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A RECIPROCITY RELATION FOR FLUID-LOADED ELASTIC PLATES THAT CONTAIN RIGID DEFECTS

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A reciprocity relation between the farfield behaviour of the scattered fields generated by incident waves, either flexural plate waves, or incident from the fluid, upon rigid defects embedded in a thin elastic, fluid-loaded, plate is derived. This reciprocity result is then illustrated upon model problems for which the explicit solution can be determined and the relation demonstrated.

1. INTRODUCTION

The diffraction of acoustic, flexural or leaky waves from inhomogeneities embedded in elastic plates or shells is important in any description of scattering by a fluid-loaded structure. The waves scattered from these defects generate sound in the fluid, and scattered flexural plate or leaky waves are also responsible for further sound generation via interaction with other material inhomogeneities. Numerical and analytical studies of these problems are often complicated by geometrical considerations and edge conditions that are required at sharp structural changes. Our aim here is to derive, and apply, a reciprocity relation that should be a useful tool for both checking results, and for reducing computational effort in parametric studies.

Reciprocity theorems have a long history in acoustics, electromagnetism and elasticity notably initiated by Helmholtz and Rayleigh amongst others. Many of these reciprocity theorems involve two scattering problems found by interchanging the position of a source and receiver; thus relations between the two states are deduced, and these are particularly useful in structural acoustics, say, scattering from an elastic sphere or cylinder [1, pp. 376].

A closely related reciprocity result is often used in acoustics, and in a more complicated guise in elasticity. In the latter case, several different body waves (both shear and compression), surface waves and mode conversion at interfaces often lead to complicated analysis; it is well-worth having subsidiary results to act as check. If one is interested in scattering by an obstacle, of arbitrary shape or cross-section, say, a crack or void, then

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reciprocity relations have been deduced for obstacles in an infinite isotropic elastic domain [2, 3], an elastic half-space [4–6] or coupled fluid-solid media [7]. Typically, these reciprocity theorems are concerned with relating one aspect of the solution of one problem to another aspect of a second problem. It is usual to explore how one relates the scattered farfield angular coefficient (the directivity) associated with a scattered cylindrical wave, or the amplitude of a scattered surface wave, that is generated by one type of incident wave (the first problem) to the scattered farfield coefficient generated by another incident wave (the second problem); the incident wave in isotropic elasticity could be either plane compressional or shear body waves, surface Rayleigh waves or interfacial Stoneley waves.

A typical relation would emerge from analyzing the scattered shear directivity, say, generated by an incident plane compressional wave upon a defect to the scattered compressional directivity generated by an incident plane shear wave. If we have a half-space or joined elastic media then other relations occur with, and between, the other waves of interest, that is, surface or interfacial waves. One can construct several different inter-relations each of which forms a useful non-trivial check upon any numerical or analytical work. The theorems are usually quite general and hold for obstacles of arbitrary number, orientation and shape, provided they are compact, that is, in so far as the farfield is concerned they are all clustered near to the origin. These results are particularly useful for checking numerical or analytical results that involve complicated subsidiary calculations, say, the evaluation of Green's functions and solution of coupled integral equations for scattering by sub-surface cracks [7]. In structural acoustics there are several numerical techniques available, some based upon solving integral equations [8, 9] and others based upon coupled boundary and finite elements (for instance, see reference [10]) and that could be used to tackle problems for which the reciprocity relations provide a checking mechanism. As the reciprocity formula arise from finding equivalences between two different scattering problems, this can also substantially reduce the number of calculations in a parametric study.

In a similar vein to the elastodynamic studies we now consider a compressible fluid overlying a thin elastic plate; the plate contains embedded obstacles, cracks, or other scatterers. A practical example might be a fluid-loaded elastic plate containing a finite array of parallel reinforcing ribs. This is in many ways analogous to elastic half-space problems in that we now have both a compressional fluid wave (a body wave) and a flexural plate wave (a surface wave). This elastic plates coupled to an overlying compressible fluid support a subsonic flexural wave, and many studies (for instance, see references [11–14]) in structural acoustics are concerned with the mechanisms whereby model defects scatter these waves; a substantial proportion of vibrational energy in a structure is transmitted into a fluid via such interactions [5]. Our aim is to deduce the relation that exists between the scattered farfield directivity associated with the scattered cylindrical wave in the fluid due to a flexural wave obliquely incident (in the plane occupied by the plate) upon this collection of defects to the amplitude of a scattered flexural wave created by an incoming fluid compressional plane wave also incident upon those defects. To demonstrate the manner in which the relation should be applied we briefly consider two model geometries for which analytical solutions can be derived and the reciprocity relation verified.

The present analysis is designed to complement the so-called "optical theorems". These arise from power balance considerations and are also useful in scattering problems. Recent work along these lines in structural acoustics and fluid–solid coupled media are contained in references [7, 16, 17].

2. BASIC EQUATIONS

We consider a single elastic plate, with one-sided fluid loading, containing embedded rigid strips, or line defects and joints; a typical geometrical configuration is shown in Figure 1.

Time-harmonic vibrations of frequency ω are assumed, thus all physical variables have an $e^{-i\omega t}$ dependence; this is considered understood, and is henceforth suppressed. The problem is three dimensional with an inviscid, compressible fluid lying in the half-space $z > 0$ and $-\infty < x, y < \infty$. With this assumed time dependence the fluid pressure $\hat{p}(x, y, z)$ satisfies the Helmholtz equation in $z > 0$,

$$(\nabla^2 + k_0^2)\hat{p}(x, y, z) = 0, \tag{2.1}$$

and k_0 the acoustic wavenumber, is related to the sound speed of the fluid, c_0, via $k_0 = \omega/c_0$. The displacement in the z direction on the plate, $\hat{\eta}(x, y)$, is related to the fluid pressure via $\rho\omega^2\hat{\eta}(x, y) = \hat{p}_z(x, y, 0)$.

The elastic plate lies in the plane $z = 0$ and is potentially separated by a number, j, of embedded rigid defects; these defects occupy domains $\mathscr{D}_j$ where $a_j \leqslant x \leqslant b_j$. To model the elastic plate we adopt the classical thin plate equation [1] in the form

$$B\nabla_h^4\hat{\eta}(x, y) - m\omega^2\hat{\eta}(x, y) = -\hat{p}(x, y, 0), \tag{2.2}$$

where ∇_h^2 is the horizontal Laplacian,

$$\nabla_h^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2.3}$$

The plate separates fluid in the region $z > 0$ from a vacuum in $z < 0$. The bending stiffness, B, and mass per unit area, m, of the plate are related to the physical properties of the elastic plate through $B = Eh^3/12(1 - \nu^2)$ and $m = \rho h$, with E, h, ν, and ρ the Young's modulus, plate thickness, the Poisson ratio and mass density of the elastic material respectively. In order to minimize the number of parameters that occur later, we introduce the *in vacuo* flexural wavenumber κ_p, "Mach" number Ω and fluid loading parameter ε as

$$\kappa_p \equiv \frac{\omega^2 m}{B}, \qquad \varepsilon = \frac{\rho}{m}\left(\frac{B}{mc_0^2}\right)^{1/2} \quad \text{and} \quad \Omega = \left(\frac{k_0}{\kappa_p}\right)^2 = \frac{\omega}{c_0\rho}m\varepsilon. \tag{2.4}$$

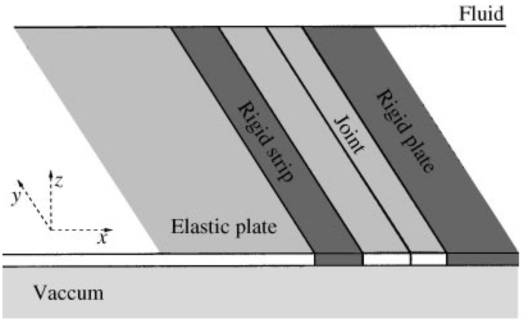

Figure 1. The geometry of a general problem showing typical rigid defects, involving rigid strips and plates, and line joints.

Here Ω, the square of the ratio of the *in vacuo* plate wave speed to that of the fluid, provides a dimensionless frequency and the fluid loading parameter, ε, provides a frequency independent measure of fluid loading.

At plate edges, joints, or defects various edge conditions can be adopted (as the displacement is directly related to p_z we give the conditions in terms of the latter quantity); we take $x = 0$ to be the edge of a rigid plate extending along $0 < x < \infty$, say, and then $x = 0^-$ is the line along which the edge condition is to be applied, for instance:

Clamped edges: Both the displacement and rotation vanish at $x = 0^-$, i.e.,

$$\hat{p}_z(0^-, y, 0) = \hat{p}_{zx}(0^-, y, 0) = 0. \tag{2.5}$$

Hinged edges. The displacement and force are zero at $x = 0^-$, i.e.,

$$\hat{p}_z(0^-, y, 0) = 0, \quad \left(\frac{\partial^2}{\partial x^2} + v\frac{\partial^2}{\partial y^2}\right)\hat{p}_z(0^-, y, 0) = 0. \tag{2.6}$$

2.1. NON-DIMENSIONALIZATION

To proceed, we first non-dimensionalize the equations and adopt the non-dimensional space variable $\tilde{\mathbf{x}} = k_0\mathbf{x}$ based on the acoustic wavenumber; henceforth we drop the tilde and hat decoration. For convenience, the pressure is scaled so that the amplitude of the incident waves is unity.

The governing equation is now

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + 1\right)p(x, y, z) = 0, \tag{2.7}$$

subject to the non-dimensional thin plate equation

$$\left[\Omega^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 - 1\right]\frac{\partial p}{\partial z}(x, y, 0) + \frac{\varepsilon}{\Omega}p(x, y, 0) = 0 \tag{2.8}$$

for all x on $z = 0$ excluding $x \in \mathscr{D}_j$. In addition, the scattered field decays as $z \to \infty$. For $x \in \mathscr{D}_j$ the rigid plate condition translates to $p_z = 0$.

2.2. INCIDENT WAVE STRUCTURE

Several different incident wavefields could be considered, incident flexural plate, leaky and acoustic waves are the more common, although we could also treat "end-fire" waves [11], and we briefly discuss the flexural, leaky and acoustic waves.

2.2.1. *Flexural waves*

An elastic plate can support a flexural wave, of unit amplitude on the plate, that takes the form

$$p^{(inc)}(x, y, z) = \exp[-(\Gamma_1^2 - 1)^{1/2}z + \mathrm{i}\xi_1 x + \mathrm{i}\kappa y], \tag{2.9}$$

the superscript (*inc*) denotes that this is an incident wave. This surface wave decays exponentially with distance into the fluid and is localized close to the plate. The total plate wavenumber Γ_1, defined from $\Gamma_1^2 = \xi_1^2 + \kappa^2$, is found from the positive real root (for Γ) of the dispersion relation $\mathscr{K}(\xi)$ [1, p. 237; 18]:

$$\mathscr{K}(\xi) = \left(\Gamma^4(\xi) - \frac{1}{\Omega^2}\right) - \frac{\varepsilon}{\Omega^3}\frac{1}{(\Gamma^2(\xi) - 1)^{1/2}}. \tag{2.10}$$

The total plate wavenumber Γ_1 is greater than unity and this indicates that the flexural plate waves are subsonic relative to the acoustic wavespeed. Associated with the flexural plate wave is an angle of propagation θ_1 (see Figure 2) such that $\xi_1 = \Gamma_1 \cos\theta_1$ and $\kappa = \Gamma_1 \sin\theta_1$.

2.2.2. *Leaky waves*

Depending upon the precise choice of branch cuts for $(\Gamma^2(\xi) -)^{1/2}$ in equation (2.10) then the dispersion relation $\mathscr{K}(\xi)$ has, in addition to two real solutions at $\pm\,\xi_1$, complex roots that are also potentially important. In particular, when the fluid loading is *light*, that is, the dimensionless frequency is large, $\Omega \gg 1$, and the fluid loading parameter is small, $\varepsilon \ll 1$, then the *in vacuo* flexural plate waves (which, as $\Omega > 1$, have supersonic velocities relative to the acoustic wavespeed) are perturbed by the presence of the overlying fluid and shed energy into the fluid along characteristic angles [12]. In terms of the dispersion relation these waves emerge from complex roots, with small imaginary part, at $\xi = \pm\,\xi_{leaky}$, where

$$\xi_{leaky} \sim (\Omega_j^{-1} - \kappa^2)^{1/2} + \frac{\mathrm{i}\varepsilon_j}{4\Omega_j^2}(\Omega_j^{-1} - \kappa^2)^{1/2}(1 - \Omega_j^{-1})^{-1/2}. \tag{2.11}$$

The incident field of a leaky wave is then

$$p^{(inc)}(x, y, z) = \exp\left[-(\Gamma_{leaky}^2 - 1)^{1/2} z + \mathrm{i}\xi_{leaky} x + \mathrm{i}\kappa y\right]; \tag{2.12}$$

here we take κ real, and note that a leaky wave is a piece of the wave spectrum that can be identified explicitly, but cannot exist in isolation, the wave decays exponentially with both

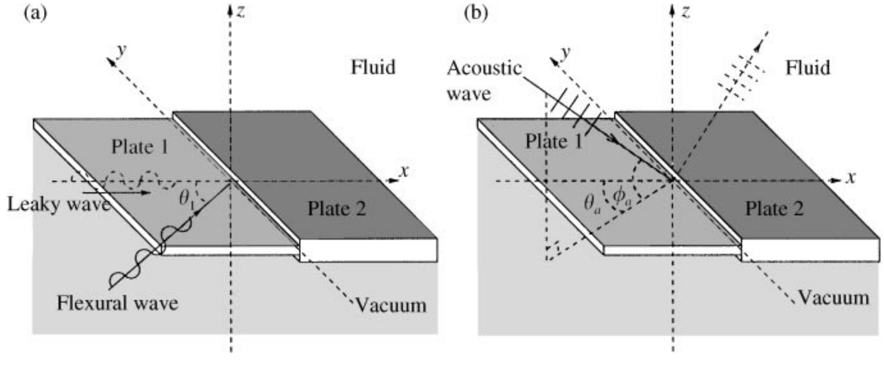

Figure 2. The geometry of the problem and the incident wave fields under consideration. In panel (a) the obliquely incident flexural wave of section 2.2.1 is shown; the angle of incidence, θ_1, is the angle made between the wavenumber vector and the horizontal normal to the plate junction (which lies along the line $x = z = 0$). In addition an incident leaky wave section 2.2.2 is also illustrated. In panel (b) an incident acoustic wave (section 2.2.3) from the fluid, along an angle, θ_a, to the horizontal normal and an angle, ϕ_a, to the vertical normal of the plate junction is shown together with the wave that would be reflected from a defect-free elastic plate.

distance into the fluid and distance along the plate; they are discussed in reference [12]. In addition, if $\Omega > 1/\kappa^2$ then the leaky wave root no longer has the imaginary component, and this can significantly affect the scattered far field [19].

2.2.3. *Acoustic plane wave*

Alternatively, we could have incoming acoustic waves and associated angles of incidence with this wave; one angle in the $z = 0$ plane giving the angle of incidence on the plate relative to the joint, and the other giving the angle of incidence within the fluid relative to the $z = 0$ plane. Thus, we define an angle of incidence θ_a in the $z = 0$ plane (on the plate) as the angle subtended between the incoming wave and the x-axis and angle ϕ_a as the angle subtended between the incoming wave and the plate (see Figure 2).

Taking the incident field to have unit amplitude then that acoustic wave has the form

$$p^{(inc)}(x, y, z) = \exp[(\Gamma_a^2 - 1)^{1/2} z + \mathrm{i}\xi_a x + \mathrm{i}\kappa y]. \tag{2.13}$$

In terms of the angles of incidence we define $\Gamma_a^2 = (\xi_a^2 + \kappa^2) = \cos^2\phi_a$ (so the square root terms is $[\Gamma_a^2 - 1]^{1/2} \equiv -\mathrm{i}\sin\phi_a$), with $\xi_a = \cos\theta_a \cos\phi_a$ and $\kappa = \sin\theta_a \cos\phi_a$; note the wavenumber in the y direction, κ has $|\kappa| < 1$ always, this is in contrast to the case of incoming flexural plate waves, where κ due to the subsonic nature of the flexural wave can be greater than unity. In order to deduce a relation between the flexural and acoustic waves we shall require κ to be identical for both problems, that we restrict our attention to $\kappa < 1$.

Later in this paper, we shall require the solution for a plane wave reflected from a defect free elastic plate; this is shown in $x > 0$ in Figure 2. In this regard, we define a plate reflection coefficient, R, as

$$R = \frac{\bar{\mathscr{K}}(\xi_a)}{\mathscr{K}(\xi_a)} \quad \text{with} \quad \bar{\mathscr{K}}(\xi) = \left(\Gamma^4(\xi) - \frac{1}{\Omega^2}\right) + \frac{\varepsilon}{\Omega^3} \frac{1}{(\Gamma^2(\xi) - 1)^{1/2}} \tag{2.14}$$

and the reflected field, denoted by a superscript (*ref*), is

$$p^{(ref)}(x, y, z) = R\exp[-(\Gamma_a^2 - 1)^{1/2} z + \mathrm{i}\xi_a x + \mathrm{i}\kappa y]. \tag{2.15}$$

Thus, for an acoustic wave incident upon a defect-free plate, the full field is

$$p(x, y, z) = p^{(inc)}(x, y, z) + p^{(ref)}(x, y, z). \tag{2.16}$$

2.3. FARFIELD WAVE STRUCTURE

We assume the defects, ribs, joints or plates are all clustered within a non-dimensional distance d of the origin, and that we observe the far field such that $x, r \gg d$.

The scattered field, denoted by the superscript (*sc*), falls into two distinguishable pieces in the farfield. Firstly, one generates scattered flexural plate waves that propagate to $x \to \pm\infty$, these are characterized by amplitude coefficients $H^{\pm}$:

$$p^{(sc)}(x, y, z) \sim H^{\pm} \exp[\pm \mathrm{i}\xi_1 x + \mathrm{i}\kappa y - (\Gamma_1^2 - 1)^{1/2} z]. \tag{2.17}$$

Secondly, we can also excite acoustic waves that propagate in the fluid. In the farfield these are cylindrical waves, and are better described in a cylindrical polar co-ordinate system

(r, ϑ, y) whose axis lies along the line $x = 0$ ($x = r\cos\vartheta$, $z = r\sin\vartheta$). These waves have the farfield form

$$p^{(sc)}(r, \vartheta, y) \sim \sqrt{\frac{2}{\pi\lambda r}}\, G(\vartheta)\, \mathrm{e}^{\mathrm{i}\lambda r - \mathrm{i}\pi/4 - \mathrm{i}\kappa y}, \tag{2.18}$$

characterized by an angular directivity $G(\vartheta)$, where $\lambda = \sqrt{1-\kappa^2}$ is the radial wavenumber of the acoustic wave. For scattered cylindrical waves we require $\kappa < 1$; if $\kappa > 1$, these waves are evanescent and there are no scattered acoustic waves in the far field (see references [19, 20, pp. 280] for further details.) Equivalently, in order to excite acoustic waves, the component of the flexural wave in the y direction must travel supersonically. To deduce our reciprocity relation we restrict our attention to $\kappa < 1$. Additionally, if the fluid loading is light we can distinguish a response due to a leaky wave; we attach amplitude coefficients $L^{\pm}$ to this response and take.

$$p^{(sc)}(x, y, z) \sim L^{\pm} \exp[\pm \mathrm{i}\xi_{leaky} x + \mathrm{i}\kappa y - (\Gamma_{leaky}^2 - 1)^{1/2} z], \tag{2.19}$$

as $x \to \pm\infty$.

We assume that each defect does not vary spatially in the y direction, thus the wavenumber in the y direction is unaltered during the scattering from a defect and this $\mathrm{e}^{\mathrm{i}\kappa y}$ dependence can be incorporated throughout. This is, we take

$$p^{(sc)}(x, y, z) = p^{(sc)}(x, z)\mathrm{e}^{\mathrm{i}\kappa y} \tag{2.20}$$

with a similar form for the incident fields, and henceforth we omit the $\mathrm{e}^{\mathrm{i}\kappa y}$ term and consider this exponential y dependence as understood.

Given a collection of defects on the plane $z = 0$ with $x \in \mathscr{D} = \bigcup_j \mathscr{D}_j$ then the scattered field can be written down as an integral over $\mathscr{D}$. The scattered pressure field is given in terms of an unknown distribution of point forces along the elastic plate. This requires a Green's function that follows from solving the Helmholtz equation with boundary condition

$$\mathscr{L}(\partial_x)p_z^G(x, z; x') + \mathscr{M}(\partial_x)p^G(x, z; x') = \delta(x - x') \tag{2.21}$$

on $z = 0$. The resulting solution is found as the inverse Fourier transform

$$p^G(x, z; x') = -\frac{1}{2\pi}\int_C \frac{\exp[-\mathrm{i}\xi(x - x') - (\Gamma^2 - 1)^{1/2} z]}{(\Gamma^2 - 1)^{1/2}\mathscr{K}(\xi)}\,\mathrm{d}\xi. \tag{2.22}$$

The path C in the inverse Fourier transform runs from $-\infty$ to $+\infty$ and is indented above (below) and singularities occurring on the negative (positive) real axis. Thus, the scattered field is

$$p^{(sc)}(x, z) = \int_{\mathscr{D}} [\mathscr{L}(\partial_{x'})p_z^{sc}(x', 0) + \mathscr{M}(\partial_{x'})p^{(sc)}(x', 0)]\,p^G(x, z; x')\,\mathrm{d}x'. \tag{2.23}$$

Using $\partial_{x'}$ to denote the partial derivative with respect to x', the operators $\mathscr{L}(\partial_x)$ and $\mathscr{M}(\partial_x)$ are

$$\mathscr{L}(\partial_x) = (\partial_x^2 + \kappa^2)^2 - 1/\Omega^2, \qquad \mathscr{M}(\partial_x) = \varepsilon/\Omega^3. \tag{2.24}$$

Each integral within equation (2.23) must be interpreted as an integral over $a_j < x < b_j$ together with a contribution from the edges a_j and b_j, that is,

$$\int_{\mathscr{D}_j} [\mathscr{L}(\partial_{x'})p_z^{(sc)}(x',0) + \mathscr{M}(\partial_{x'})p^{(sc)}(x',0)]p^G(x,z;x')\,\mathrm{d}x'$$

$$\equiv \int_{a_j}^{b_j} [\mathscr{L}(\partial_{x'})p_z^{(sc)}(x',0) + \mathscr{M}(\partial_{x'})p^{(sc)}(x',0)]p^G(x,z;x')\,\mathrm{d}x'$$

$$+ p^G(x,z;x')\partial_{x'}^3 p_z^{(sc)}(x',0)|_{a_j}^{b_j} - \partial_{x'}p^G(x,z;x')\partial_{x'}^2 p_z^{(sc)}(x',0)|_{a_j}^{b_j} \tag{2.25}$$

$$+ \partial_{x'}^2 p^G(x,z;x')\partial_{x'} p_z^{(sc)}(x',0)|_{a_j}^{b_j} - \partial_{x'}^3 p^G(x,z;x') p_z^{(sc)}(x',0)|_{a_j}^{b_j}$$

$$+ 2\kappa^2 [\partial_{x'}p^G(x,z;x')p_z^{(sc)}(x',0)|_{a_j}^{b_j} - p^G(x,z;x')\partial_{x'}p_z^{(sc)}(x',0)|_{a_j}^{b_j}].$$

We now substitute the Green's function (2.22) into equation (2.23) to deduce the farfield scattered pressure. Analyzing the asymptotic form of the Fourier integral that appears in (2.23) using residues, or a saddle-point analysis, we explicitly identify the characteristic farfield coefficients (2.17, 2.18, 2.19) as:

$$\mathrm{i}(\Gamma_1^2 - 1)^{1/2}\mathscr{K}'(\xi_1)H^{\pm} = \int_{\mathscr{D}} [\mathscr{L}(\partial_{x'})p_z^{(sc)}(x',0) + \mathscr{M}(\partial_{x'})p^{(sc)}(x',0)]\mathrm{e}^{\mp \mathrm{i}\xi_1 x'}\,\mathrm{d}x', \tag{2.26}$$

the prime on $\mathscr{K}'(\xi)$ denotes the differential with respect to ξ,

$$2\mathrm{i}\mathscr{K}(\lambda\cos\vartheta)G(\vartheta) = \int_{\mathscr{D}} [\mathscr{L}(\partial_{x'})p_z^{(sc)}(x',0) + \mathscr{M}(\partial_{x'})p^{(sc)}(x',0)]\mathrm{e}^{-\mathrm{i}x'\lambda\cos\vartheta}\,\mathrm{d}x', \tag{2.27}$$

and

$$\mathrm{i}(\Gamma_{leaky}^2 - 1)^{1/2}\mathscr{K}'(\xi_{leaky})L^{\pm} = \int_{\mathscr{D}} [\mathscr{L}(\partial_{x'})p_z^{(sc)}(x',0) + \mathscr{M}(\partial_{x'})p^{(sc)}(x',0)]\mathrm{e}^{\mp \mathrm{i}\xi_{leaky}x'}\,\mathrm{d}x'. \tag{2.28}$$

The domain $\mathscr{D}$ incorporates the edge and as in equation (2.25) the edge conditions are included in these expressions. This is illustrated for a single joint in section 4.1.

3. THE RECIPROCITY RELATION

In this section, we extract a reciprocity relation between the scattered fields generated by the different incident waves under consideration in this article. In order to achieve this we use the reciprocity relation

$$\int_S [p^{(f)}p_{n_i}^{(a)} - p^{(a)}p_{n_i}^{(f)}]\,n_i\,\mathrm{d}S = 0 \tag{3.1}$$

(which follows from Green's theorem applied to $\int_V (p^{(f)}[\nabla^2 + k^2]p^{(a)} - p^{(f)}[\nabla^2 + k^2]p^{(a)})\,\mathrm{d}V = 0$) for two independent states (f) and (a) (the choice of superscript will become transparent) in a source-free domain bounded by a surface S (with outward pointing normal n_i), for which both states satisfy the Helmholtz equation. Taking the assumed y dependence and using this result in the x, z plane with S bounded by a semi-circular arc at infinity and running parallel to, and just above, the plate, furthermore,

$$\int_S ([\mathscr{L}(\partial_x)p_{n_i}^{(f)} + \mathscr{M}(\partial_x)p^{(f)}]p_{n_i}^{(a)} - [\mathscr{L}(\partial_x)p_{n_i}^{(a)} + \mathscr{M}(\partial_x)p^{(a)}]p_{n_i}^{(f)})\, n_i\, \mathrm{d}S = 0, \tag{3.2}$$

provided, that is, $[\mathscr{L}(\partial_x)p_z^{(f,a)} + \mathscr{M}(\partial_x)p^{(f,a)})] = 0$ as $x \to \pm\infty$ and $p^{(f,a)}$ decays at infinity. Furthermore, we require that the operators $\mathscr{L}$ and $\mathscr{M}$ contain only even derivatives of x; for the plate theory used here they are defined in equation (2.24). Now we manipulate this.

Let state (f) be the scattered field due to an incoming flexural plate wave, equation (2.9),

$$p^{(f\ inc)}(x, z) = \exp\,[\mathrm{i}\xi_1 x - (\Gamma_1^2 - 1)^{1/2} z]. \tag{3.3}$$

and state (a) is that due to an incident acoustic wave together with its reflection from an unblemished elastic plate (2.16), that is,

$$p^{(a\ inc)}(x, z) = \exp\,[\mathrm{i}\xi_a x + (\Gamma_a^2 - 1)^{1/2} z] + R \exp\,[\mathrm{i}\xi_a x - (\Gamma_a^2 - 1)^{1/2} z]. \tag{3.4}$$

Both incident fields have $\mathscr{L}(\partial_x)p_z^{(f,a\ inc)}(x, z) + \mathscr{M}(\partial_x)p^{(f,a\ inc)}(x, z) = 0$ on $z = 0$. Thus, relation (3.2) gives

$$\int_{\mathscr{D}} [\mathscr{L}(\partial_x)p_z^{(f\ sc)} + \mathscr{M}(\partial_x)p^{(f\ sc)}]p_z^{(a\ inc)}\, \mathrm{d}x = \int_{\mathscr{D}} [\mathscr{L}(\partial_x)p_z^{(a\ sc)} + \mathscr{M}(\partial_x)p^{(a\ sc)}]p_z^{(f\ inc)})\, \mathrm{d}x, \tag{3.5}$$

where we have used the rigid boundary condition $p_z^{(sc)} + p_z^{(inc)} = 0$ on $\mathscr{D}$. We have also exploited the edge conditions (2.5) and (2.6); taking into account the $\exp(\mathrm{i}\kappa y)$ dependence these translate to: clamped, $p_z(0^-, 0) = 0$, $p_{zx}(0^-, 0) = 0$; hinged, $p_z(0^-, 0) = 0$, $p_{zxx}(0^-, 0) = 0$. The direct relations with p_z and its derivatives mean that these edge conditions can be easily incorporated.

Inserting the respective incident fields into equation (3.5), and furthermore noticing the similarity to the $H^{\pm}$ and G formulae, equations (2.26) and (2.27), yield

$$(\Gamma_1^2 - 1)\mathscr{K}'(\xi_1)H^{(a)-}(\theta_a, \phi_a) = \frac{4\varepsilon}{\Omega^3} G^{(f)}(\vartheta, \theta_1). \tag{3.6}$$

We have now appended some further decoration to the flexural wave amplitude H^- and directivity G, this is to make it plain that this relation holds for specific angles of incidence, angles of observation and types of wave incidence.

- $H^{(a)-}(\theta_a, \phi_a)$ is the amplitude of the scattered flexural wave travelling to $x = -\infty$ due to to an incoming plane wave, state (a), from θ_a, ϕ_a
- $G^{(f)}(\vartheta_a, \theta_1)$ is the directivity due to an incident flexural wave travelling, state (f), from $x = -\infty$. This travels along an angle θ_1 to the x-axis; the directivity is evaluated at angle ϑ:

$$\vartheta = \pi - \cos^{-1}\left[\frac{1}{\lambda}\cos\theta_a \cos\phi_a\right]. \tag{3.7}$$

The wavenumber in the y direction, κ, is identical for both incident waves; this leads to the relation $\Gamma_1 \sin\theta_1 = \sin\theta_a \cos\phi_a$, and we recall that $\lambda = \sqrt{1 - \kappa^2}$; if $\kappa = 0$, that is, normal

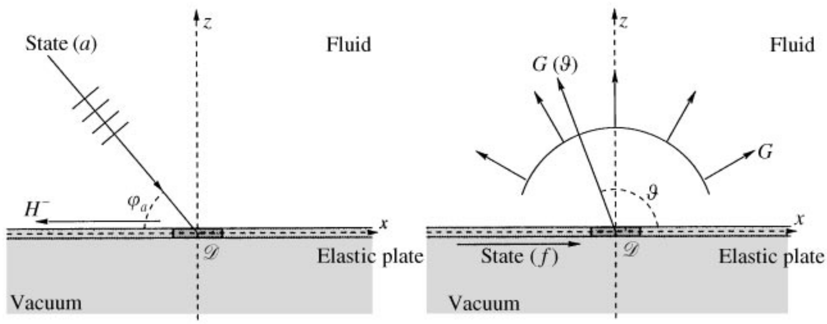

Figure 3. Illustration of the reciprocity relation for states (f) and (a). $\vartheta + \phi_a \leqslant \pi$.

incidence then these formulae simplify with $\vartheta = \pi - \phi_a$. One then observes the directivity along the same angle, given the definitions of these angles, upon which the acoustic wave in state (a) is incoming; the effect of altering the angle of incidence (in the plate) of the flexural wave is to remove this simple relation.

At this point it is worthwhile to draw the readers attention to the precise angular behaviour in the x, z plane and a sketch of the two relations is shown in Figure 3.

4. ILLUSTRATIVE EXAMPLES

The main application of the reciprocity relation is in numerical studies; it is also valuable for analytic work and we briefly demonstrate the manner in which it can be applied for a line joint or semi-infinite plate.

4.1. A SINGLE LINE JOINT

A single line joint is probably the simplest example upon which to illustrate the reciprocity relation as we have a single defect of vanishing width so that a_1 and b_1 are 0_- and 0_+ respectively.

The farfield coefficients then involve the jumps in scattered field across the joint (denoted by $[\![\;]\!]$), for instance, the amplitude of the scattered flexural wave, H^- is

$$\mathrm{i}(\Gamma_1^2 - 1)^{1/2}\mathscr{K}'(\xi_1)H^- = [\![p_{zxxx}^{(sc)}(x',0)]\!] - \mathrm{i}\xi_1[\![p_{zxx}^{(sc)}(x',0)]\!] - \xi_1^2[\![p_{zx}^{(sc)}(x',0)]\!] + \mathrm{i}\xi_1^3[\![p_z^{(sc}(x',0)]\!]. \tag{4.1}$$

This is crucially dependent upon the edge conditions, for instance see references [21, 13]. Following that analysis, then for a clamped joint

$$p_z^{(sc)}(x,z) = -\frac{1}{2\pi}\int_C \frac{(E_1\xi + E_0)}{\mathscr{K}(\xi)}\exp\left[-\mathrm{i}\xi x - (\Gamma^2 - 1)^{1/2}z\right]\mathrm{d}\xi \tag{4.2}$$

with constants E_1 and E_0 determined from the incident field. We now consider two states (f) and (a) as those due to incoming flexural and acoustic waves respectively. The constants E_1 and E_0 take different forms in both cases and we distinguish them as $E_0^{(f,a)}$ and $E_1^{(f,a)}$. Applying the edge conditions which are that the jump in p_z and p_{zx} across the joint are zero one finds that

$$E_0^{(f,a)} = -\mathscr{A}^{(f,a)} \left[\frac{1}{2\pi} \int_C \frac{\mathrm{d}\xi}{\mathscr{K}(\xi)} \right]^{-1} = -\frac{\mathscr{A}^{(f,a)}}{I_0}, \tag{4.3}$$

$$E_1^{(f,a)} = \xi^{(f,a)} \mathscr{A}^{(f,a)} \left[\frac{1}{2\pi} \int_C \frac{\xi^2}{\mathscr{K}(\xi)} \mathrm{d}\xi \right]^{-1} = \frac{\xi^{(f,a)} \mathscr{A}^{(f,a)}}{I_1}. \tag{4.4}$$

The constants $\mathscr{A}^{(f,a)}$ are

$$\mathscr{A}^{(f)} = (\Gamma_1^2 - 1)^{1/2}, \qquad \mathscr{A}^{(a)} = (R-1)(\Gamma_a^2 - 1)^{1/2} \tag{4.5}$$

and the $\xi^{(f,a)}$ are $\xi^{(f)} = \xi_1$ and $\xi^{(a)} = \xi_a$. We have also defined I_q [21] as

$$I_q = \frac{1}{2\pi} \int_C \frac{\xi^{2q}}{\mathscr{K}(\xi)} \mathrm{d}\xi, \tag{4.6}$$

where the path C is defined following equation (2.22).

Asymptotic considerations of the inverse Fourier transform for $p^{(sc)}(x, z)$ (which follows from equation (4.2)) give the farfield coefficients required for equation (3.6) as

$$H^{(a)-}(\theta_a, \phi_a) = \mathrm{i} \frac{[E_1^{(a)} \xi_1 + E_0^{(a)}]}{\mathscr{K}'(\xi)(\Gamma_1^2 - 1)^{1/2}} \equiv \mathrm{i} \frac{(R-1)(\Gamma_a^2 - 1)^{1/2}}{\mathscr{K}'(\xi_1)(\Gamma_1^2 - 1)^{1/2}} [\xi_a \xi_1 I_1^{-1} - I_0^{-1}], \tag{4.7}$$

$$G^{(f)}(\vartheta, \theta_1) = -\mathrm{i}[-E_1^{(f)} \lambda \cos\vartheta + E_0^{(f)}] \frac{(\Gamma_1^2 - 1)^{1/2}}{2\mathscr{K}(\lambda \cos\vartheta)}$$

$$\equiv -\mathrm{i} \frac{(\Gamma^2 - 1)^{1/2}}{2\mathscr{K}(\lambda \cos\vartheta)} [\xi_a \xi_1 I_1^{-1} - I_0^{-1}], \tag{4.8}$$

for ϑ given by equation (3.7). After substitution into equation (3.6) one sees that the reciprocity relation holds.

Other edge conditions upon the joint can be considered, and the analysis is then more complicated as the integrals which appear in an analogous manner to those in equation (4.8) can be apparently divergent [21]. Nonetheless, one can pursue the analysis and obtain similar results.

4.2. A SEMI-INFINITE RIGID PLATE

The reciprocity relation is valid even when the rigid defect covers the half-plane on $z = 0$ for $x > 0$, $-\infty < y < \infty$ and we now turn to this slightly more involved example, that is, an elastic plate on $z = 0$ in $x < 0$ connected to a co-planar rigid plate on $x > 0$. This can be solved using Fourier transforms and the Wiener–Hopf technique, for instance see reference [18]. One can approach this either by constructing an integral equation by manipulating equation (2.23), or directly from the governing equations and boundary conditions; we follow the latter route.

We define the Fourier transform of the scattered pressure

$$P(\xi, z) = \int_{-\infty}^{\infty} p^{(sc)}(x, z) e^{i\xi x} \, dx = P_+(\xi, z) + P_-(\xi, z), \tag{4.9}$$

where $P_\pm$ denote the half-range transforms of $p^{(sc)}(x, z)$:

$$P_+(\xi, z) = \int_0^{\infty} p^{(sc)}(x, z) e^{i\xi x} \, dx \qquad P_-(\xi, z) = \int_{-\infty}^{0} p^{(sc)}(x, z) e^{i\xi x} \, dx \tag{4.10}$$

the same notation is used for the half-range transforms of $p_z^{(sc)}(x, z)$ which are $P_{\pm z}$. The inverse transform is defined by

$$p^{(sc)}(x, z) = \frac{1}{2\pi} \int_C P(\xi, z) e^{-i\xi x} \, d\xi, \tag{4.11}$$

where the path C is defined after equation (2.22). The subscripts $+$ and $-$ attached to the half-range transforms denote that these functions are analytic in the $+$ and $-$ regions; these denote the regions of the complex ξ-plane above and below C; we loosely refer to these two regions as the "upper" and "lower" halves of the complex ξ-plane. In what follows, we shall mainly deal with the transforms along the plate, $z = 0$, and we shall shorten $P_+(\xi, 0)$ to $P_+(\xi)$ henceforth, and similarly for the other half-range transforms on $z = 0$.

We generate a functional relation between half-range transforms that are unknown. This relation is then unravelled using the Wiener–Hopf technique to identify the unknowns and deduce the full solution. Along the way we are required to satisfy the edge conditions; for problems in structural acoustics these edge conditions are slightly awkward to incorporate.

The incident flexural and acoustic waves can be treated simultaneously: we let state (f) be that associated with an incident flexural wave (2.9), and (a) be that associated with an incident acoustic wave (2.16). Using the rigid boundary condition $p_z^{(inc)} + p_z^{(sc)} = 0$ on $z = 0$ and $x > 0$ then for the states (f) and (a) we have the relation, $P_{+z}^{(f,a)}(\xi) = i\mathscr{A}^{(f,a)}/(\xi + \xi^{(f,a)})_+$. The terms involving the superscript (f, a) take different values depending upon whether we are dealing with state (f) or state (a). The representation of $P_{+z}^{(f,a)}(\xi)$ simply state that the transform of $p_z^{sc}(x, 0)$ is known, and the subscript $+$ we have attached to the last term is to remind us that the pole at $-\xi^{(f,a)}$ is taken to lie in the plus region, and we must indent the inversion contour, and take account of this in the analysis, accordingly.

The constants $\mathscr{A}^{(f,a)}$ are given in equation (4.5) an the $\xi^{(f,a)}$ are again $\xi^{(f)} = \xi_1$ and $\xi^{(a)} = \xi_a$.

We follow the usual Wiener–Hopf recipe [22] and the functional equation emerges as

$$\mathscr{K}(\xi) \left[P_{-z}^{(f,a)}(\xi) - \frac{\mathscr{A}^{(f,a)}}{i(\xi^{(f,a)} + \xi)_+} \right] = \left[\Gamma^4(\xi) - \frac{1}{\Omega^2} \right] P_{+z}^{(f,a)}(\xi) + \frac{\varepsilon}{\Omega^3} P_+^{(f,a)}(\xi) - \mathscr{R}^{-(f,a)}(\xi). \tag{4.12}$$

This relates the transform of the unknown pressure on the rigid plate, $P_+(\xi)$, to the transform of the unknown displacement of the elastic plate, effectively $P_{-z}(\xi)$; these are clearly different depending upon the incident field. The edge behaviour of the plates is completely captured in the term

$$\mathscr{R}^-(\xi) = [p_{zxxx}^{(sc)}(0^-, 0) - i\xi p_{zxx}^{(sc)}(0^-, 0) - (2\kappa^2 + \xi^2) p_-^{(sc)}(0^-, 0) + i\xi(2\kappa^2 + \xi^2) p_z^{(sc)}(0^-, 0)]. \tag{4.13}$$

Our most valuable player here is the Wiener–Hopf technique, in essence one separates the functional equation into a piece that is analytic in the $+$ region and a piece that is analytic in the $-$ region. These two pieces are equal along a common line and therefore both are equal to the same entire function $E(\xi)$, using analytic continuation this is extended to the whole complex ξ-plane. Edge behaviour is then used with Liouville's theorem to fix the form of $E(\xi)$.

Technically, we require the split of the function $\mathscr{K}(\xi)$ into a product of $+$ and $-$ functions. That is, we require $\mathscr{K}(\xi)=\mathscr{K}_+(\xi)\mathscr{K}_-(\xi)$; this is discussed in detail in references [19] and [14] so is not repeated here, splitting is most easily performed in terms of some quadratures. It is worth mentioning that $\mathscr{K}_-(+\xi)=\mathscr{K}_+(-\xi)$). For our purposes here it is only necessary to note that one can do the factorization and we proceed formally.

This factorization and subsequent rearrangement of the functional equation expresses the equality of a $+$ and $-$ function, and utilizing Liouville's theorem and an estimate of the growth behaviour of the unknown transforms we may deduce that our entire function, $E^{(f,a)}(\xi)$, is $O(\xi)$ when $|\xi|\to\infty$, for all edge conditions; this leads to the transform of the unknown p_z along the plate as

$$P^{(f,a)}_{-z}(\xi)=\frac{E_1^{(f,a)}\xi+E_0^{(f,a)}}{\mathscr{K}_-(\xi)}+\frac{\mathscr{A}^{(f,a)}}{\mathrm{i}(\xi^{(f,a}+\xi)_+}\left[1-\frac{\mathscr{K}_+(\xi^{(f,a)})}{\mathscr{K}_-(\xi)}\right]. \tag{4.14}$$

Both $E_{0,1}^{(f,a)}$ are unknown and must be determined from the edge conditions; we shall consider clamped and hinged cases. Hence, the scattered pressure field is ultimately

$$p^{(sc\ f,a)}(x,z)=-\frac{1}{2\pi}\int_C\left[E_1^{(f,a)}\xi+E_0^{(f,a)}-\frac{\mathscr{A}^{(f,a)}\mathscr{K}_+(\xi^{(f,a)})}{\mathrm{i}(\xi+\xi^{(f,a)})_+}\right]$$

$$\times\frac{\exp[-(\Gamma^2(\xi)-1)^{1/2}z-\mathrm{i}\xi x]}{\mathscr{K}_-(\xi)[\Gamma^2(\xi)-1]^{1/2}}\,\mathrm{d}\xi \tag{4.15}$$

and the farfield coefficients follow from asymptotic considerations of this integral. Consequently, the coefficients for incident waves (f) (flexural) and (a) (acoustic), are

$$H^{(f,a)-}=\frac{-\mathrm{i}\mathscr{K}_+(\xi_1)}{(\Gamma_1^2-1)^{1/2}\mathscr{K}'(\xi_1)}\left[E_1^{(f,a)}\xi_1+E_0^{(f,a)}-\frac{\mathscr{A}^{(f,a)}\mathscr{K}_+(\xi^{(f,a)})}{\mathrm{i}(\xi_1+\xi^{(f,a)})}\right], \tag{4.16}$$

$$L^{(f,a)-}=\frac{-\mathrm{i}\mathscr{K}_+(\xi_{leaky})}{(\Gamma_{leaky}^2-1)^{1/2}\mathscr{K}'(\xi_{leaky})}\left[E_1^{(f,a)}\xi_{leaky}+E_0^{(f,a)}-\frac{\mathscr{A}^{(f,a)}\mathscr{K}_+(\xi^{(f,a)})}{\mathrm{i}(\xi_{leaky}+\xi^{(f,a)})}\right] \tag{4.17}$$

and

$$G^{(f,a)}(\vartheta)=\frac{-\mathrm{i}}{2\mathscr{K}_+(\lambda\cos\vartheta)}\left[-E_1^{(f,a)}\lambda\cos\vartheta+E_0^{(f,a)}-\frac{\mathscr{A}^{(f,a)}\mathscr{K}_+(\xi^{(f,a)})}{\mathrm{i}(\xi^{(f,a)}-\lambda\cos\vartheta)}\right]. \tag{4.18}$$

If required, the coefficients for incident leaky waves may be similarly deduced; they are closely related to incident flexural waves replacing Γ_1 with Γ_{leaky} in $\mathscr{A}^{(f)}$ and $\xi^{(f)}$.

Clearly the terms $G^{(f)}(\vartheta,\theta_1)$ and $H^{(a)-}(\theta_a,\phi_a)$ required for the reciprocity result (3.6) appear similar, at least they have a similar structure, but we still have the edge conditions to incorporate; it is at first sight unclear that these components too are correctly related.

If we take the edge to be clamped, it transpires that $E_1^{(f,a)} = E_0^{(f,a)} \equiv 0$ and upon noting the choice of ϑ for the reciprocity relation is $\lambda \cos \vartheta = -\xi_a$ then the relation (3.6) is immediately satisfied. However, more complicated edge conditions have non-zero E's associated with them. In general, to satisfy relation (3.6) the constants must satisfy

$$\mathscr{A}^{(f)}\mathscr{K}_+(\xi^{(f)})[E_1^{(a)}\xi^{(f)} + E_0^{(a)}] = \mathscr{A}^{(a)}\mathscr{K}_+(\xi^{(a)})[E_1^{(f)}\xi^{(a)} + E_0^{(f)}]. \tag{4.19}$$

If we now take the edge to have the hinged condition (2.6), these edge conditions are incorporated by taking the limit in the Fourier transform $P_{-z}^{(f,a)}(\xi)$ that $\xi \to \infty$ which after inversion corresponds to $x \to 0^-$. That is, we explicitly determine p_z along the elastic plate and then enforce the edge conditions.

To enforce the chosen edge condition we require the expansion of the split function $\mathscr{K}_-(\xi)$ as $\xi \to \infty$ which is

$$\mathscr{K}_-(\xi) \sim \xi^2 + k_1\xi + \cdots \tag{4.20}$$

where k_1 is independent of ξ, and for our purposes is a constant found using quadratures. Inserting this result into $P_{-z}^{(f,a)}$ and inverting term by term to obtain that

$$p_z(x,z) = p_0 + xp_1 + x^2p_2 + \cdots \tag{4.21}$$

for constants p_0, p_1 and p_2. This result is for the total pressure now and not only the scattered piece of the pressure. On applying the hinged conditions we find that $E_1^{(f,a)} = 0$ and use the equation

$$0 = 2p_2 - \nu\kappa^2 p_0 = -\mathrm{i}[-E_0^{(f,a)}k_1 + \mathrm{i}\mathscr{A}^{(f,a)}\mathscr{K}_+(\xi^{(f,a)})] \tag{4.22}$$

to determine that

$$E_0^{(f,a)} = \frac{\mathscr{A}^{(f,a)}\mathscr{K}_+(\xi^{(f,a)})}{\mathrm{i}k_1} \tag{4.23}$$

and thus the constants are determined.

Substitution into equations (4.16) and (4.18) leads to the farfield coefficients

$$H^{(a)-}(\theta_a, \phi_a) = -\frac{\mathscr{K}_+(\xi_1)\mathscr{K}_+(\xi_a)(R-1)(\Gamma_a^2-1)^{1/2}}{(\Gamma_1^2-1)^{1/2}\mathscr{K}'(\xi_1)}\left(\frac{1}{k_1} - \frac{1}{(\xi_1+\xi_a)}\right) \tag{4.24}$$

and

$$G^{(f)}(\vartheta, \theta_1) = -\frac{(\Gamma_1^2-1)^{1/2}\mathscr{K}_+(\xi_1)}{2K_+(\lambda\cos\vartheta)}\left(\frac{1}{k_1} - \frac{1}{(\xi_1 - \lambda\cos\vartheta)}\right). \tag{4.25}$$

Noting that the choice of ϑ in equation (3.7) ensures that $\xi_a = -\lambda\cos\vartheta$, and some minor manipulations, these too satisfy the reciprocity relations (3.6) and (4.19)

5. CONCLUDING REMARKS

A reciprocity relation has been identified for rigid plates lying upon an infinite elastic plate that should, besides being of independent interest, be of value in numerical studies involving, say, arrays of rigid ribs, plates and other rigid defects; it provides a non-trivial check. It therefore complements other results, such as extensions of the optical scattering theorem [16, 23, 17]. The two analytic examples demonstrate how the result should be applied.

In addition, the reciprocity result we have given can be generalized in a straightforward manner to rigid plates on, say, a membrane, or rigid cylindrical shells on an elastic cylindrical shell; the general methodology outlined here should be useful in those contexts. However, the replacement of the rigid plate by an elastic plate (of differing material properties to the plate which extends to infinity) and higher order edge conditions leads to further difficulties.

There are two additional reciprocity results, that is, involving two incident waves of the same type, that is, both flexural waves or both acoustic, and then interrelating the scattering coefficients; the resulting relations are then obvious, so we have not given upon these cases. Relations between flexural (or acoustic) waves and incident, scattered leaky waves can also be deduced. For instance, if we have an incident leaky wave with the form (2.12) then the amplitudes (2.17) and (2.19) are connected via

$$(\Gamma_1^2 - 1)\mathscr{K}'(\xi_1)H^-(\theta_{leaky}) = (\Gamma_{leaky}^2 - 1)\mathscr{K}'(\xi_{leaky})L^-(\theta_1), \tag{5.1}$$

where we take the two states to be

- $H^-(\theta_{leaky})$ is the amplitude of the flexural wave travelling to $x = -\infty$ due to an incoming leaky wave from θ_{leaky}.
- $L^-(\theta_1)$ is the amplitude of the leaky wave travelling to $x = -\infty$ due to an incoming flexural wave from θ_1.

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