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RELAXATION-TIME APPROXIMATION FOR ANALYTICAL EVALUATION OF TEMPERATURE FIELD IN THERMOACOUSTIC STACK

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The problem of analytical description of temperature fields and heat fluxes in thermoacoustic devices (such as refrigerators and prime-movers) is discussed. It is demonstrated that for the precise analysis of the thermoacoustic process near the edges of the stacks and the heat exchangers, and also for the prediction of the heat fluxes between the stack and the heat exchangers, it is necessary to avoid the traditional "mean-field" approximation. In other words, on the spatial scale of the order of a particle displacement in the standing acoustic wave, hydrodynamical (advective) transport of heat cannot be described as a diffusional transport with an effective (depending on the acoustic wave power) diffusivity. In order to get insight into the non-linear phenomena, related to axial (along the stack) advective transport of heat, the simplified description of the transverse heat exchange between the gas and the stack (the relaxation-time approximation) has been adopted in the present investigation. The analytical descriptions obtained of the temperature distribution and of the heat flux predict, in particular, that in some cases the thermoacoustic heat flux between two stacks separated by an adiabatic gap can increase with the increasing width of the gap.

1. INTRODUCTION

The classical thermoacoustic refrigerator [1] is based on a half-wavelength resonator driven by a loudspeaker (see Figure 1). A system of parallel solid plates (i.e., the so-called thermoacoustic stack) is installed in the resonator. As a consequence of the thermal interaction of the acoustic waves with the stack there is an additional phase shift between the oscillations of particle velocity and temperature in the standing acoustic wave. This results in a directional heat flux along the stack and in heating of one of its terminations (edges) and in cooling of the other. Installation of additional stacks (i.e., the so-called heat exchangers) near the edges of the basic stack (see Figure 1) provides an opportunity to extract this heat flux from the resonator (through the hot heat exchanger) and use this thermoacoustic engine to cool the other systems which are connected to the cold heat exchanger.

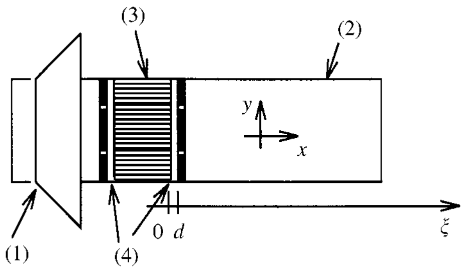

Figure 1. The qualitative scheme of the thermoacoustic heat pump (refrigerator): 1—the loud-speaker, 2—the acoustic resonator, 3—the porous thermoacoustic stack, 4—the porous heat exchangers.

The basic physical problem to be solved concerning thermoacoustic refrigeration is the description of the temperature and heat flux distributions inside and at the edges of the stacks, and the heat fluxes between the basic stack and the heat exchangers. This solution is necessary for the optimization of the performances of practical thermoacoustic engines. In the following the fundamental equations and approximations used for its analytical treatment are briefly discussed. In particular, attention is concentrated on the limitations of the traditional "mean-field" approximation in which the convective heat transport is described as a diffusional process. After that, the results of an analytical approach which precisely treats convective heat transport avoiding the "mean-field" approximation are presented.

2. "MEAN-FIELD" APPROXIMATION IN THERMOACOUSTICS

The equation for the energy transport in the gas can be presented in the form [2]

$$\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = \frac{1}{\rho T}\left[\sigma'_{ij}\frac{\partial v_i}{\partial x_j} + \nabla(k\nabla T)\right], \tag{1}$$

where s is the entropy, $\mathbf{v}$ is the particle velocity (with the components v_i), ρ and T are the density and the temperature of the gas, σ'_{ij} is the viscous part of the stress tensor, and k is the gas thermal conductivity. Traditionally, in the analytical approaches, the energy release due to viscous losses is neglected (simultaneously with a possibility of the formation of vortices near the stack edges) [1, 3]. The experimental results on vortex shedding at the edges of the plates and the results of computer modelling of this phenomenon can be found in references [4, 5] and in the references therein. As the goal of the present investigation is not the analysis of these effects, we accept in the following the approximation of an inviscid fluid where these effects are completely absent. If the viscosity is neglected and the thickness of the stack plates is assumed to be much less than the separation distance between the plates (the limiting case of the infinitely thin plates), then the velocity field is approximately one-dimensional $\mathbf{v} = \mathbf{i}v_x \equiv \mathbf{i}v$ and the operator describing the hydrodynamic advective transport of entropy on the left-hand side (l.h.s.) of equation (1) becomes $v\,\partial s/\partial x$. Here and in the following the x-axis is directed along the axis of the resonator, while the y-axis is perpendicular to the plates of the thermoacoustic stack (see Figure 1).

In the thermoacoustic devices designed for applications around room temperature, the relative variations of temperature are usually small and, consequently, the right-hand side (r.h.s.) of equation (1) can be linearized relative to some characteristic state (ρ_0, T_0) of the gas. It is suitable to associate the characteristic state of the gas with its state in the absence of the stack and the heat exchangers. Thus, equation (1) takes the form

$$\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial x} = \frac{k_0}{\rho_0 T_0}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right)T', \tag{2}$$

where T' denotes the deviation of the temperature from T_0. Upon applying the thermodynamic relation $\mathrm{d}s = (c_p/T)\,\mathrm{d}T - (\beta/\rho)\,\mathrm{d}p$ (where c_p is the isobaric heat capacity, β is the thermal expansion coefficient and p is the gas pressure), equation (2) is transformed into

$$\frac{\partial T'}{\partial t} + v\frac{\partial T'}{\partial x} = \frac{\beta T}{\rho_0 c_p}\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \frac{k_0}{\rho_0 c_p(T_0)}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right)T'. \tag{3}$$

If ω is the cyclic frequency of harmonic acoustic oscillations, then the following estimations are valid: $\partial p/\partial t \propto \omega p$ and $v \propto \omega U$ (where U is the particle displacement in the acoustic wave). As $\partial p/\partial x \propto p/\lambda$ (where λ is the acoustic wavelength), then the second term on the r.h.s. of equation (3) can be neglected in comparison with the first one, because it contains an additional small parameter $U/\lambda \ll 1$. Finally, the equation for the temperature is presented in the form

$$\frac{\partial T'}{\partial t} + v\frac{\partial T'}{\partial x} = \frac{\beta T_0}{\rho_0 c_p}\frac{\partial p}{\partial t} + D_0\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right)T', \tag{4}$$

where β, c_p and the thermal diffusivity of the gas $D_0 \equiv k_0/\rho_0 c_p$ can be evaluated in the characteristic state of the gas and, consequently, are considered to be constants.

For the analysis of thermoacoustic refrigeration, the fields of particle velocity and pressure can be approximated by their distribution in the resonator in the absence of the temperature gradients (i.e., by neglecting inverse influence of the thermoacoustic processes on acoustic field) [1, 3, 6]. In fact, in this approximation it is assumed that all the installations in the resonator (i.e., both the stack and the heat exchangers) are always acoustically thin and, as a consequence, the travelling-wave components of the acoustic field in the resonator are negligibly small in comparison with the standing-wave component. We choose the following description of the acoustic field:

$$U \equiv -u_A\sin(2\pi x/\lambda)\cos(\omega t) \equiv -u\cos(\omega t),$$

$$v = \omega u\sin(\omega t), \quad p = \rho_0(\lambda\omega^2/2\pi)u_A\cos(2\pi x/\lambda)\cos(\omega t). \tag{5}$$

Here u_A denotes the maximum amplitude of the particle displacement and u denotes the local amplitude of the particle displacement amplitude in the acoustic field. The acoustic field described in equations (5) introduces in the standing-wave thermoacoustic effects, a characteristic axial scale in space equal to the acoustic wavelength λ [1]. The other characteristic scales in space are introduced, in general, by the presence of the stack and of the heat exchangers (these scales are equal to the stack length and the separation between the stack and the heat exchanger, etc.). Though it is assumed here that the latter scales do not influence the acoustic field in equations (5) (in particular, because thermoacoustic installations are usually acoustically thin and also because of neglecting of the viscosity here) these scales can manifest themselves in the distribution of the temperature as the installations inside the acoustic resonator can significantly modify the conditions for the temperature variations (because heat conduction is active in equation (4)). Although in thermoacoustics the scale related to the installation of the stack is usually much shorter than the acoustic wavelength [1] there is an even shorter scale in the problem under consideration which is equal to the local particle displacement amplitude u in the acoustic field. It is also a commonly accepted idea that the separation between the stack and the heat exchanger should not exceed an order of u for the effective heat transport between these elements [1].

The primary goal of the analysis presented here is the investigation of thermoacoustic phenomena at the scale much less than λ near the stack edges and in a vicinity of the gap separating the stack and the heat exchanger. To achieve this purpose we simplify equation (4) by explicitly taking into account that the variation of v and p with co-ordinate x in equations (5) is much slower than the spatial variation of temperature in equation (4) in the region of the stack termination. Introducing the dimensionless temperature $\theta = T'/T_c$ ($T_c \equiv -(\beta T_0/c_p)(\lambda\omega^2/2\pi)u_A \cos(2\pi x/\lambda)$) and the dimensionless variables $\tau = \omega t$, $\zeta = x/u$ and $\eta = y/y_0$ (where y_0 is the distance between the plates in the stack) and neglecting the contribution from the scale of the order of λ to the derivatives over axial co-ordinate, one can simplify equation (4) to

$$\theta_\tau + \theta_\zeta \sin\tau = \sin\tau + \Delta\theta,$$

where

$$\Delta \equiv (D_0/y_0^2\omega)\partial^2/\partial\eta^2 + (D_0/u^2\omega)\partial^2/\partial\zeta^2. \tag{6}$$

Note that the solution of equation (6) depends implicitly (through the normalization of the function θ and the variable ζ) on the position of the stack inside the acoustic resonator. It should be mentioned that equation (6) can be applied for the description of thermoacoustic phenomena at any scale that is significantly less than the acoustic wavelength λ (in particular, for the description of temperature distribution inside an acoustically thin stack) as long as the local acoustic field parameters are used. The solutions of equation (6) are related via the boundary conditions to the solutions of the equation for the diffusional thermal conduction in the plates [1, 3]. Until now the only way used to solve this system of coupled equations analytically was by means of its additional simplification in the "mean-field" approximation (see the review article [1] and the references therein). For a recent application of the mean-field approximation in thermoacoustics see reference [7], for example.

The complexity of the problem of the analytical solution of equation (6) is, in fact, in the presence of the differential term $\theta_\zeta \sin\tau$ with a time-dependent coefficient. This term, describing the hydrodynamic [2] (or convective (advective) [8]) transport of heat, is responsible for the thermoacoustic effect. The idea of the "mean-field" approximation is in the derivation from equation (6) of the equation for the time-averaged physical functions. In practice, the temperature field is presented in the form of a superposition of an oscillating periodic part $\tilde\theta$ and the field $\langle\theta\rangle$ averaged over the period of oscillations:

$$\theta = \tilde\theta + \langle\theta\rangle, \quad \langle\theta\rangle \equiv (2\pi)^{-1}\int_{-\pi}^{\pi}\theta\,\mathrm{d}\tau, \quad \langle\tilde\theta\rangle = 0. \tag{7}$$

In accordance with equations (7) $\partial\langle\theta\rangle/\partial\tau = 0$. Then the substitution of equations (7) into equation (6) provides

$$\tilde\theta_\tau + (\langle\theta\rangle\sin\tau + \tilde\theta\sin\tau)_\zeta = \sin\tau + \Delta(\langle\theta\rangle + \tilde\theta). \tag{8}$$

The averaging of equation (8) over a period of the process leads to

$$\langle\tilde\theta\sin\tau\rangle_\zeta = \Delta\langle\theta\rangle. \tag{9}$$

In accordance with equation (9), to close the equation for the average temperature $\langle\theta\rangle$, we need to derive an equation for the sine component $\langle\tilde\theta\sin\tau\rangle$ of the first harmonic of the oscillating temperature field. In fact, the second order heat flux per unit area in a thermoacoustic device can be defined as [1]

$$J \equiv \rho_0 c_p \langle vT' \rangle = \rho_0 c_p T_c u\omega \langle \tilde{\theta} \sin\tau \rangle \equiv \rho_0 c_p T_c u\omega J_\omega. \tag{10}$$

In the standing acoustic wave under consideration, it coincides with the second order enthalpy flux per unit area [1]. Consequently, equation (9) has a clear physical meaning: for the description of the temperature distribution it is necessary to know spatial variation of acoustically induced heat flux per unit area. In order to find $\langle \tilde{\theta} \sin\tau \rangle$, equation (9) is subtracted from equation (8) with the result

$$\tilde{\theta}_\tau + (\langle\theta\rangle \sin\tau + \tilde{\theta}\sin\tau - \langle\tilde{\theta}\sin\tau\rangle)_\zeta = \sin\tau + \Delta\tilde{\theta}. \tag{11}$$

Multiplying equation (11) by $\cos\tau$ and averaging over a period, one finds

$$\langle\tilde{\theta}\sin\tau\rangle + (1/2)\langle\tilde{\theta}\sin 2\tau\rangle_\zeta = \Delta\langle\tilde{\theta}\cos\tau\rangle. \tag{12}$$

Multiplying equation (11) by $\sin\tau$ and averaging over a period, one has

$$-\langle\tilde{\theta}\cos\tau\rangle + (1/2)(\langle\theta\rangle - \langle\tilde{\theta}\cos 2\tau\rangle)_\zeta = 1/2 + \Delta\langle\tilde{\theta}\sin\tau\rangle. \tag{13}$$

In order to solve equations (12) and (13) for the first harmonic of the oscillating temperature field, one first has to find the second-harmonic components $\langle\tilde{\theta}\sin 2\tau\rangle$ and $\langle\tilde{\theta}\cos 2\tau\rangle$. Of course, the equations for the second harmonic are readily obtained by multiplying each term of equation (11) by $\cos 2\tau$ ($\sin 2\tau$) and then averaging, but their solution requires a knowledge of the third harmonic, and so on. At any step of this procedure one has an open set of equations. Thus, this problem is nearly identical to the problem of closing the moment hierarchy in the theory of fluid turbulence, which is usually referred to as the "closure problem" and is the underlying problem of turbulence theory [8].

To the best of our knowledge, in thermoacoustic theory until now the "closure problem" has been solved in the most simple way, just by neglecting the second harmonic of the temperature field in equations (12) and (13). Thus, system (9), (12), (13) becomes a closed system of equations for $\langle\theta\rangle$, $\langle\tilde{\theta}\sin\tau\rangle$ and $\langle\tilde{\theta}\cos\tau\rangle$. In particular, this system can be reduced to the following single equation for the average temperature distribution:

$$[1 + \Delta^2 - (1/2)\partial^2/\partial\zeta^2]\Delta\langle\theta\rangle = 0. \tag{14}$$

However, a traditional way of analytical evaluation of the temperature fields in a thermoacoustic stack [1, 3] avoids direct solution of equation (14) (or the truncated system of equations (9), (12), (13), i.e., with the second harmonic being neglected). Instead, the term $(\tilde{\theta}\sin\tau - \langle\tilde{\theta}\sin\tau\rangle)_\zeta$ (responsible for the generation of the hierarchy of the harmonics) is neglected already in equation (11), resulting in

$$\tilde{\theta}_\tau + \sin\tau\langle\theta\rangle_\zeta = \sin\tau + \Delta\tilde{\theta}. \tag{15}$$

Equation (15) should be solved simultaneously with equation (9).

Before discussing the solution methods for the problem described by equations (9), (15), let us establish the formal physical criteria for the validity of the "mean-field" approximation. This can be achieved by examining the spectrum of harmonics in the solutions of the equation $\theta_\tau + \sin\tau\ \theta_\zeta = \sin\tau$, which describes an adiabatic process (and follows from equation (6) when molecular heat diffusion is neglected). Deleting the adiabatic harmonic oscillation of the temperature by means of the transformation $\theta = \theta' - \cos\tau$, one derives the equation

$$\theta'_\tau + \sin\tau\ \theta'_\zeta = 0.$$

This equation has an exact analytical solution

$$\theta' = f[\zeta - (1 - \cos\tau)]$$

for any initial temperature distribution $\theta'(\tau = 0) = f(\zeta)$. Let us choose a function with a single characteristic spatial scale ζ_c, for example, $f(\zeta) = \exp(\zeta/\zeta_c)$. Thus, the solution for the temperature is

$$\theta' = \exp[(\zeta - 1)/\zeta_c] \exp(\cos\tau/\zeta_c).$$

Then one derives

$$\langle \theta' \cos\tau \rangle = \exp[(\zeta - 1)/\zeta_c]\, \mathrm{I}_1(1/\zeta_c), \quad \langle \theta' \cos 2\tau \rangle = \exp[(\zeta - 1)/\zeta_c]\, \mathrm{I}_2(1/\zeta_c),$$

where I_1, I_2 are the modified Bessel functions of integer order. Consequently, one can estimate that in the adiabatic version of equation (13) (i.e., in the absence of molecular heat transport) the contribution $(1/2)\langle \theta' \cos 2\tau \rangle_\zeta$ from the second harmonic is negligible in comparison with the contribution $\langle \theta' \cos\tau \rangle$ from the first harmonic, if $(1/2\zeta_c)\mathrm{I}_2(1/\zeta_c) \ll \mathrm{I}_1(1/\zeta_c)$. This inequality holds only if $\zeta_c \gg 1$: that is, if the spatial scale $x_c \equiv \zeta_c u$ of the axial variation of the temperature field significantly exceeds the amplitude u of particle displacement in the acoustic field. Consequently, one can conclude that the "mean-field" approximation is asymptotically valid only for the description of temperature fields that vary slowly at the scale of the acoustic displacement ($x_c \gg u$).

The classical (and the simplest) solution of system (9), (15) is obtained by assuming that the average temperature of the gas inside the stack is equal to the average temperature of the plate [1] at each point of the ζ-axis and, consequently, there is no average transverse heat flux between them (i.e., $\langle \theta \rangle_\eta = 0$) . Under this condition, equation (14) has an important simple precise solution $\langle \theta \rangle_\zeta \equiv const$, and, thus, this classical approximation is also known as an approximation of constant axial temperature gradient in the thermoacoustic stack [1]. When $\langle \theta \rangle_\zeta$ is constant, the oscillating temperature field can be easily found from equation (15) and then the acoustically induced temperature flux (10) can be evaluated.

In the absence of average transverse heat exchange between the gas and the plates inside the stack, the theory predicts the heat exchange to be delta-localized at the terminations of the plates. However, recent computer simulations of the heat transport in the vicinity of an isolated stack in the acoustic field (which were based on the numerical solution of a complete set of non-linear hydrodynamical equations) [6] do not support this conclusion. The results of reference [6] demonstrate that, though the effective heat exchange between the gas and the plates really takes place in the vicinity of the plate termination (at distances not exceeding the particle displacement from the plate termination), it is not delta-localized.

In an attempt to confirm the numerical predictions of reference [6] analytically, the possibility of the transverse energy transport inside the stack ($\langle \theta \rangle_\eta \neq 0$) has been taken into account in reference [3]. The predictions of numerical analysis [6] have been confirmed in reference [3]. But, because of the complicated transverse structure of the acoustically induced heat flux and the temperature field, this confirmation has been obtained only by combining analytical methods with numerical ones. As for the design and optimization of thermoacoustic devices more simple (and preferably, purely analytical) models are highly desirable, so it was proposed in references [3, 9] to overcome the complexity of the description of transverse heat transfer by the differential operator $\partial^2\langle \theta \rangle/\partial\eta^2$ via modelling its action by a term proportional to the difference of some characteristic average temperatures of the gas and the plate ($\propto -h(\langle \theta \rangle - \langle \theta \rangle_{plate})$). The characteristic temperatures can be defined by a procedure of averaging across the stack cross-section, while the coefficient h, in general, depends on the distance y_0 between the plates, the thermophysical parameters of the gas and the plates, and on the frequency [3, 9]. In fact, this approximation consists in the modelling of the differential operator by an effective relaxational operator, since the operator $-h(\langle \theta \rangle - \langle \theta \rangle_{plate})$ just describes a physically clear tendency of the system in the absence of acoustic oscillations to relax to the state with $\langle\theta\rangle = \langle\theta\rangle_{plate}$. The description of the transverse heat transfer by a term proportional to the difference of the temperatures between the moving fluid and the solid reminds one of the application of Newton's law of cooling in the general theory of the heat transfer in convection mode [2, 10]. It should be mentioned, however, that Newton's law of cooling is not a solution for the problem of transverse heat exchange between the gas and the plates, but is rather its reformulation, because the precise determination of the heat exchange coefficient h demands itself the solution of the heat transfer problem (see, for example, reference [3]). But, concentrating attention on the physics of the non-linear phenomena of the axial heat transfer, we will treat the parameter h (and, as a consequence, the relaxation parameter R in the next section) as phenomenological. The relaxational approximation discussed above is also quite frequently used in the analysis of the cyclic flow regenerators [11, 12]. Under this approximation, a completely analytical description of the temperature fields and the heat fluxes around an isolated stack and also near a close contact of two different stacks was obtained in reference [9].

However, the results obtained in references [3, 9], of course, do not reproduce the computer simulation [6] precisely. Among the reasons for the discrepancy can be the use of the "mean-field" approximation. Consequently, it is necessary to evaluate the preciseness of the "mean-field" theory and the limits of its validity. For this purpose, we will extend the application of relaxational operator (introduced in references [3, 9] for the description of the edge effects in averaged transverse heat exchange) also to the description of transverse heat exchange in an oscillating temperature field. It should be noted that the relaxation operators have been used many times before for the analysis of the periodic and transient processes (for the latest publications see, for example, references [11–15]). However, here this approach will be applied for the first time to the completely non-linear analysis of the thermoacoustic edge effects at the scale of the order of particle displacement in acoustic field. Exact analytical solutions for the non-linear problem will be presented also for the first time.

3. "RELAXATION-TIME" APPROXIMATION IN THERMOACOUSTICS

In the case of an isolated stack with infinite thermal conductivity [6], which is kept at the initial temperature ($\langle\theta\rangle_{plate} = 0$) and, consequently, plays itself the role of a heat exchanger with external to resonator systems, we change the operator $(D_0/y_0^2\omega)\partial^2/\partial\eta^2$ in equations (9), (12)–(15) to $(-1/R)$. Here R is a relaxation parameter $R \equiv \omega\tau_R$, while τ_R is the characteristic relaxation time, whose dependence on y_0, ω, etc. should be modelled by solving the problem of the transverse heat conduction [3, 9, 10]. When $\tau_R \to \infty$, the acoustic oscillations inside the stack are adiabatic, while in the limiting case $\tau_R \to 0$ they are isothermal. It should be also mentioned that now both $\langle\theta\rangle$ and $\tilde{\theta}$ in equations (9), (12)–(15) are considered as averages over a cross-section of the stack.

In the "relaxation-time" approximation adopted in the following, equation (14) takes the form

$$\left\{\left(1+\frac{1}{R^2}\right)-\frac{1}{2}\frac{\partial^2}{\partial\zeta^2}\left[1+\frac{4D_0}{Ru^2\omega}-2\left(\frac{D_0}{u^2\omega}\right)^2\frac{\partial^2}{\partial\zeta^2}\right]\right\}\left\{-\frac{1}{R}+\frac{D_0}{u^2\omega}\frac{\partial^2}{\partial\zeta^2}\right\}\langle\theta\rangle = 0. \quad (16)$$

In the currently operating thermoacoustic engines, the molecular heat transport in the direction of the x-axis is usually negligible in comparison with acoustically induced heat flux [1, 3, 16, 17]. The terms related to molecular heat transport in equation (16) (i.e., those proportional to D_0) can be omitted if

$$4D_0/Ru^2\omega \ll 1, \quad 2(D_0/u^2\omega)^2\partial^2/\partial\zeta^2 \ll 1, \quad (D_0/u^2\omega)\partial^2/\partial\zeta^2 \ll 1/R. \tag{17}$$

The last inequality from equation (17) corresponds to the case where the transverse molecular heat transport is more important than the axial molecular heat transport. Under conditions (17), equation (16) becomes

$$\frac{R}{2(1+R^2)}\frac{\partial^2}{\partial\zeta^2}\langle\theta\rangle - \frac{\langle\theta\rangle}{R} = 0.$$

This equation demonstrates that the acoustic oscillations induce in the stack an effective thermal diffusivity

$$D_{eff} \equiv \frac{R}{2(1+R^2)}u^2\omega \equiv \frac{2R}{(1+R^2)}D_{ac}.$$

This acoustically induced thermal diffusivity is the highest for $R = 1$ ($D_{eff}(R=1) = D_{ac}$). It diminishes both when the relaxation parameter increases ($D_{eff} \approx (2/R)D_{ac} \approx u^2/(4\tau_R) \ll D_{ac}$, in the quasi-adiabatic regime $R \gg 1$) and when the relaxation parameter diminishes ($D_{eff} \approx 2RD_{ac} \approx (u\omega)^2\tau_R/2 \ll D_{ac}$ in the quasi-isothermal regime $R \ll 1$).

The general solution of the simplified equation is

$$\langle\theta\rangle = C_1 \exp[(\sqrt{2(1+R^2)}/R)\zeta] + C_2 \exp[-(\sqrt{2(1+R^2)}/R)\zeta]. \tag{18}$$

Consequently, conditions (17) for the validity of solution (18) can be rewritten as

$$D_0/u^2\omega \ll \min\{R/4,\ 2/(1+R^2)\} \quad \text{or } D_0/D_{ac} \ll \min\{R, 8/(1+R^2)\}, \tag{19}$$

which can be always satisfied for a sufficiently high level of the acoustic oscillations in the resonator. As $\min\{R/4, 2/(1+R^2)\}$ is always less than 1/2, then solution (18) is valid only when the acoustically induced diffusivity $D_{ac} \propto u^2\omega$ significantly exceeds molecular thermal diffusivity ($D_{ac} \gg D_0$). In accordance with equation (18) the characteristic normalized scale of spatial variation of temperature field is $\zeta_0 \equiv R/\sqrt{2(1+R^2)}$ and it is always less than $1/\sqrt{2}$ (the value corresponding to the adiabatic case). Thus, the characteristic spatial scale $x_0 = \zeta_0 u$ is always less than the particle displacement amplitude u in the acoustic field. Consequently, in a stack with a length exceeding a few particle displacement amplitudes, the description of the temperature field distribution near the terminations can be obtained independently near each termination. For example, near the right termination ($\zeta = 0$) of a stack positioned in the region $\zeta \leqslant 0$ (Figure 1) the temperature distribution is described by

$$\langle\theta\rangle = C_1 \exp[(\sqrt{2(1+R^2)}/R)\zeta]. \tag{20}$$

To determine the constant C_1 it is necessary to use the condition of the absence of an acoustically induced heat flux through the stack termination $J_\omega(\zeta = 0) = 0$ (this condition holds because the gas oscillations in the region $\zeta \geqslant 0$ are adiabatic due to the absence of the plates).

Under condition (19), equation (15) is reduced to

$$\tilde{\theta}_\tau + \tilde{\theta}/R = (1 - \langle\theta\rangle_\zeta)\sin\tau$$

with a solution

$$\tilde{\theta} = [R/(1+R^2)](1 - \langle\theta\rangle_\zeta)(\sin\tau - R\cos\tau),$$

leading to the following description of the acoustically induced dimensionless (normalized) heat flux per unit area:

$$J_\omega \equiv \langle \tilde{\theta} \sin \tau \rangle = [R/2(1+R^2)](1 - \langle \theta \rangle_\zeta). \tag{21}$$

In accordance with equations (20) and (21), the acoustically induced heat flux per unit area far from the stack termination ($\zeta \leqslant -2$) is equal to $J_\omega(\zeta = -\infty) = R/2(1+R^2)$, with a maximum value equal to 1/4 in the case of a stack with an optimal relaxation parameter $R = 1$. Thus, the optimal conditions for heat pumping are in the intermediate regime between the isothermal and the adiabatic regimes (in full correlation with classical predictions [1]). Hydrodynamical temperature flux falls down both in a quasi-adiabatic stack $J_\omega(R \gg 1) \propto 1/(2R) \ll 1$ and in a quasi-isothermal stack $J_\omega(R \ll 1) \propto R/2 \ll 1$.

From the condition $J_\omega(\zeta = 0) = 0$ one finds that the amplitude of average temperature variation is

$$C_1 = \langle \theta(\zeta = 0) \rangle = R/\sqrt{2(1+R^2)}. \tag{22}$$

From equations (22) and (20) it follows that the stack termination temperature diminishes in the quasi-isothermal regime ($\langle \theta(\zeta = 0) \rangle = R/\sqrt{2}$ when $R \ll 1$) and saturates in the quasi-adiabatic regime ($\langle \theta(\zeta = 0) \rangle = 1/\sqrt{2}$ when $R \gg 1$).

It is interesting to note that the above-assumed "mean-field" approximation can be formally applied for the description of the average temperature field even in the adiabatic regions of the resonator. For example, the complete solution for the temperature distribution near the right termination ($\zeta = 0$) of an isolated stack (i.e., in the absence of the right heat exchanger (see Figure 1)) can be found:

$$\langle \theta(\zeta \leqslant 0) \rangle = [R/(2\sqrt{(1+R^2)})] \exp[(\sqrt{2(1+R^2)}/R)\zeta],$$
$$\langle \theta(\zeta \geqslant 0) \rangle = [R/(2\sqrt{(1+R^2)})] \exp[-\sqrt{2}\zeta].$$

However, in accordance with equation (21), as $J_\omega(R = \infty) = 0$, the "mean-field" approximation fails to describe the hydrodynamic (advective) heat transport through the adiabatic gap that should take place if the distance between the different stacks does not exceed the maximum particle displacement in the acoustic field. Thus, in the analysis of the heat transport between the stack and the heat exchangers, the "mean-field" approximation should be avoided.

We have found that, indeed, in the "relaxation-time" approximation, the analytical solution of some important thermoacoustic problems can be obtained beyond the "mean-field" approximation.

4. BEYOND "MEAN-FIELD" APPROXIMATION IN THERMOACOUSTICS

In the "relaxation-time" approximation, under the conditions of equation (19) (i.e., when the acoustically induced heat transport dominates), equation (6) for the total temperature takes the form

$$\theta_\tau + \sin \tau \, \theta_\zeta = -\theta/R + \sin \tau, \tag{23}$$

and can be solved analytically. Then the average temperature field $\langle \theta \rangle$ and the heat flux per unit area $J_\omega \equiv \langle \theta \sin \tau \rangle$ can be evaluated by the averaging procedure defined in equation (7).

It is suitable, by introducing a new function $\theta' = \theta + \cos \tau$, to delete from the total temperature field the adiabatic oscillations which do not contribute to $\langle \theta \rangle$ and $J_\omega \equiv \langle \theta \sin \tau \rangle$. It is also suitable to transfer equation (23) to a system of co-ordinates moving together with the oscillating particle, by considering $\zeta = \zeta_0 + (1 - \cos\tau)$, where ζ_0 is the co-ordinate of a particle at the moment of time $\tau = 0$. In the Lagrange co-ordinates (τ, ζ_0), equation (23) becomes

$$\theta'_\tau = -\theta'/R + \cos\tau/R. \tag{24}$$

Equation (24) describes a variation of the temperature of a particle with initial co-ordinate ζ_0 when it oscillates in the acoustic field. As an example, we evaluate, by solving equation (24), the temperature field and heat fluxes in the vicinity of an adiabatic gap separating two stacks. The first stack is positioned as before in the region $\zeta \leqslant 0$, and the second is positioned in the region $\zeta \geqslant d$, where d is the width of an adiabatic gap normalized by the local particle displacement amplitude (see Figure 2). Both of them are kept via heat exchange with the external to the resonator systems at $\langle\theta\rangle_{plate} = 0$. The system of equations to be solved is

$$\theta'_\tau = -\theta'/R_1 + \cos\tau/R_1 \ (\zeta \leqslant 0), \quad \theta'_\tau = 0 \ (0 \leqslant \zeta \leqslant d),$$
$$\theta'_\tau = -\theta'/R_2 + \cos\tau/R_2 \ (\zeta \geqslant d), \tag{25}$$

subjected to the conditions of the continuity of the particle temperature when a particle crosses the boundaries $\zeta = 0$ and d. Only the particles with initial co-ordinates satisfying the inequality $-2 + d \leqslant \zeta_0 \leqslant 0$, which are initially in the left stack and can penetrate into the right stack during a period of oscillation, can contribute to heat flux across the gap $(0 \leqslant d \leqslant 2)$. A particle with an initial co-ordinate satisfying the above inequality leaves the first (left) stack at the time moment $\tau_+^{(1)} = \arccos(1 + \zeta_0)$, reaches the second (right) stack at the moment $\tau_+^{(2)} = \arccos(1 + \zeta_0 - d)$, leaves the second stack at $\tau_-^{(2)} = 2\pi - \tau_+^{(2)}$, and returns in the first stack at the moment $\tau_-^{(1)} = 2\pi - \tau_+^{(1)}$ (which, due to the periodicity of the process considered, is equivalent to the moment $-\tau_+^{(1)}$). The variation of particle temperature when it moves inside the first stack $(-\tau_+^{(1)} \leqslant \tau \leqslant \tau_+^{(1)})$ is described by the solution of equation (25) as

$$\theta'_1 = C_1 \exp(-\tau/R_1) + [1/(1 + R_1^2)](\cos\tau + R_1 \sin\tau). \tag{26}$$

Similarly, when a particle moves inside the second stack $(\tau_+^{(2)} \leqslant \tau \leqslant 2\pi - \tau_+^{(2)})$ its temperature is described by

$$\theta'_2 = C_2 \exp(-\tau/R_2) + [1/(1 + R_2^2)](\cos\tau + R_2 \sin\tau). \tag{27}$$

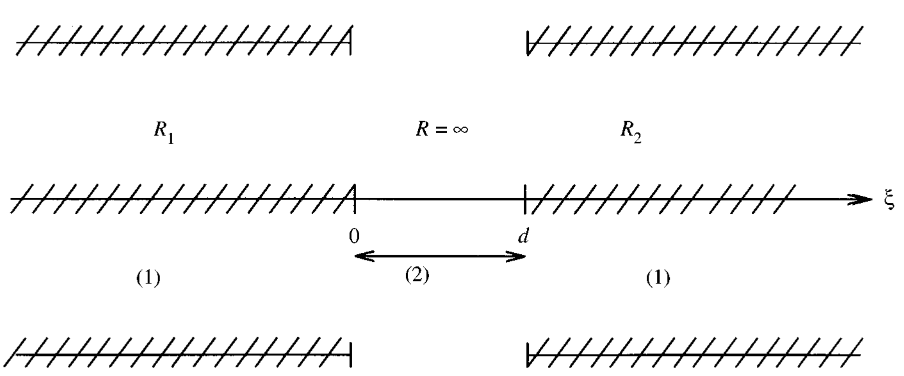

Figure 2. The configuration of the plates assumed for the analysis of the thermoacoustic heat flux across the adiabatic gap (1—the regions occupied by the stacks, 2—the adiabatic gap between the stacks).

Because the temperature of a particle, in accordance with equation (25), does not change when it crosses the adiabatic region $0 \leqslant \zeta \leqslant d$, the constants in equations (26) and (27) can be found from the conditions

$$\theta'_1(\tau_+^{(1)}) = \theta'_2(\tau_+^{(2)}), \quad \theta'_1(-\tau_+^{(1)}) = \theta'_2(\tau_-^{(2)}). \tag{28}$$

After that, the average temperature and the average temperature flux at any point can be found by the integration of the contributions from all particles reaching this point during the period of oscillations (and, of course, by dividing by 2π).

Let us start the analysis with the simplest limiting case where the distance d between the stacks exceeds the maximal particle displacement ($d > 2$) and, consequently, there is no heat transport across the gap. In this situation the particles with initial co-ordinates $-2 \leqslant \zeta_0 \leqslant 0$ contribute to the variation of $\langle\theta\rangle$ near the stack termination. The temperature θ' of the particle inside the first stack is described by equation (26), subjected to the boundary condition $\theta'_1(\tau_+^{(1)}) = \theta'_1(-\tau_+^{(1)})$. The final presentation of the average temperature in the region $-2 \leqslant \zeta \leqslant 0$ (note that $\langle\theta(\zeta \leqslant -2)\rangle = 0$) is

$$\langle\theta\rangle = \frac{R_1}{1+R_1^2}\frac{1}{\pi}\int_0^{\arccos(-1-\zeta)} \frac{\sin\tau'}{\sinh(\tau'/R_1)}\cosh(\tau/R_1)\,\mathrm{d}\tau, \quad \tau' = \arccos(\cos\tau + \zeta). \tag{29}$$

Consequently, the temperature of the stack termination ($\zeta = 0$) is described by

$$\langle\theta(\zeta = 0)\rangle = \frac{R_1}{1+R_1^2}\frac{1}{\pi}\int_0^{\pi} \frac{\sin\tau}{\tanh(\tau/R_1)}\,\mathrm{d}\tau. \tag{30}$$

The analysis of solution (30) demonstrates that this solution confirms the asymptotic behaviour of $\langle\theta(\zeta = 0)\rangle$ predicted in the "mean-field" approximation (see equation (22)), that is, it confirms the saturation of $\langle\theta(\zeta = 0)\rangle$ in the adiabatic limit ($R_1 \gg 1$) and its diminishing ($\propto R_1 \ll 1$) in the isothermal limit. However, the average temperature of the stack termination, predicted by the "mean-field" theory (equation (22)), is approximately 10–20% higher than the value, predicted by solution (30) based on the precise description of hydrodynamical transport. The dependence of the average temperature of the stack termination on the relaxation parameter R_1 described in the "mean-field" approximation by equation (22) and by the solution (30) (which avoids this approximation) are presented in Figure 3. In Figure 4, we present the spatial distribution of average temperature $\langle\theta\rangle$ near the stack termination described by equation (29) for different values of $R_1 \equiv R$. Note that $\langle\theta\rangle$ in Figure 4 is additionally normalized to $\langle\theta(\zeta = 0)\rangle$ (equation (30)) in order to appreciate the significant diminishing of the characteristic spatial scale of the distribution when the relaxation parameter R diminishes. It should be pointed out that, in the "relaxation-time" approximation, the transverse heat exchange between the gas and the plate is (by assumption) proportional to $\langle\theta\rangle$ (in the case $\langle\theta\rangle_{plate} = 0$). Consequently, Figure 4 simultaneously provides the distributions of the transverse heat fluxes evaluated in references [3, 6] by other methods. The results presented in Figure 4 are in agreement with those results of the numerical evaluation of the problem [6], which have been established for negligible Prandtl numbers. Finally, in Figure 5 we compare the predictions of the "mean-field" approximation (equations (20), (22)) and of the solution (30) for the temperature distribution in the important (optimal) regime $R_1 = 1$. These results provide an idea of the preciseness of the "mean-field" approximation for thermoacoustics. It can be concluded that the "mean-field" approximation, due to its simplicity, and due to the sufficiency for the estimations preciseness is definitely a very suitable mathematical tool for thermoacoustics. However, for the description of heat transport between adiabatically isolated elements of the thermoacoustic devices, the "mean-field" approximation should be avoided.

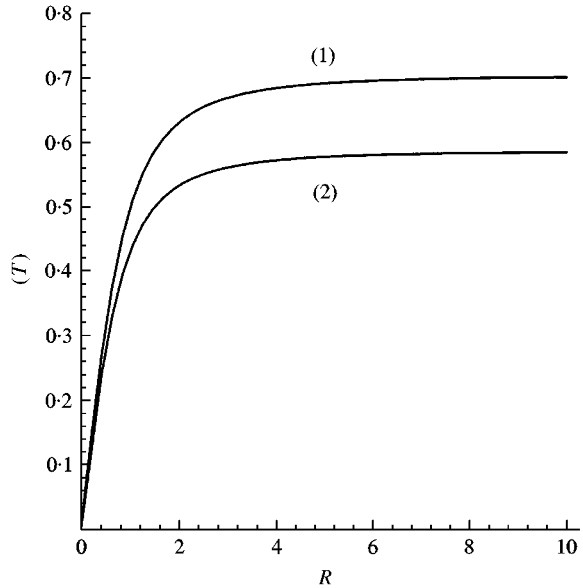

Figure 3. The dependence of the average dimensionless temperature $\langle\theta(\zeta=0)\rangle$ of the stack termination on the dimensionless relaxation parameter R (curve 1—the prediction of the "mean-field" theory, curve 2—the prediction of the theory based on the "relaxation-time" approximation).

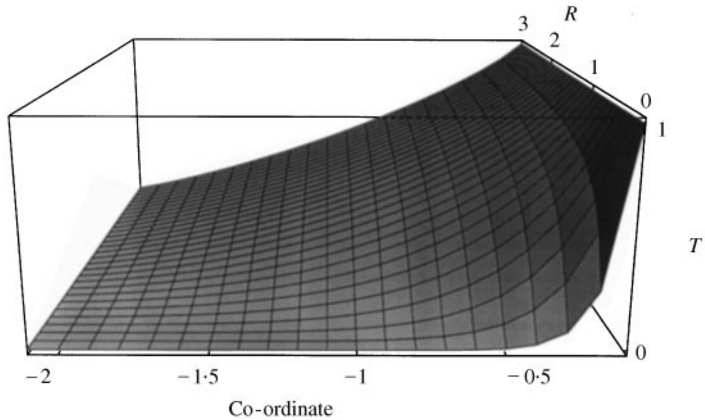

Figure 4. The distribution of the normalized average temperature $\langle\theta\rangle/\langle\theta(\zeta=0)\rangle$ near the stack termination as a function of the relaxation parameter R and the dimensionless co-ordinate ζ predicted in the "relaxation-time" approximation.

The derived solution (26)–(28) leads to the following description of the normalized heat flux per unit area $\langle\theta(\zeta=0)\sin\tau\rangle$ across the adiabatic gap:

$$J_\omega(\zeta=0)=\frac{1}{\pi}\left\{\frac{R_1}{2(1+R_1^2)}\left[\arccos(d-1)+(1-d)\sqrt{1-(1-d)^2}\right]\right.$$

$$\left.-\int_0^{\arccos(d-1)} C_1(\tau)\sinh\left(\frac{\tau}{R_1}\right)\sin\tau\,\mathrm{d}\tau\right\},$$

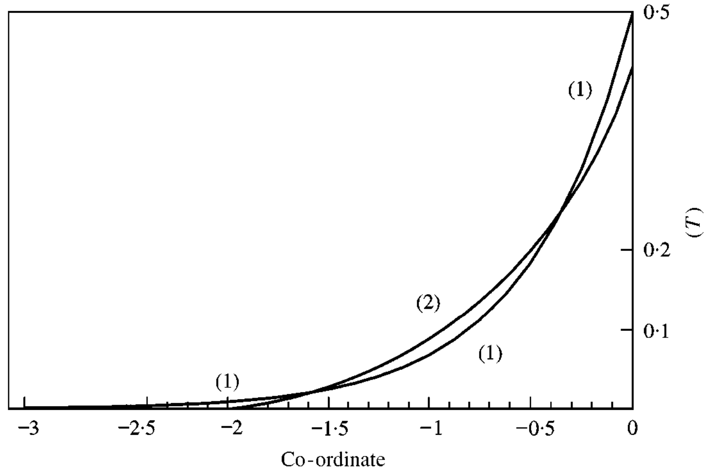

Figure 5. The distribution of the average dimensionless temperature $\langle\theta\rangle$ (in the case of the optimal value $R_1 = 1$ of the relaxation parameter) predicted by the "mean-field" theory (curve 1) and by the "relaxation-time" model (curve 2) as a function of the dimensional co-ordinate ζ.

$$C_1(\tau) = \frac{\begin{bmatrix} (1/(1+R_2^2)\cos\tau' - 1/(1+R_1^2)\cos\tau)\sinh(\tau'/R_2 - \pi/R_2) \\ +\,(R_2/(1+R_2^2)\sin\tau' - R_1/(1+R_1^2)\sin\tau)\cosh(\tau'/R_2 - \pi/R_2) \end{bmatrix}}{\sinh(\tau'/R_2 - \pi)/R_2 - \tau/R_1},$$

$$\tau' = \arccos(\cos\tau - d). \tag{31}$$

The dependence of the heat flux per unit area on the normalized width d of the adiabatic gap and on the relaxation parameter $R_2 \equiv R$ of the second stack in the case where the relaxation parameter of the first stack is equal to an optimal value ($R_1 = 1$) is presented in Figure 6. In accordance with the obtained results (see Figure 6), in the absence of the gap ($d = 0$) the maximal heat flux between the stacks is achieved when they are matched (i.e., when they have equal relaxation parameters $R_2 = 1 = R_1$). An interesting and rather unexpected feature of the description obtained (equation (31), Figure 6) is the prediction that for $R_1 = 1$, $R_2 \equiv R \leqslant \sqrt{2}$ the heat flux increases when the separation distance d between the stacks increases from $d = 0$ to some finite value $d < 2$. This effect corresponds to the diminishing of the temperature of the first stack termination observed in Figure 7, where $\langle\theta(\zeta = 0)\rangle$ is presented as a function of d and $R_2 \equiv R$ (for $R_1 = 1$), as predicted by an analytical solution

$$\langle\theta(\zeta=0)\rangle = \frac{1}{\pi}\left\{\int_0^{\arccos(d-1)} C_1(\tau)\cosh\left(\frac{\tau}{R_1}\right)\mathrm{d}\tau + \int_{\arccos(d-1)}^{\pi} C_1(\tau, R_2 = \infty)\cosh\left(\frac{\tau}{R_1}\right)\mathrm{d}\tau\right\} \tag{32}$$

(which is valid, in fact, for an arbitrary value of R_1). Here the function $C_1(\tau, R_1, R_2)$ is described in equation (31). The second integral in equation (32) accounts for the contribution to the average temperature from the particles with initial co-ordinates in the region $2 \leqslant \zeta_0 \leqslant -2 + d$. They do not contribute to the heat flux per unit area (equation (31)) but they do contribute to the average temperature in equation (32).

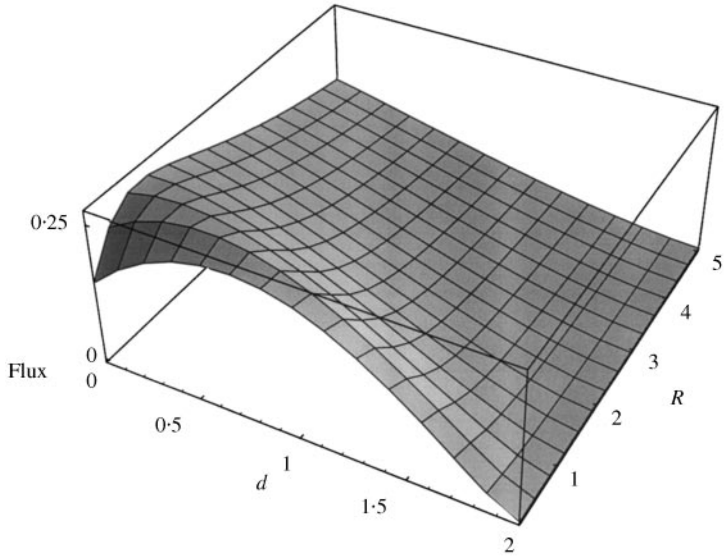

Figure 6. The dependence of the dimensionless heat flux per unit area J_ω across the adiabatic gap on the dimensionless width d of the gap and on the relaxation parameter $R_2 \equiv R$ of the second stack. The relaxation parameter of the first stack is optimized ($R_1 = 1$).

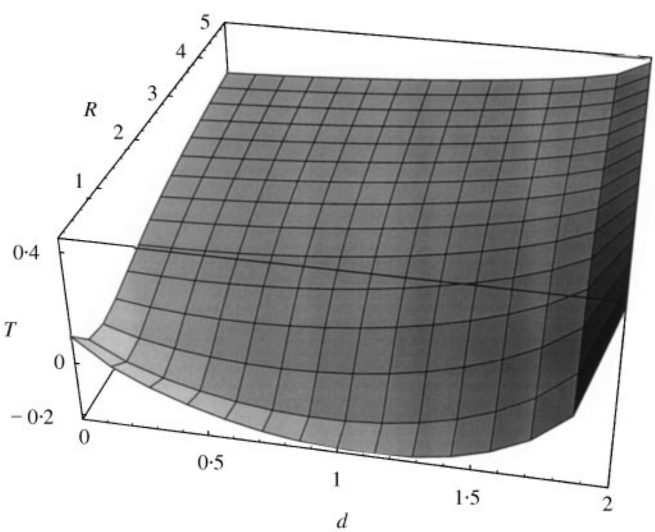

Figure 7. The dependence of the average dimensional temperature $\langle \theta(\zeta = 0) \rangle$ of the first stack termination on the dimensionless width d of the gap and on the relaxation parameter $R_2 \equiv R$ of the second stack. The relaxation parameter of the first stack is optimized ($R_1 = 1$).

The effect of the thermoacoustic heat flux increasing with increasing separation distance between the stacks (Figure 6) is, in fact, important only for $R_2 \equiv R \leqslant 1$, $d \leqslant 0{\cdot}5$. This observation provides an opportunity to propose the following qualitative explanation for the effect considered. In the case, $R_1 = 1$, $R_2 < 1$, $d = 0$ the total process of the heat exchange between the oscillating fluid element and the stacks is too fast in comparison with the optimal regime $R_2 = R_1 = 1, d = 0$. When the stacks with $R_1 = 1, R_2 < 1$ are a little bit separated in space, this introduces additional adiabaticity in the total process of the fluid motion (because the separation gap is adiabatic). Thus the system shifts in the direction of the optimal regime. In other words, if one defines some characteristic relaxation parameter R_c (averaged over all fluid elements, which are able to carry heat across the gap, and averaged over the period of the oscillations with accounting of the fact that each of these elements spends a part of the period inside the stack with $R = R_1$ and another part inside the stack with $R = R_2$) then it will appear that in the case $R_1 = 1, R_2 < 1$ the characteristic parameter R_c is less than an optimal one $R_c^{opt} = 1$. Thus, it is clear that by "adding" adiabaticity (for example, by introducing and optimizing the width of the adiabatic gap) the thermoacoustic heat flux can be increased.

The possibility of an application of the predicted effect for the optimization of the hydrodynamical heat transfer between the thermoacoustic stack and the heat exchangers deserves detailed investigation and will be a subject for the future research.

5. CONCLUSIONS

It has been demonstrated that by extending the application of the "relaxation-time" approximation to the description of the transverse heat exchange in the total temperature field it is possible to avoid the use of the "mean-field" approximation in thermoacoustics. The analytical results obtained, based on the exact description of the axial hydrodynamical (advective) transport of heat in the standing acoustic wave, are not just more precise for temperature field evaluation than those of the "mean-field" approximation. Importantly, the approach developed provides the description of the heat transport between the thermoacoustic elements separated by an adiabatic gap, which is absolutely impossible to obtain in the "mean-field" approximation.

The perspectives for further developments of the proposed method consist in taking into account the gas viscosity in order to investigate the dependence of the above-described phenomena on the Prandtl number [6]. For the solution of this problem and also for the evaluation of the relaxation time $\tau_R = \tau_R(\omega, y_0, \text{etc.})$ the methods developed in the acoustics of porous materials [18] could be very useful.

Another extension of the theory may include an account of the possibility that the local phasing between the oscillations of the pressure and the oscillations of the velocity deviates from the one in the standing wave described by equation (5). In particular, the relaxation time approximation can be applied for the analysis of travelling-wave thermoacoustic devices as well.

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