERS OF TUNED LIQUID COLUMN DAMPER FOR SUPPRESSING PITCHING VIBRATION OF AN UNDAMPED STRUCTURE

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In order to evaluate the control performance of the tuned liquid column damper in suppressing pitching vibration of structures, an optimal parametric study of the damper is carried out for an undamped structure. The structure is assumed to be subjected to harmonic excitation in the analysis. The optimum tuning ratio (or the optimum liquid length) and the optimum head loss coefficient of the damper are determined using Den Hartog's method. The analytical formulas of the optimum TLCD parameters for the undamped structure are derived. The optimum peak amplitudes for the structure and the liquid are also obtained. Based on the developed analytical formulas, the practical solution procedures for finding the optimum parameters are proposed. The presented example indicates that the optimum TLCD parameters can be easily calculated from the developed formulas. With the help of this study, the understanding of TLCD behavior with respect to its optimum parameters is enhanced.



## 1. INTRODUCTION

Tuned liquid column-damper (TLCD), proposed by Sakai *et al.* [1], was developed mainly for suppressing horizontal motion of structures. TLCD has definite advantages over other types of damping devices, such as relatively simple mathematical modelling of the damper–structure interaction, easy tuning of frequency in practice, easy manufacturing and installation, low cost, and almost free maintenance, etc. Therefore, it is preferable device for vibration control of large structures.

In the past few years, the researches on TLCD are only concerned with its control performance and applications in the suppression of structural horizontal vibrations [2–10]. Recently, the possibility and effectiveness of using TLCD to reduce pitching vibration of structures were investigated by the authors [11, 12]. A mathematical model of TLCD–structure interaction under pitching motion was developed. This model was further verified by a series of free and forced vibration experiments. It was demonstrated that TLCD can effectively reduce structural pitching vibrations as well. Incorporating the developed theoretical model with Scanlan's wind force model for bridge decks, it has been proved that the TLCD can also be successfully used for wind-induced torsional vibration control of suspension bridge decks [13]. The studies indicated that the control effectiveness could be significantly influenced by the parameters of the tuned liquid column damper. Therefore, a necessary optimal parameter design should be required in the practical application of the damper.

In this paper, in order to enhance the understanding of the TLCD performance and its behavior in the mitigation of pitching vibration of structures, an optimal parametric study is carried out for the damper. The structure is assumed to be undamped and be subjected to harmonic excitation for facilitating the analysis.

The method used in this paper is similar to that described by Den Hartog [14] in determining the optimum vibration absorber parameters for an undamped single-degree-of-freedom main system, where closed-form expressions for the optimum damper parameters were derived with the assumption that no damping was presented in the main mass. Den Hartog's procedure has been employed by many other authors. Snowdon [15] extended it to include different types of absorber damping instead of the viscous damping considered by Den Hartog. Introducing the concept of effective mass and effective stiffness and representing the structural response by a single relevant mode, the optimum parameters for absorbers, which are attached to elastic bodies like beams, plates, and cylindrical shells, etc., were investigated by Jacquot [16], Warburton and Ayorinde [17], Ayorinde and Warburton [18]. The extension of Den Hartog's method to the main systems with two or multiple degrees of freedom was outlined by Warburton [19] with both narrow-band and broad-band optimizations. Warburton [20] also derived closed-form expressions for optimum absorber parameters for undamped main systems subjected to harmonic and white-noise random excitations.

However, the above investigations are only limited to the applications of determining the optimum parameters of vibration absorbers. No relevant research has been found on the determination of the optimum parameters of the tuned liquid column dampers with the use of Den Hartog's method. Therefore, an effort is made in this paper to investigate the optimum parameters of the tuned liquid column damper in suppressing pitching vibration of undamped structures. First, a TLCD-structure model subjected to pitching motion is presented. The TLCD-structure interactive equations under harmonic excitation are solved in a frequency domain. Then, assuming the structure to be undamped, the optimum tuning ratio and the optimum head loss coefficient of the tuned liquid column damper are studied with the help of Den Hartog's method. The analytical formulas of the optimum TLCD parameters are derived. The optimum peak amplitudes for the structure and the liquid are also obtained. Finally, based on the developed analytical formulas, the practical solution procedures for finding the optimum TLCD parameters are proposed, and an example is presented to illustrate the results.

## 2. TLCD-STRUCTURE MODEL

Figure 1 shows a model of a primary structure equipped with a tuned liquid column damper. The structure is modelled as a single-degree-of-freedom system subjected to pitching motion only. It has parameters of the mass moment of inertia $I_s$, damping coefficient $C_s$ and torsional stiffness $K_s$. The tuned liquid column damper is a U-shaped container filled with liquid. It is characterized by the density of liquid $\rho$, the cross-sectional area $A$, the total length of liquid $L$, and the horizontal width $B$. The orifice installed inside the column tube of TLCD is used to provide the resistance to the liquid motion, thus increasing the additional damping to the structure. $H$ is the distance from the centerline of the bottom tube of TLCD to the rotational axis of the structure.

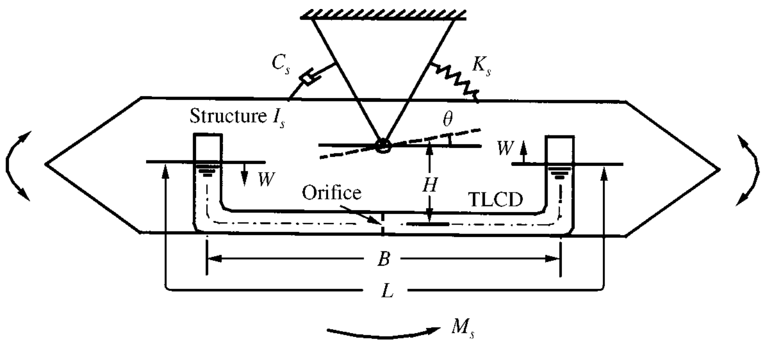


Figure 1. TLCD–structure interaction model.

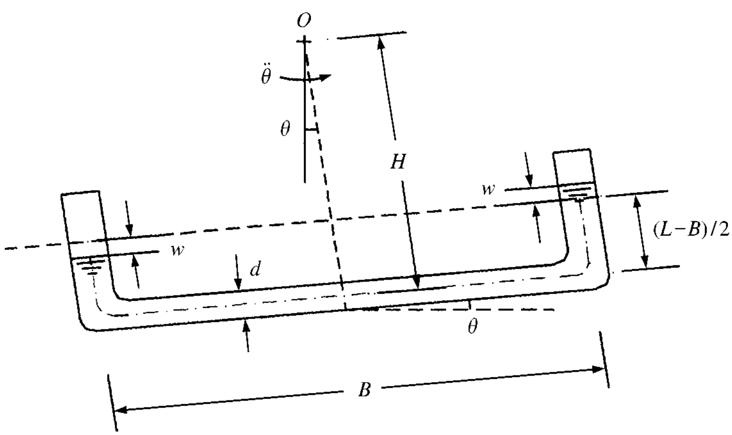


Figure 2. TLCD under pitching motion.

When an external moment $M_s$ is applied, the structure will vibrate around its rotational axis. The tuned liquid column damper will move together with the structure, resulting in the sloshing of the liquid. The liquid motion in the TLCD will produce a restoring force to counteract the external moment acting on the structure, while the orifice in the TLCD will induce damping force that dissipates energy. Therefore, the structure–damper system behaves interactively during the vibration. If the parameters of the TLCD are properly selected according to the design requirements, the vibrational response of the structure will be mitigated.

The motion of TLCD during pitching vibration is shown in Figure 2. The TLCD-structure interactive equations have been developed by the author as follows [13]:

$$\begin{bmatrix} I_s + I_d & G \\ G & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{W} \end{Bmatrix} \begin{bmatrix} C_s & 0 \\ 0 & (\rho A/2)\delta|\dot{W}| \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{W} \end{Bmatrix} + \begin{bmatrix} K_s + mgH & mg\alpha \\ mg\alpha & 2\rho Ag \end{bmatrix} \begin{Bmatrix} \theta \\ W \end{Bmatrix} = \begin{Bmatrix} M_s \\ 0 \end{Bmatrix}, \quad (1)$$

where $G = \rho AB(H + (L - B)/2)$; $m = \rho AL$ is the mass of the liquid; and $\alpha = B/L$ defines the ratio of liquid horizontal length to its total length.

$I_d$ represents the moment of inertia of the liquid column of TLCD around the rotational axis of the structure, which is expressed as

$$I_d = \rho AB(H^2 + B^2/12) + \rho A(L - B)[H^2 + B^2/4 - H(L - B)/2 + (L - B)^2/12] \quad (2)$$

and $M_s$ is the external moment acting on the structure; $W$ is the relative motion of liquid to the column tube of TLCD; $\theta$ is the rotational displacement of the structure; the dot over them denotes the time derivative; $g$ is the acceleration of gravity; and the constant $\delta$ is defined as the coefficient of head loss governed by the opening ratio of orifice. The natural circular frequency of the liquid motion is given by $\omega_d = \sqrt{2g/L}$, which depends only on the length of the oscillating liquid mass.

In order to keep equation (1) valid, the liquid motion in TLCD has to satisfy

$$W \leqslant \frac{L - B}{2} - \frac{d}{2}, \quad (3)$$

where $d$ is the thickness of the liquid column (inner dimension of the TLCD tube).

It is noticed that the combined system of the structure with TLCD is nonlinear due to the nonlinear damping property of the tuned liquid column damper. For a study in the frequency domain, the nonlinear equations can be linearized by some equivalent linearization techniques. Taking an equivalent linear damping force $C_{eq}\dot{W}$ to represent the non-linear damping force $\frac{1}{2}\rho A\delta|\dot{W}|\dot{W}$ and supposing the liquid motion in TLCD to be sinusoidal under harmonic excitation, the equivalent damping coefficient $C_{eq}$ and the equivalent damping ratio $\xi_d$ of the TLCD are found by a energy equivalent principle [13]

$$C_{eq} = \frac{4}{3\pi}\rho A\delta\bar{W}_0\omega, \quad (4)$$

$$\xi_d = \frac{\sqrt{2}\delta}{3\pi\sqrt{gL}}\bar{W}_0\omega, \quad (5)$$

where $\bar{W}_0$ is the amplitude of liquid motion under harmonic vibration and $\omega$ is the frequency of oscillation.

The linearized equations for TLCD–structure interaction can be written as

$$\begin{bmatrix} 1+\mu & G/I_s \\ G/m & 1 \end{bmatrix}\begin{Bmatrix} \ddot{\theta} \\ \ddot{W} \end{Bmatrix} + \begin{bmatrix} 2\xi_s\omega_s & 0 \\ 0 & 2\xi_d\omega_d \end{bmatrix}\begin{Bmatrix} \dot{\theta} \\ \dot{W} \end{Bmatrix} + \begin{bmatrix} \omega_s^2 + mgH/I_s & mg\alpha/I_s \\ \alpha g & \omega_d^2 \end{bmatrix}\begin{Bmatrix} \theta \\ W \end{Bmatrix} = \begin{Bmatrix} M_s/I_s \\ 0 \end{Bmatrix}, \quad (6)$$

where $\mu = I_d/I_s$ is the ratio of moment of inertia of TLCD to structure; $\omega_s = \sqrt{K_s/I_s}$ is the natural circular frequency of structure; and $\xi_s = C_s/2I_s\omega_s$ represents the damping ratio of structure.

Under harmonic excitations, the structure dynamic magnification factor (DMF), i.e., the ratio of dynamic response to static response and the liquid response amplitude $\bar{W}_0$ can be determined as

$$DMF = \sqrt{\frac{(\lambda^2 - \beta^2)^2 + 4\xi_d^2\lambda^2\beta^2}{E^2 + F^2}}, \quad (7)$$

$$\bar{W}_0 = \frac{|G\beta^2 - mg\alpha/\omega_s^2|}{m\sqrt{E^2 + F^2}}\left(\frac{M_0}{K_s}\right), \quad (8)$$

where

$$E = [1 + mgH/K_s - (1+\mu)\beta^2](\lambda^2 - \beta^2) - 4\xi_s\xi_d\lambda\beta^2 - \frac{(G\beta^2 - mg\alpha/\omega_s^2)^2}{mI_s}, \qquad (9)$$

$$F = 2\xi_s\beta(\lambda^2 - \beta^2) + 2\xi_d\lambda\beta[1 + mgH/K_s - (1+\mu)\beta^2], \qquad (10)$$

and $\lambda = \omega_d/\omega_s$ represents the tuning ratio of TLCD frequency to that of structure; $\beta = \omega/\omega_s$ stands for the ratio of excitation frequency to that of structure; $M_0$ is the excitation force amplitude.

It should be noted that the value of equivalent damping ratio $\xi_d$ of the TLCD depends on the amplitude of liquid motion $\bar{W}_0$, while $\bar{W}_0$ is further a function of $\omega$, $M_0$ and $\xi_d$. Therefore, the equivalent damping ratio $\xi_d$ has to be determined by iteration from equations (5) and (8) for each specific excitation frequency $\omega$ and excitation force amplitude $M_0$. For different amplitudes of excitations, the equivalent damping ratio has different values. Therefore, it varies over the excitation frequency ranges.

## 3. OPTIMUM TUNING RATIO

It has been shown that for tuned liquid column dampers, the most important parameters affecting the control performance are the tuning frequency ratio, which controls the total length of liquid of TLCD, and the head loss coefficient, which dominates the damping values induced by TLCD.

When structural damping ratio $\xi_s = 0$, the dynamic magnification factor (DMF) of the structure can be reduced to the following form:

$$DMF = \sqrt{\frac{A_1 + B_1\xi_d^2}{C_1 + D_1\xi_d^2}}, \qquad (11)$$

where

$$A_1 = (\lambda^2 - \beta^2)^2, \qquad (12a)$$

$$B_1 = 4\lambda^2\beta^2, \qquad (12b)$$

$$C_1 = \left\{[1 + mgH/K_s - (1+\mu)\beta^2](\lambda^2 - \beta^2) - \frac{(G\beta^2 - mg\alpha/\omega_s^2)^2}{mI_s}\right\}^2, \qquad (12c)$$

$$D_1 = 4\lambda^2\beta^2[1 + mgH/K_s - (1+\mu)\beta^2]^2 \qquad (12d)$$

$\xi_d$ is the equivalent damping ratio of TLCD. The magnitude of $\xi_d$ is determined by the value of head loss coefficient $\delta$ of the TLCD.

Figure 3 shows some typical plots of frequency response curves under different values of head loss coefficients for an undamped structure. Two extreme conditions for head loss coefficients $\delta = 0$ and $\delta = \infty$ are plotted together with those of $\delta = 8{\cdot}0$ and $20{\cdot}0$. It is observed that with $\delta = 0$, the peak response is infinite, and with $\delta = \infty$ the peak response is again infinite. The condition for $\delta = 0$ and $\infty$ can be easily examined from equation (11). With $\delta = 0$ and $\infty$, we have $\xi_d = 0$ and $\infty$. Equation (11) is thus reduced to $DMF = \sqrt{A_1/C_1}$ and $DMF = \sqrt{B_1/D_1}$ respectively, from which the condition for infinite peak response can be easily determined. Since the objective of installing the damper to the structure is to bring the resonant peak down to its lowest possible value, somewhere

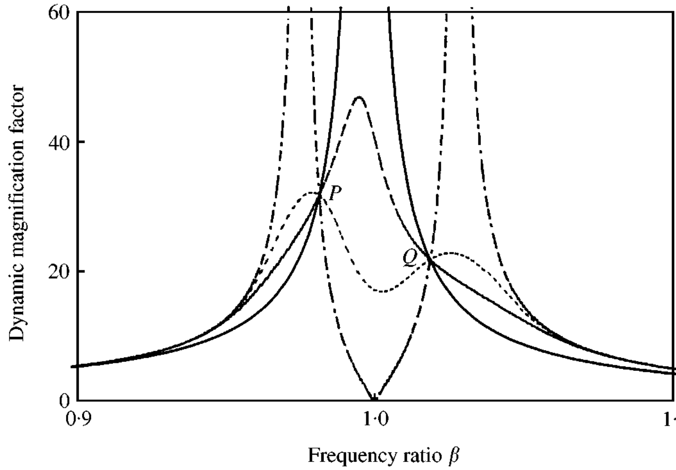


Figure 3. Frequency response curves of undamped structure with different head loss coefficients. - - - -, $\delta = 0{\cdot}0$; - - - - - -, $\delta = 8{\cdot}0$; – – –, $\delta = 20{\cdot}0$; ——, $\delta = \infty$.

between $\delta = 0$ and $\delta = \infty$ there must be a value of $\delta$ for which the peak becomes a minimum

As indicated by Den Hartog [14], Figure 3 also reveals a fact that all four curves of different $\delta$ intersect at two fixed points, P and Q. This is not accidental. Studies for other cases also indicate that all curves pass through these two points independent of the coefficients of head loss, or the equivalent damping of the tuned liquid column damper. Obviously, the most favorable curve under a given $\delta$ is the one which passes with a horizontal tangent through the highest of the two fixed points P and Q. The ordinate at that point gives the best obtainable amplitude of the response peak, and the corresponding head loss coefficient is the optimum head loss coefficient for the given system. In addition, by changing the tuning frequency ratio $\lambda$, the two fixed points P and Q can be shifted up and down, one point going up and the other going down. Clearly, the most favorable case is such that first by a proper choice of the tuning ratio $\lambda$, the two fixed points are adjusted to equal heights, and second by a proper choice of the head loss coefficient $\delta$, the curve is adjusted to pass with a horizontal tangent through one of them.

First, let us find the optimum tuning ratio. The relation that is independent of the equivalent damping ratio can be found from equation (11) if $A_1/C_1 = B_1/D_1$, or written out fully

$$\frac{\lambda^2 - \beta^2}{[1 + \mathrm{mg}H/K_s - (1+\mu)\beta^2](\lambda^2 - \beta^2) - (G\beta^2 - mg\alpha/\omega_s^2)^2/mI_s} = \pm \frac{1}{1 + mgH/K_s - (1+\mu)\beta^2}. \tag{13}$$

With plus sign, we have

$$\frac{(G\beta^2 - mg\alpha/\omega_s^2)^2}{mI_s} = 0 \quad \text{or} \quad G\beta^2 - mg\alpha/\omega_s^2 = 0. \tag{14}$$

This gives a zero amplitude of liquid motion, i.e., $\bar{W} = 0$ (see equation (8)), thus giving a zero equivalent damping, $\xi_d = 0$ (see equation (5)). Therefore, it is a trivial result which we are not interested in.

With the minus sign, equation (13) becomes

$$2(\lambda^2 - \beta^2)[1 + mgH/K_s - (1 + \mu)\beta^2] - \frac{(G\beta^2 - mg\alpha/\omega_s^2)^2}{mI_s} = 0. \tag{15}$$

Expanding equation (15) leads to a quadratic equation in $\beta^2$

$$\left[2(1 + \mu) - \frac{G^2}{mI_s}\right]\beta^4 - 2[1 + (1 + \mu)\lambda^2 + (mgH - Gg\alpha)/K_s]\beta^2$$

$$+ 2\lambda^2(1 + mgH/K_s) - \frac{mg^2\alpha^2}{K_2\omega_s^2} = 0 \tag{16}$$

which has two positive roots of $\beta_1$ and $\beta_2$, representing the horizontal co-ordinates of the fixed points P and Q in Figure 3, respectively. From equation (16), we have the following relation:

$$\beta_1^2 + \beta_2^2 = \frac{2[1 + (1 + \mu)\lambda^2 + (mgH - Gg\alpha)/K_s]}{2(1 + \mu) - (G^2/mI_s)}. \tag{17}$$

It has been mentioned that at the two fixed points the value of the dynamic magnification factor (DMF) is independent of the damping $\xi_d$. This happens for $\xi_d = \infty$, whereby equation (11) becomes

$$DMF = \sqrt{\frac{B_1}{D_1}} = \frac{1}{|1 + mgH/K_s - (1 + \mu)\beta^2|}, \tag{18}$$

which is infinite at $\beta = \sqrt{(1 + mgH/K_s)/(1 + \mu)}$. Adjusting the two fixed points to equal amplitudes gives

$$\frac{1}{1 + mgH/K_s - (1 + \mu)\beta_1^2} = -\frac{1}{1 + mgH/K_s - (1 + \mu)\beta_2^2}, \tag{19}$$

where a minus sign is added because the horizontal co-ordinates of the two fixed points $\beta_1$ and $\beta_2$ are on different sides of the point of infinity, $\beta = \sqrt{(1 + mgH/K_s)/(1 + \mu)}$.

From equation (19), we have

$$\beta_1^2 + \beta_2^2 = \frac{2(1 + mgH/K_s)}{1 + \mu}. \tag{20}$$

Combining equations (17) and (20) leads to

$$\lambda^2(1 + \mu)^2 = (1 + \mu)[1 + (mgH + Gg\alpha)/K_s] - (1 + mgH/K_s)\,\frac{G^2}{mI_s}. \tag{21}$$

If other parameters are known, the optimum tuning ratio $\lambda_{opt}$ can be easily calculated from equation (21). However, it should be noted that the parameters $\lambda$, $m$, and $G$ are all function of the liquid length $L$. Therefore, we have to determine the optimum liquid length $L_{opt}$ first, then the other optimum parameters can be calculated from $L_{opt}$.

The parameter $G$ can be expressed in terms of $L$ as

$$G = \rho AL\alpha\left(H + \frac{1-\alpha}{2}L\right) = m[H\alpha + \tfrac{1}{2}\alpha(1-\alpha)L], \tag{22}$$

therefore we have

$$\frac{G}{m} = H\alpha + \frac{1}{2}\alpha(1-\alpha)L, \tag{23}$$

where $\alpha = B/L$ is the ratio of liquid horizontal length to its total length; $m = \rho AL$ is the mass of liquid column.

Similarly, the mass moment of inertia of liquid column can be expressed in terms of $L$ as

$$I_d = \mu I_s = \rho AL\alpha\left(H^2 + \frac{\alpha^2}{12}L^2\right) + \rho AL(1-\alpha)\left[H^2 + \frac{\alpha^2}{4}L^2 - H\frac{1-\alpha}{2}L + \frac{(1-\alpha)^2}{12}L^2\right]$$

$$= \frac{m}{12}[(-3\alpha^3 + 6\alpha^2 - 3\alpha + 1)L^2 - 6H(1-\alpha)^2 L + 12H^2], \tag{24}$$

thus we have

$$m = \frac{12\mu I_s}{c_{11}L^2 - c_{12}L + 12H^2}, \tag{25}$$

where

$$c_{11} = -3\alpha^3 + 6\alpha^2 - 3\alpha + 1, \tag{26a}$$

$$c_{12} = 6H(1-\alpha)^2. \tag{26b}$$

Noticing the following relation

$$\lambda^2 = \omega_d^2/\omega_s^2; \quad \omega_d^2 = 2g/L; \quad \omega_s^2 = K_s/I_s. \tag{27}$$

Equation (21) can be rewritten as

$$\left[\frac{2g(1+\mu)^2}{L} - (1+\mu)\omega_s^2\right]\frac{I_s}{m} - (1+\mu)\left[gH + g\alpha\left(\frac{G}{m}\right)\right] + [\omega_s^2 + mgH/I_s]\left(\frac{G}{m}\right)^2 = 0. \tag{28}$$

Substituting equations (23) and (25) into equation (28) gives

$$\left(\frac{A_{11}}{L} - A_{12}\right)(c_{11}L^2 - c_{12}L + 12H^2) + \left(\omega_s^2 + \frac{A_{13}}{c_{11}L^2 - c_{12}L + 12H^2}\right)$$

$$\times (H\alpha + A_{14}L)^2 - A_{15}L - A_{16} = 0, \tag{29}$$

where

$$A_{11} = \frac{g(1+\mu)^2}{6\mu}, \tag{30a}$$

$$A_{12} = \frac{(1+\mu)\omega_s^2}{12\mu}, \tag{30b}$$

$$A_{13} = 12\mu g H, \tag{30c}$$

$$A_{14} = \tfrac{1}{2}\alpha(1-\alpha), \tag{30d}$$

$$A_{15} = \frac{g\alpha^2(1+\mu)(1-\alpha)}{2}, \tag{30e}$$

$$A_{16} = gH(1+\mu)(1+\alpha^2). \tag{30f}$$

Equation (29) gives an implicit formula for the optimum liquid length. Solving this equation by iterative procedure, the optimum liquid length $L_{opt}$ can be determined. Then, the optimum tuning ratio is given by

$$\lambda_{opt} = \omega_{dopt}/\omega_s, \tag{31}$$

where $\omega_{dopt} = \sqrt{2g/L_{opt}}$ is the optimum natural frequency of the tuned liquid column damper.

The optimum liquid mass $m_{opt}$ can be calculated from equation (25) in terms of $L_{opt}$. The optimum cross-sectional area of TLCD is given by $A_{opt} = m_{opt}/\rho L_{opt}$, and the optimum parameter $G_{opt}$ can be determined from equation (22).

## 4. OPTIMUM HEAD LOSS COEFFICIENT

The optimum tuning ratio gives the result that makes the heights of the two fixed points equal. Substituting the optimum parameters to equation (16), the two horizontal co-ordinates, $\beta_1$ and $\beta_2$, of the fixed points P and Q at the optimum tuning are solves as

$$\beta_{1,2}^2 = \frac{1}{2B_{11}}\left[-B_{12} \mp \sqrt{B_{12}^2 - 4B_{11}B_{13}}\right], \tag{32}$$

where

$$B_{11} = 2(1+\mu) - \frac{G_{opt}^2}{m_{opt}I_s}, \tag{33a}$$

$$B_{12} = -2[1 + (1+\mu)\lambda_{opt}^2 + (m_{opt}gH - G_{opt}g\alpha)/K_s], \tag{33b}$$

$$B_{13} = 2\lambda_{opt}^2(1 + m_{opt}gH/K_s) - \frac{m_{opt}g^2\alpha^2}{K_s\omega_s^2}. \tag{33c}$$

The optimum equivalent damping ratio $\xi_{dopt}$ of TLCD at optimum tuning can be found by setting the slope of the curve of equation (11) equal to zero at the fixed co-ordinates $\beta_1$ and $\beta_2$, i.e.

$$\left.\frac{\mathrm{d}(DMF)}{\mathrm{d}\beta}\right|_{\beta=\beta_1,\beta_2} = 0. \tag{34}$$

Expressing equation (11) in the following form

$$DMF = \sqrt{\frac{X(\beta)}{Y(\beta)}} \tag{35}$$

then the condition that satisfies $\mathrm{d}(DMF)/\mathrm{d}\beta = 0$ is given by

$$Y(\beta)\frac{\mathrm{d}X(\beta)}{\mathrm{d}\beta} - X(\beta)\frac{\mathrm{d}Y(\beta)}{\mathrm{d}\beta} = 0, \tag{36}$$

where

$$X(\beta) = (\lambda_{opt}^2 - \beta^2)^2 + 4\xi_d^2\lambda_{opt}^2\beta^2, \tag{37}$$

$$Y(\beta) = E(\beta)^2 + F(\beta)^2, \tag{38}$$

$$E(\beta) = [1 + m_{opt}gH/K_s - (1+\mu)\beta^2](\lambda_{opt}^2 - \beta^2) - \frac{(G_{opt}\beta^2 - m_{opt}g\alpha/\omega_s^2)^2}{m_{opt}I_s}, \tag{39}$$

$$F(\beta) = 2\xi_d\lambda_{opt}\beta[1 + m_{opt}gH/K_s - (1+\mu)\beta^2]. \tag{40}$$

Differentiating $X(\beta)$ and $Y(\beta)$ with respect to $\beta$ gives

$$\frac{\mathrm{d}X(\beta)}{\mathrm{d}\beta} = -4\beta(\lambda_{opt}^2 - \beta^2) + 8\beta\lambda_{opt}^2\xi_d^2, \tag{41}$$

$$\frac{\mathrm{d}Y(\beta)}{\mathrm{d}\beta} = 2E(\beta)\frac{\mathrm{d}E(\beta)}{\mathrm{d}\beta} + 2F(\beta)\frac{\mathrm{d}F(\beta)}{\mathrm{d}\beta}$$

$$= 2E(\beta)E_{11}\beta + 2F_{11}(\beta)F_{12}(\beta)\xi_d^2, \tag{42}$$

where

$$E_{11}(\beta) = 4\left(1 + \mu - \frac{G_{opt}^2}{m_{opt}I_s}\right)\beta^3 - 2[1 + (1+\mu)\lambda_{opt}^2 + (m_{opt}gH - 2G_{opt}g\alpha)/K_s]\beta, \tag{43}$$

$$F_{11}(\beta) = 2\lambda_{opt}\beta[1 + m_{opt}gH/K_s - (1+\mu)\beta^2], \tag{44}$$

$$F_{12}(\beta) = -6\lambda_{opt}(1+\mu)\beta^2 + 2\lambda_{opt}(1 + m_{opt}gH/K_s). \tag{45}$$

Substituting the above results into equation (36) leads to a quadratic equation in $\xi_d^2$,

$$D_{11}(\beta)\xi_d^4 + D_{12}(\beta)\xi_d^2 + D_{13}(\beta) = 0, \tag{46}$$

where

$$D_{11}(\beta) = 32\lambda_{opt}^3 \beta^4 (1+\mu) F_{11}(\beta), \tag{47}$$

$$D_{12}(\beta) = 8E(\beta)\lambda_{opt}^2 \beta [E(\beta) - \beta E_{11}(\beta)] - 2F_{11}(\beta)(\lambda_{opt}^2 - \beta^2) \times [2\beta F_{11}(\beta) + (\lambda_{opt}^2 - \beta^2) F_{12}(\beta)], \tag{48}$$

$$D_{13}(\beta) = -2E(\beta)(\lambda_{opt}^2 - \beta^2)[2\beta E(\beta) + (\lambda_{opt}^2 - \beta^2) E_{11}(\beta)]. \tag{49}$$

Solving equation (46) corresponding to the horizontal co-ordinate $\beta_1$ at the fixed point P gives the equivalent damping ratio $\xi_{d1}$ that makes the curve pass horizontally through the point P (but not horizontally at point Q):

$$\xi_{d1} = \sqrt{\frac{1}{2D_{11}(\beta_1)}\left[-D_{12}(\beta_1) + \sqrt{D_{12}(\beta_1)^2 - 4D_{11}(\beta_1)D_{13}(\beta_1)}\right]}. \tag{50}$$

Similarly, solving equation (46) corresponding to the horizontal co-ordinate $\beta_2$ at the fixed point Q gives the equivalent damping into $\xi_{d2}$ that makes the curve pass horizontally through the point Q (but not horizontally at point P):

$$\xi_{d2} = \sqrt{\frac{1}{2D_{11}(\beta_2)}\left[-D_{12}(\beta_2) - \sqrt{D_{12}(\beta_2)^2 - 4D_{11}(\beta_2)D_{13}(\beta_2)}\right]}. \tag{51}$$

Then, the optimum head loss coefficients, $\delta_1$ and $\delta_2$, corresponding to $\xi_{d1}$ and $\xi_{d2}$ respectively, can be obtained from equations (5) and (8) as

$$\delta_1 = \frac{3\pi m_{opt}\sqrt{gL_{opt}}}{\sqrt{2}} \frac{\sqrt{E(\beta_1)^2 + F_{11}(\beta_1)^2 \xi_{d1}^2}}{|G_{opt}\beta_1^2 - m_{opt} g\alpha/\omega_s^2|} \left(\frac{K_s}{M_0}\right) \frac{1}{\beta_1 \omega_s} \xi_{d1} \tag{52}$$

$$\delta_2 = \frac{3\pi m_{opt}\sqrt{gL_{opt}}}{\sqrt{2}} \frac{\sqrt{E(\beta_2)^2 + F_{11}(\beta_2)^2 \xi_{d2}^2}}{|G_{opt}\beta_2^2 - m_{opt} g\alpha/\omega_s^2|} \left(\frac{K_s}{M_0}\right) \frac{1}{\beta_2 \omega_s} \xi_{d2}. \tag{53}$$

It is noticed that $\delta_1$ and $\delta_2$ define the optimum conditions for points P and Q separately. In practical design, the optimum head loss coefficient $\delta_{opt}$ for the system can be evaluated by the average value between $\delta_1$ and $\delta_2$:

$$\delta_{opt} = (\delta_1 + \delta_2)/2. \tag{54}$$

## 5. OPTIMAL PEAK AMPLITUDES

At the optimal condition, the two fixed points P and Q have equal heights, and the response curve passes horizontally through either point P or point Q. Therefore, the optimal peak amplitude of the structure can be taken as the height of either point.

Since the amplitude at points P or Q is independent of the damping (or the head loss coefficient) of the tuned liquid column damper, it can be taken as the simple form as shown in equation (18). Substituting one of the roots, $\beta_1$ and $\beta_2$, into equation (18) gives the peak amplitude at the optimum condition

$$DMF_{opt} = \frac{1}{|1 + m_{opt} gH/K_s - (1+\mu)\beta_i^2|} \quad (i = 1, 2). \tag{55}$$

The corresponding amplitude of liquid motion at the optimum condition can be found by substituting the optimum parameters to equation (8). The amplitudes of liquid motion for points P and Q have also equal magnitudes if the related optimum parameters for the two points are used correspondingly,

$$\bar{W}_{opt} = \frac{|G_{opt}\beta_i^2 - m_{opt} g\alpha/\omega_s^2|}{m_{opt}\sqrt{E(\beta_i)^2 + F_{11}(\beta_i)^2\xi_{di}^2}}\left(\frac{M_0}{K_s}\right) \quad (i = 1, 2). \tag{56}$$

It should be noted that equations (55) and (56) are based on an assumption that the head loss coefficients (or equivalent damping ratio) of TLCD take different optimum values for each point. However, for a practical structure system, the tuned liquid column damper is designed with a specific orifice opening, which gives a fixed value of head loss coefficient. If the average value given by equation (54) is used for the optimum head loss coefficient of the damper, the response curve will not pass points P or Q horizontally, therefore, the amplitudes given by equations (55) and (56) will not be the maximum amplitudes for the structure and the liquid. In this case, the optimal peak amplitudes can be simply found from numerical results of the frequency responses using the suggested optimum TLCD parameters. In general, equations (55) and (56) can give a good approximation for the true peak amplitudes of the structure and the liquid.

As indicated by equation (3), in the practical design of the tuned liquid column damper, the predicted maximum liquid amplitude $\bar{W}_{max}$ has to satisfy

$$\bar{W}_{max} \leqslant [(1-\alpha)L - d]/2, \tag{57}$$

so that the liquid motion keeps valid during the vibration.

## 6. SOLUTION PROCEDURE AND EXAMPLE

Based on the developed theoretical formulas, the optimum TLCD parameters for an undamped structure can be determined by the procedures suggested as follows:

(1) Input basic parameters: $\rho$, $g$, $\mu$, $\alpha$, $H$, $\omega_s$, $I_s$, $K_s$, $M_0$.
(2) Determine optimum liquid length $L_{opt}$ from equation (29).
(3) Calculate optimum tuning ratio $\lambda_{opt}$ and other optimum parameters: $m_{opt}$, $G_{opt}$, and $A_{opt}$.
(4) Calculate the horizontal co-ordinates $\beta_1$ and $\beta_2$ of the two fixed points P and Q from equation (32).
(5) Calculate optimum equivalent damping ratio $\xi_{d1}$ and $\xi_{d2}$ from equations (50) and (51).
(6) Determine the optimum head loss coefficient $\delta_1$, $\delta_2$ and $\delta_{opt}$ from equations (52)–(54).
(7) Calculate the optimal amplitudes of structure and liquid from equations (55) and (56).
(8) Check equation (57) to ensure it is not violated.

According to above solution procedures, a computer program has been developed to assist the analysis. In order to demonstrate the method, we present a numerical example taken from a practical suspension bridge deck. Figure 4 shows a typical deck section equipped with a tuned liquid column damper. The TLCD is used to control the torsional vibration of the bridge deck. The cross-sectional properties of the deck section is characterized by the total deck width $b = 30$ m; the inner deck width at the bottom $b' = 20$ m; and the deck height $h = 3{\cdot}5$ m. The mass moment of inertia of the deck section per unit span is $I_s = 4{\cdot}34 \times 10^6$ kg m$^2$/m, and the torsional stiffness of the deck section per unit span is $K_s = 3{\cdot}86 \times 10^6$ N m/m/rad. The damping ratio of the deck is very low, and is

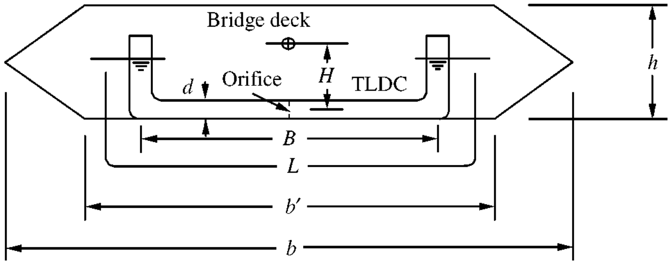


Figure 4. Geometric property of Bridge Deck with TLCD.

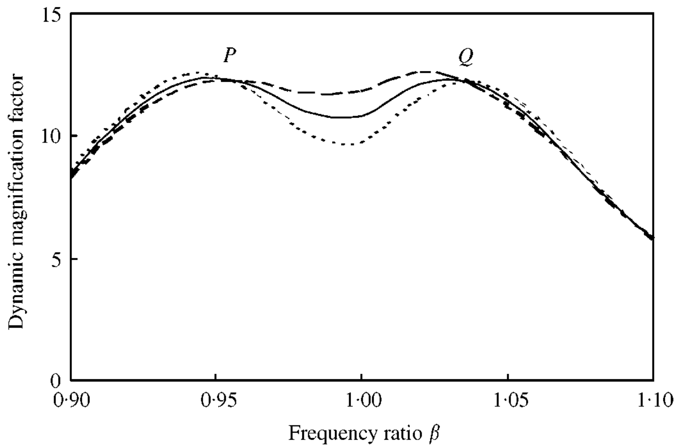


Figure 5. Frequency response curve of undamped structure under optimum condition. - - - -, horizontally pass P; - - - - - -, horizontally pass Q; ——, average curve.

therefore neglected in the analysis. The natural frequency of the deck torsional motion is found by $f_s = 0{\cdot}15$ Hz ($\omega_s = 0{\cdot}943$ rad/s). The TLCD is filled with water. Design the TLCD with following parameters: $\rho = 1000\,\mathrm{kg/m^3}$, $g = 9{\cdot}8\,\mathrm{m/s^2}$, $\mu = 1\%$, $\alpha = 0{\cdot}85$, $H = 1{\cdot}0$ m, and $M_0 = 4{\cdot}0 \times 10^3$ N m/m.

Using the described method, the optimum liquid length is determined as $L_{opt} = 22{\cdot}13$ m, which gives an optimum tuning ratio of $\lambda_{opt} = 99{\cdot}85\%$. The optimum cross-sectional area of the TLCD is found by $A_{opt} = 0{\cdot}05\,\mathrm{m^2/m}$. The two horizontal coordinates for points P and Q are found, respectively, $\beta_1 = 0{\cdot}9554$ and $\beta_2 = 1{\cdot}0359$, and the same amplitude of 12·35 is found for the two points. The optimum equivalent damping ratios of the TLCD for the two points are calculated as $\xi_{d1} = 7{\cdot}39\%$ and $\xi_{d2} = 6{\cdot}59\%$. The corresponding optimum head loss coefficients are found to be $\delta_1 = 14{\cdot}121$ and $\delta_2 = 11{\cdot}611$ respectively, which gives an average head loss coefficient of 12·866. Using the individual optimum parameters for points P and Q, respectively, a same liquid amplitude of 0·571 m is found for the two points.

Figure 5 shows the structural response curves determined from the optimum TLCD parameters, where three curves are drawn, which are respectively, the curve passing horizontally through point P; the curve passing horizontally through point Q; and the curve using the average optimum head loss coefficient given by equation (54). The corresponding liquid response for the optimum TLCD parameters are also illustrated in Figure 6. It is seen that the average curve gives a good approximation for the two optimum curves, either for the response of structure or for the response of liquid.

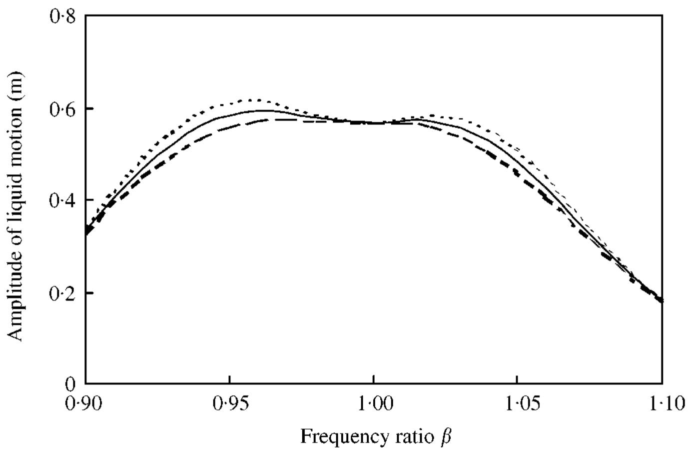


Figure 6. Frequency response curve of liquid under optimum condition. – – – –, $\delta_1 = 14{\cdot}121$; - - - - -, $\delta_2 = 11{\cdot}611$; ——, $\delta = 12{\cdot}866$.

In the condition of using the average head loss coefficient, the true peak amplitudes of the response curves are found respectively, to be 12·42 for the structure, and 0·596 m for the liquid. It is seen that for both the structure and the liquid, the differences between the peak amplitudes using the average head loss coefficient and the amplitudes at points P and Q are very small. Therefore, the amplitude at points P and Q can give a good evaluation of the true peak amplitude. Using the optimum parameters obtained, the horizontal width of the TLCD is determined as $B = \alpha L_{opt} = 18{\cdot}81$ m. Taking the thickness of the TLCD tube to be $d = 0{\cdot}6$ m, it is seen that the liquid vibration is within the valid range. It is also evident that the designed TLCD can be suitably accommodated within bridge decks.

## 7. CONCLUSIONS

The main objective of this paper is to develop some analytical formulas for determining the optimum parameters of the tuned liquid column damper in suppressing pitching vibration of undamped structures. Using Den Hartog's method, the optimum tuning ratio and the optimum head loss coefficient of the damper were investigated. The analytical formulas of the optimum TLCD parameters for the undamped structure were derived. The optimum peak amplitudes for the structure and the liquid were also obtained. Based on the developed analytical formulas, the practical solution procedures for finding the optimum parameters were proposed. The presented example indicated that the optimum TLCD parameters can be easily calculated from the developed formulas. With the help of this study, the understanding of TLCD behavior with respect to its optimum parameters could be enhanced. It is expected that the analytical formulas developed in this paper will give a good estimation of the optimum TLCD parameters to low damped structures.

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