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EVALUATION OF THE NON-LINEAR DYNAMIC RESPONSE TO HARMONIC EXCITATION OF A BEAM WITH SEVERAL BREATHING CRACKS

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The aim of this article is to present a technique capable of evaluating the dynamic response of a beam with several breathing cracks perpendicular to its axis and subjected to harmonic excitation. The method described is based on the assumption of periodic response and that cracks open and close continuously. In this way, a non-linear system of algebraic equations can be defined and solved iteratively, with the advantage over direct numerical integration of the equation of motion of being easier and therefore faster to compute.

In this article, the vibrational response to harmonic force of a cantilever beam with cracks of different size and location is analyzed using this "harmonic balance" approach and the results are compared with those obtained through numerical integration.

1. INTRODUCTION

Currently, the research community is demonstrating considerable interest in techniques which, by processing the dynamic response of the structure under test, can identify its damaged state. During the past few years, this progress has led to the development of a large number of methods most of which, however, have the limitation of being only valid for linear structures [1, 2].

Various studies performed over the last decade have indicated that a beam with a breathing crack, i.e., one which opens and closes during oscillation, shows non-linear dynamic behaviour because of the variation in the structural stiffness which occurs during the response cycle. This phenomenon was observed initially by Gudmunson [3] through experimental tests aimed at correlating the location and extent of a crack with the variation in natural frequencies.

The main result obtained by several researchers is that the shift in natural frequencies due to a breathing crack cannot be determined through mathematical models if it is assumed that the crack is always open during the motion of the beam. In particular, a beam with a breathing crack has natural frequencies which are intermediate between natural frequencies of the undamaged and of the faulty beam with crack always open [4].

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It is clear that in these cases vibration-based inspection methods based on the hypothesis that the structure under analysis behaves linearly may lead to incorrect conclusions regarding the state-of-damage. Correspondingly, it would seem appropriate to study the non-linear dynamic behaviour of beams with breathing cracks in order to investigate the potential for developing more generally applicable vibration-based inspection techniques.

1.1. PREVIOUS STUDIES

Several researchers have addressed the problem of a beam with a transverse closing crack from the analytical standpoint. Zastrau [5] demonstrated the bilinear behaviour by using the finite element method to determine the dynamic response of a simply supported beam. Qian *et al.* [6] observed that the difference between the forced vibration amplitude of cracked and uncracked beams becomes lower if the model of a breathing crack is considered. Ibrahim *et al.* [7] applied a method based on a lumped parameter model to analyze the frequency response function of a cantilever beam and in reference [8] performed some experimental tests to validate their method. Collins *et al.* [9] used direct numerical integration to study forced vibrations of a beam with breathing crack.

In reference [10] Friswell and Penny studied the non-linear behaviour of a beam with a closing crack and analyzed its forced response to harmonic excitation for a frequency near the first natural frequency of the beam, such that it can be considered as a simple one-degree-of-freedom (d.o.f.) system with bilinear stiffness. The analysis of frequency response functions and of the response to harmonic excitation, obtained through numerical integration, highlighted the presence of peaks in the response spectrum at integer multiples of the excitation frequency, a common property for non-linear systems. Similar results were obtained by Ruotolo *et al.* [11] by performing numerical integration of the equation of motion and using a finite element model of the beam, as well as by Shen and Chu [4], who tried to determine the variations in the response spectrum due to the presence of a breathing crack. In a following article [12], the latter authors used two square-wave functions representing the modification in stiffness, at low-frequency excitation, during the beam motion in order to reduce calculation times.

The non-linear behaviour of beams with breathing cracks has been also highlighted by Ruotolo *et al.* [11, 13, 14], Crespo *et al.* [15] and Pugno *et al.* [16] where the concept of higher order frequency response functions [17] has been applied to characterize the non-linearity due to the closing crack.

In reference [18] Ostachowicz and Krawczuk used the harmonic balance method to determine the response of a cantilever beam with closing crack taking advantage of the great reduction of calculation times permitted by this technique with respect to numerical integration. However, their approach to solve the problem does not permit one to consider either the out-of-phase relation between forcing term and structural response, or the fact that the non-linear nature of the system may cause appreciable distortion in the response waveform due to the higher harmonic components which in turn influence the activation of the crack.

In almost all the previous articles it is assumed that the crack opens or closes instantaneously while there is experimental evidence that the passage from closed to open crack and *vice versa* happens in a smoother way. This was demonstrated by Clark *et al.* [19] during some studies on the effect of crack closure on the accuracy of different non-destructive testing predictions of crack size. By using four-point bending specimens and measuring the crack opening, they obtained the relation opening–closing displacement

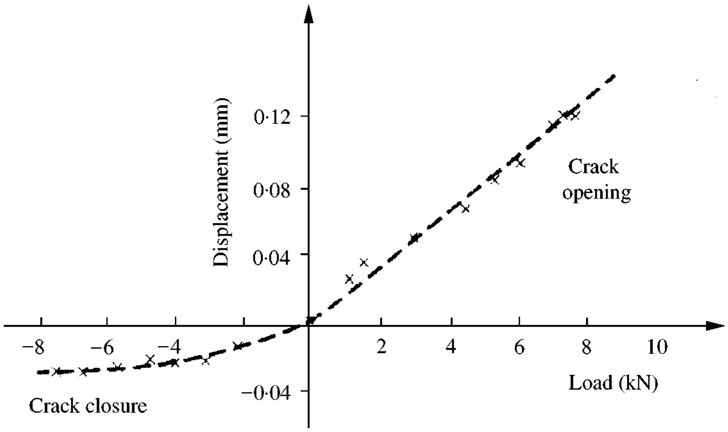

Figure 1. Clark's relation for opening–closing displacement versus applied load (after reference [2]).

versus applied load shown in Figure 1, which highlights that it should be necessary to pay great attention to the crack closure effect.

1.2. PROJECT SCOPE

Starting from this body of previous studies, research was initiated using the work by Ostachowicz and Krawczuk [18] as a basis for considering a beam with one continuously open crack, i.e., where stiffness varies linearly between two extremes assumed with the crack fully open and with the beam undamaged. In reference [16] it was shown that higher order harmonics arise in the frequency response which highlight the structure's non-linear behaviour.

In the literature surveyed, relatively few studies have dealt with the direct and/or the inverse problem of multi-cracked structures with always open cracks, for example references [20–25]; moreover, there is apparently a complete lack of documented research regarding multi-cracked structures with breathing cracks.

In order to address this aspect, the aim of the work documented in this article has been to extend the method discussed in reference [16], to the general case of several breathing cracks and, subsequently, by introducing a smooth crack closure.

As in reference [16], the dynamic response of the beam under analysis is determined by applying a numerical technique based on the harmonic balance, this being considerably more rapid to compute than by using direct numerical integration to solve a non-linear system of algebraic equations iteratively. The article is completed by numerical examples which permit the comparison of the dynamic response evaluated through the numerical integration with that obtained by applying the method presented.

The method proposed can be considered applicable to systems such as the cracked beam in which the extent of non-linear dynamic behaviour, which is related to both the characteristics of the system and the level of excitation applied, is relatively weak; as a consequence the vibrational response spectrum to harmonic forcing is assumed to contain only super-harmonic and not sub-harmonic components. In this context, despite the fact that in reference [26] the authors have reported the occurrence of period doubling albeit "a relatively rare phenomenon", the existence of sub-harmonics in the response spectrum would indicate an excitation level so high as to promote relatively strong non-linear behaviour which the method presented in this article does not attempt to address. Correspondingly, a simple model which takes into account only flexural motion and can deal with super-harmonics has been used, as in most of the articles cited previously.

2. EQUATION OF MOTION

2.1. DETERMINATION OF THE CRACK FUNCTION

In the analysis described in this section, a cantilever beam with M breathing cracks is considered. Discretizing the structure by using Euler-type finite elements with two nodes and two degrees of freedom per node, the following equation of motion is obtained:

$$[\mathbf{M}]\{\ddot{\mathbf{q}}\} + [\mathbf{D}]\{\dot{\mathbf{q}}\} + [\mathbf{K}]\{\mathbf{q}\} + \sum_{m=1}^{M} [\Delta\mathbf{K}^{(m)}] f^{(m)}(\{\mathbf{q}\})\{\mathbf{q}\} = \{\mathbf{P}\}, \tag{1}$$

where $[\mathbf{M}]$ is the mass matrix, $[\mathbf{D}]$ the damping matrix, $[\mathbf{K}] + \sum_{m=1}^{M}[\Delta\mathbf{K}^{(m)}]$ the stiffness matrix of the undamaged beam and $[\Delta\mathbf{K}^{(m)}]$ is half of the variation in stiffness introduced when the mth crack is fully open; the stiffness matrix for a cracked element can be evaluated according to references [6, 27]. Moreover, $\{\mathbf{P}\}$ is the vector of the applied forces, $\{\mathbf{q}\}$ is the vector containing the generalized displacements of the various nodes (translations and rotations) and $f^{(m)}(\{\mathbf{q}\})$, which can be called *crack function*, assumes values in the range -1 to 1.

With the approach followed, the crack function has the important role of representing the transition from closed to open crack and *vice versa*. By assuming that this transition is instantaneous and hence discontinuous, $f^{(m)}(\{\mathbf{q}\})$ corresponds to a step function and has the sign of the curvature of the mth cracked element. When all cracks are fully closed, this function is positive for each element, $f^{(m)}(\{\mathbf{q}\}) = 1$ for every crack, and the global stiffness matrix of the structure corresponds to that of the undamaged beam, i.e., to $[\mathbf{K}] + \sum_{m=1}^{M}[\Delta\mathbf{K}^{(m)}]$. With the opposite curvature, the mth crack is considered to be fully open, $f^{(m)}(\{\mathbf{q}\}) = -1$ and the global stiffness matrix is $[\mathbf{K}] - \sum_{m=1}^{M}[\Delta\mathbf{K}^{(m)}]$. As a consequence, it is clear that being the stiffness matrix of the system

$$[\mathbf{K}'(\{\mathbf{q}\})] = [\mathbf{K}] + \sum_{m=1}^{M} [\Delta\mathbf{K}^{(m)}] f^{(m)}(\{\mathbf{q}\}),$$

a function of the generalized displacements of the beam, equation (1) is non-linear.

Under these assumptions, it can be observed that the crack function will have a square-wave form versus time with frequency ω, and the procedure proposed by Ostachowicz and Krawczuk in reference [18] can be derived.

According to this model, in which the transition from open to closed crack and *vice-versa* takes place instantaneously, the applied force and displacement of the free end have a piecewise linear relation. However, since Clark *et al.* [19] demonstrated with experiments that this relation tends to be smooth rather than discontinuous, the assumption that a crack is either completely open or completely closed is probably somewhat of a slight over-simplification. Consequently, in this investigation the relation between applied force and displacement of the free end is assumed to be parabolic, such that the crack function depends linearly on the curvature of the cracked element, i.e., the crack is not considered to be only either fully open or fully closed, but the intermediate configurations with partial opening can also be taken into account.

The simplest way to define the crack function such that it ranges from -1 to $+1$ during the oscillation of the beam is to define it as the ratio between the instantaneous curvature and the maximum curvature on the cracked element during the motion.

The curvature is proportional to the difference between the rotations at the ends of a cracked element:

$$\Delta\phi^{(m)} = q_{m_k} - q_{m_h}, \tag{2}$$

where m_k and m_h denote, respectively, the rotations at the right and the left ends of the mth cracked element. As a consequence, the crack function has the following expression:

$$f^{(m)}(\{\mathbf{q}\}) = \frac{\Delta\phi^{(m)}}{\max|\Delta\phi^{(m)}|} = \Lambda_m(q_{m_k} - q_{m_h}). \tag{3}$$

It is important to note that the denominator depends on the properties of the excitation force such that it cannot be set *a priori*.

2.2. SOLUTION OF THE EQUATION OF MOTION

The difficulty in solving the differential equation (1) is mainly due to the dependence of function $f^{(m)}$ on $\{\mathbf{q}\}$ that makes this equation non-linear.

Assuming that the dynamic response is periodic, the well-known method of harmonic balance can be employed to solve equation (1). Correspondingly, the solution of each ith degree of freedom of the beam can be approximated by

$$q_i = \sum_{j=1}^{R} (A_{ij} \sin j\omega t + B_{ij} \cos j\omega t), \tag{4}$$

where R is the number of harmonics taken into consideration. To apply harmonic balance, it is necessary to express as a Fourier series both the solution, represented by equation (4), and the non-linear term $\{g^{(m)}(\{\mathbf{q}\})\} = f^{(m)}(\{\mathbf{q}\})\{\mathbf{q}\}$ in equation (1). In this way, the coefficients of $\cos j\omega t$ and $\sin j\omega t$ can be equated and the differential equation (1) can be transformed into a set of non-linear algebraic equations which can be solved using an iterative procedure.

Accordingly, from equation (3), the generic element of $\{g^{(m)}(\{\mathbf{q}\})\}$ can be written as

$$g_i^{(m)}(\{\mathbf{q}\}) = \Lambda_m(q_{m_k} - q_{m_h})q_i. \tag{5}$$

Since the terms of $\{\mathbf{q}\}$ are of period $T = 2\pi/\omega$ and the components of $g_i^{(m)}(\{\mathbf{q}\})$ will also have the same period, the following expression is valid:

$$\int_0^{\mathrm{T}} |g_i^{(m)}(t)|\, \mathrm{d}t < +\infty, \tag{6}$$

ensuring that the components of $\{g^{(m)}(\{\mathbf{q}\})\}$ can be developed in a Fourier series and approximated by

$$g_i^{(m)}(\{\mathbf{q}\}) = \sum_{j=1}^{R} (C_{ij}^{(m)} \sin j\omega t + D_{ij}^{(m)} \cos j\omega t), \tag{7}$$

with

$$C_{ij}^{(m)} = \frac{2}{T}\int_0^{\mathrm{T}} g_i^{(m)} \sin(\mathrm{j}\omega t)\,\mathrm{d}t, \tag{8}$$

$$D_{ij}^{(m)} = \frac{2}{T}\int_0^{\mathrm{T}} g_i^{(m)} \cos(\mathrm{j}\omega t)\,\mathrm{d}t. \tag{9}$$

Introducing equations (4) and (5) into equations (8) and (9) the following relations hold:

$$C_{ij}^{(m)} = \frac{2}{T}\int_0^{\mathrm{T}} \Lambda_m \left(\sum_{l=1}^{R} (A_{m_k l} \sin l\omega t + B_{m_k l} \cos l\omega t) - \sum_{l=1}^{R} (A_{m_h l} \sin l\omega t + B_{m_h l} \cos l\omega t) \right) \times$$

$$\times \sum_{l=1}^{R} (A_{il} \sin l\omega t + B_{il} \cos l\omega t) \sin \mathrm{j}\omega t\,\mathrm{d}t \tag{10}$$

and

$$D_{ij}^{(m)} = \frac{2}{T}\int_0^{\mathrm{T}} \Lambda_m \left(\sum_{l=1}^{R} (A_{m_k l} \sin l\omega t + B_{m_k l} \cos l\omega t) - \sum_{l=1}^{R} (A_{m_h l} \sin l\omega t + B_{m_h l} \cos l\omega t) \right) \times$$

$$\times \sum_{l=1}^{R} (A_{il} \sin l\omega t + B_{il} \cos l\omega t) \cos \mathrm{j}\omega t\,\mathrm{d}t. \tag{11}$$

This expression can be simplified by using the product-to-sum trigonometric formulas and by considering the properties of orthogonality between trigonometric functions, thus enabling a more compact expression for both $C_{ij}^{(m)}$ and $D_{ij}^{(m)}$ to be obtained:

$$C_{ij}^{(m)} = \frac{\Lambda_m}{2} \left(\sum_{j_1, j_2 : j_1 + j_2 = j} (z_1 + z_2) + \sum_{j_1, j_2 : j_1 - j_2 = \pm j} \pm (z_1 - z_2) \right), \tag{12}$$

$$D_{ij}^{(m)} = \frac{\Lambda_m}{2} \left(\sum_{j_1, j_2 : j_1 + j_2 = j} (-z_3 + z_4) + \sum_{j_1, j_2 : j_1 - j_2 = \pm j} \pm (z_3 - z_4) \right), \tag{13}$$

where

$$z_1 = (A_{m_k j_1} - A_{m_h j_1}) B_{ij_2},$$

$$z_2 = (B_{m_k j_1} - B_{m_h j_1}) A_{ij_2},$$

$$z_3 = (A_{m_k j_1} - A_{m_h j_1}) A_{ij_2},$$

$$z_4 = (B_{m_k j_1} - B_{m_h j_1}) B_{ij_2}. \tag{14}$$

At this stage, it is possible to apply the harmonic balance procedure to equation (1): on both sides of the equation, the corresponding sine and cosine terms for each one of the R harmonics taken into consideration can be "balanced", such that R systems of algebraic equations are obtained:

$$\begin{bmatrix} [\mathbf{K}] - j^2\omega^2[\mathbf{M}] & -\mathrm{j}\omega[\mathbf{D}] \\ \mathrm{j}\omega[\mathbf{D}] & [\mathbf{K}] - j^2\omega^2[\mathbf{M}] \end{bmatrix} \begin{Bmatrix} \{\mathbf{A}_j\} \\ \{\mathbf{B}_j\} \end{Bmatrix} + \sum_{m=1}^{M} \begin{bmatrix} [\Delta\mathbf{K}^{(m)}] & [\mathbf{0}] \\ [\mathbf{0}] & [\Delta\mathbf{K}^{(m)}] \end{bmatrix} \begin{Bmatrix} \{\mathbf{C}_j^{(m)}\} \\ \{\mathbf{D}_j^{(m)}\} \end{Bmatrix}$$

$$= \{\mathbf{F}_j\}, \quad j = 1, \ldots, R, \tag{15}$$

where

$$\{\mathbf{A}_j\}^{\mathrm{T}} = \{A_{1j}, \ldots, A_{nj}\}^{\mathrm{T}},$$

$$\{\mathbf{B}_j\}^{\mathrm{T}} = \{B_{1j}, \ldots, B_{nj}\}^{\mathrm{T}},$$

$$\{\mathbf{C}_j^{(m)}\}^{\mathrm{T}} = \{C_{1j}^{(m)}, \ldots, C_{nj}^{(m)}\}^{\mathrm{T}},$$

$$\{\mathbf{D}_j^{(m)}\}^{\mathrm{T}} = \{D_{1j}^{(m)}, \ldots, D_{nj}^{(m)}\}^{\mathrm{T}}. \tag{16}$$

and $\{\mathbf{F}_j\}$ is a null vector when $j > 1$.

It can be observed that coefficients C_{ij} and D_{ij} of equation (15) are non-linearly related to A_{ij} and B_{ij} through equations (12) and (13), making the entire system of equations non-linear: as a consequence, the problem cannot be solved through a simple inversion procedure. Instead, it is possible to use an appropriately converging iterative procedure which consists in:

1. determining a first estimate for variables A_{ij} and B_{ij}. In order to determine the response at just one excitation frequency, the estimate can be obtained by evaluating coefficients A_{ij} and B_{ij} related to the undamaged beam. Otherwise, to evaluate the dynamic response over a range of excitation frequencies, a first estimate at frequency ω_k can be obtained as the value of the coefficients for the frequency ω_{k-1};
2. using coefficients A_{ij} and B_{ij} to determine Λ_m (see equation (3)) and coefficients $C_{ij}^{(m)}$ and $D_{ij}^{(m)}$ through equations (12) and (13);
3. determining a new value for unknowns A_{ij} and B_{ij} from equation (15);
4. repeating steps (2) and (3) until the desired precision is achieved and coefficients A_{ij} and B_{ij} are determined;
5. applying equation (4) to determine the components of the approximate vector which satisfies the non-linear equation (1).

3. NUMERICAL RESULTS

For numerical simulation the beam considered is 0·7 m long with square cross-section $0{\cdot}02\ \mathrm{m} \times 0{\cdot}02\ \mathrm{m}$, discretized into 10 finite elements, of steel with a Young's modulus of $2{\cdot}06 \times 10^{11}\ \mathrm{N/m^2}$ and a density of $7850\ \mathrm{kg/m^3}$. Modal damping ζ_n of 0·02 was considered and the excitation is a concentrated harmonic force at the free end.

Table 1 lists three cases analyzed for two cracks with different depth and position (distance from the clamped end) present. In correspondence with the dependence on the size and position of the cracks, the effective non-linear behaviour of the beam increases from cases 1 to 3.

In order to demonstrate that the method presented is capable of determining a good approximation of the dynamic response of the beam with several breathing cracks, a comparative analysis was performed with results obtained through direct numerical integration (this second solution was obtained extending the code described in reference [11] to deal with, at least, two cracks).

Nevertheless, it should be borne in mind that, whereas direct integration [11] would permit the dynamic response to be simulated when each crack in the beam introduces a bilinear stiffness, the main assumption of this work is that the transition from closed to open crack, and *vice versa*, is smooth rather than discontinuous and therefore the results obtained using the two approaches are likely to be slightly different.

TABLE 1

The three considered cracked beams

	1st Crack		2nd Crack	
Case no.	Size (mm)	Pos. (mm)	Size (mm)	Pos. (mm)
1	4	50	8	500
2	6	50	8	500
3	6	50	8	350

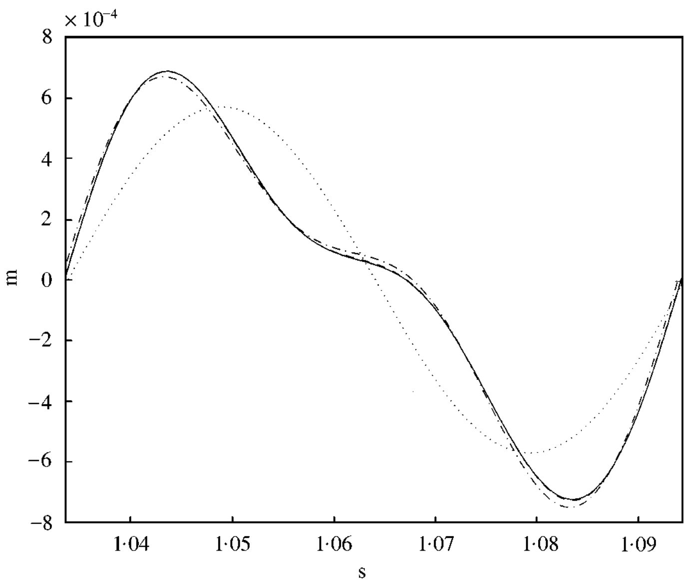

Figure 2. Comparison of the free-end dynamic response for case no. 3 obtained with 1, 2, 3 and 4 terms in the series: ·····, $R = 1$; ·–·–, $R = 2$; ––, $R = 3$; ——, $R = 4$.

The harmonic balance approach has the advantage that the stationary, periodic component of the dynamic response is identified directly, while for direct numerical integration it is necessary to compute, and then effectively discard, the initial transient part of the response.

In all these simulations the sinusoidal forcing is of 10 N amplitude with frequency,

$$\omega = \frac{1}{2}\,\omega_0 = \frac{1}{2}\,\frac{2\omega_u \omega_d^{(i)}}{(\omega_u + \omega_d^{(i)})}, \tag{17}$$

i.e., one-half of the bilinear frequency of the beam, evaluated according to reference [4] and where ω_u is the first natural frequency of the undamaged beam and $\omega_d^{(i)}$ is the first natural frequency of the beam damaged according to the ith case. In several investigations it has been shown that $\omega_0/2$ is the frequency at which the non-linear behaviour of the beam with breathing crack is clearer.

Figure 2 allows one to determine the appropriate number R of terms in the series given in equation (4), with the conclusion that three or maximum four terms are sufficient to obtain an accurate evaluation of the dynamic response at the free-end of the beam for case 3.

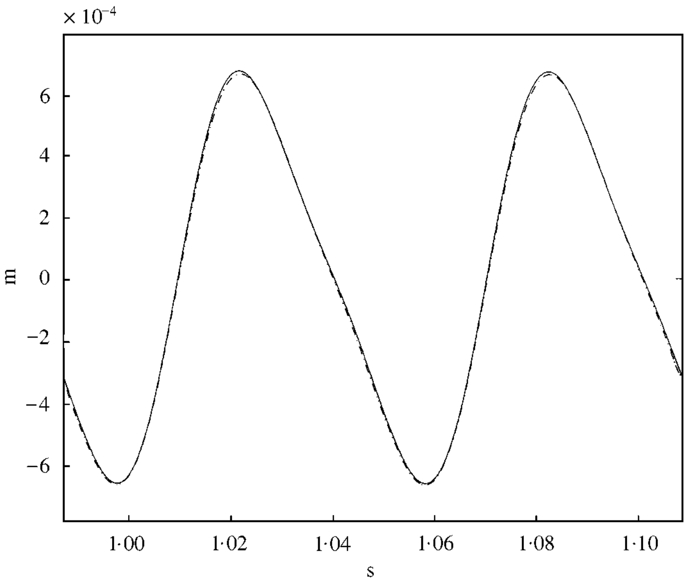

Figure 3. Comparison of the free-end dynamic response obtained with numerical integration and with harmonic balance for case no 1: ——, harmonic balance; ·–·, numerical integration.

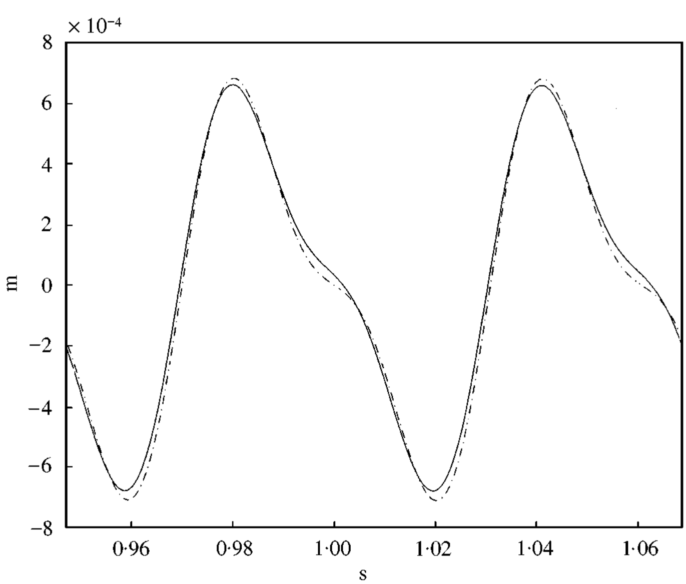

Figure 4. Comparison of the free-end dynamic response obtained with numerical integration and with harmonic balance for case no 2: ——, harmonic balance; ·–·, numerical integration.

Figures 3–5 show the comparison between time histories of the displacement of the free-end evaluated through numerical integration (dash–dot line) and the method proposed in this study (continuous line). These figures highlight that the technique proposed in this study provides very accurate results, particularly for the case with the least

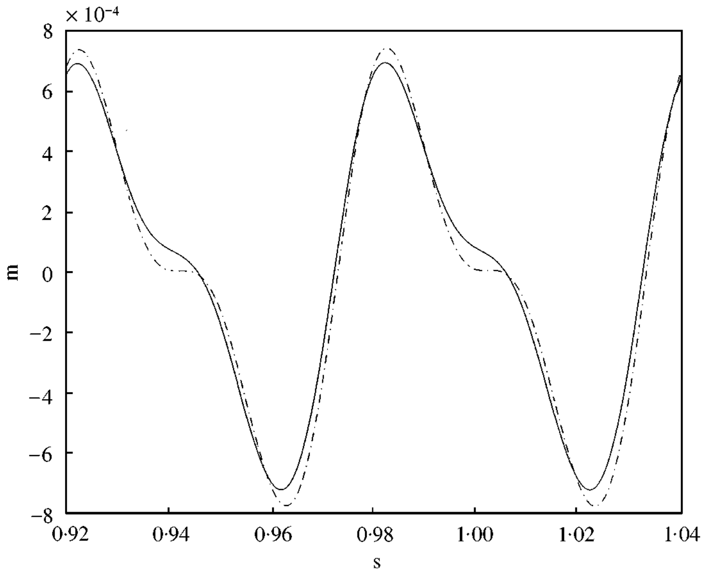

Figure 5. Comparison of the free-end dynamic response obtained with numerical integration and with harmonic balance for case no 3: ——, harmonic balance; · – ·, numerical integration.

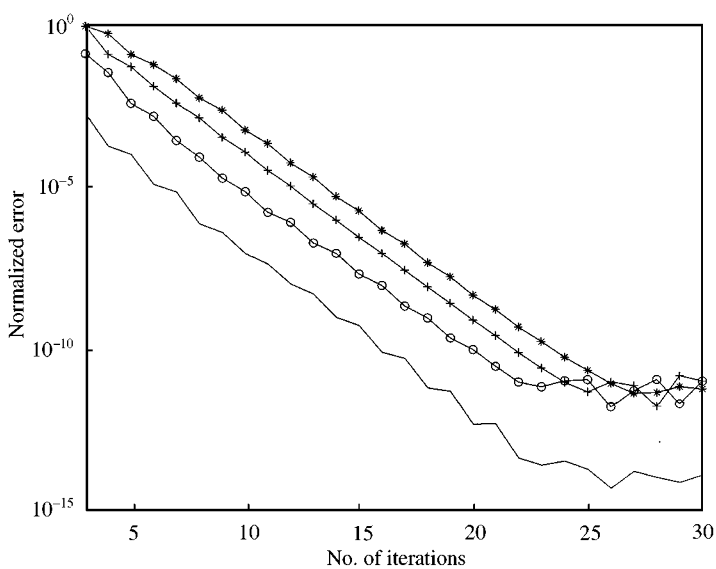

Figure 6. Normalized error for all the harmonics considered in the response for case no 1: ——, first order; –O–, second order; –×–×, third order; –✳–✳, fourth order.

significant non-linear behaviour corresponding to a crack-depth/beam-height ratio of 20%, nevertheless considerable in terms of stiffness reduction.

Figures 6–8 are related to each one of the previous comparisons and demonstrate that the normalized error decreases quickly by increasing the number of iterations, i.e., the proposed procedure is convergent. The normalized error for the jth harmonic of the

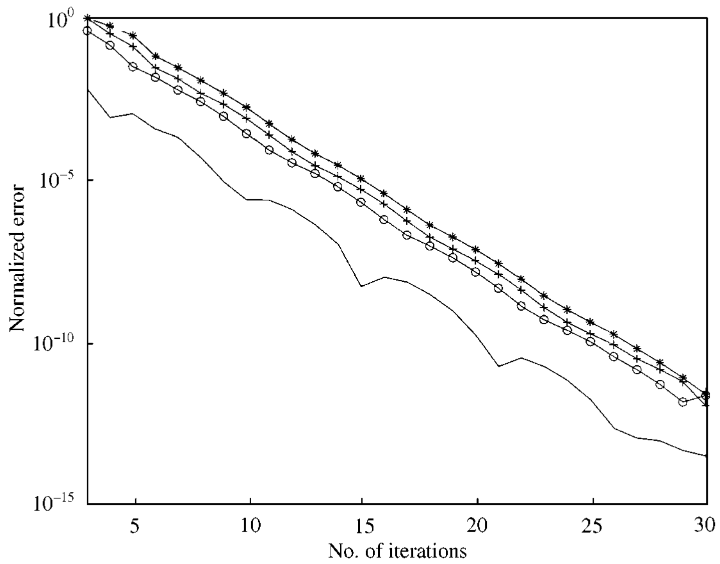

Figure 7. Normalized error for all the harmonics considered in the response for case no 2: ——, first order; –○–, second order; —+—, third order; —∗—, fourth order.

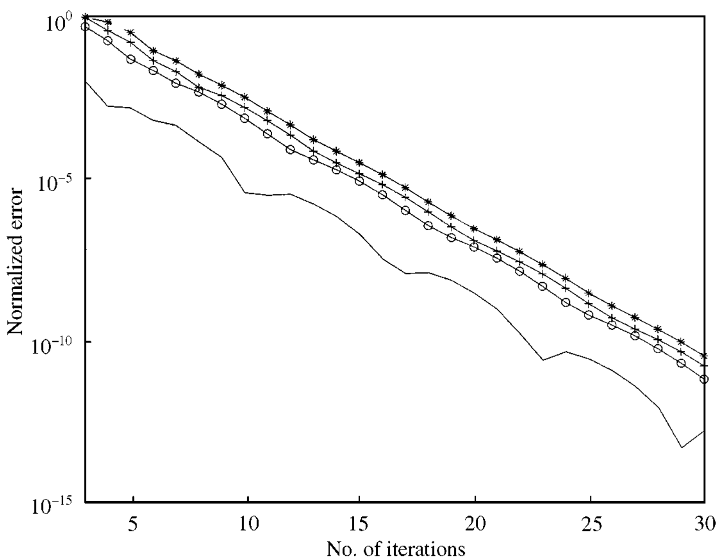

Figure 8. Normalized error for all the harmonics considered in the response for case no 3: ——, first order; –○–, second order; —+—, third order; —∗—, fourth order.

excitation frequency and at the kth iteration was evaluated as

$$\Delta_j^{(k)} = \frac{\left\| \begin{Bmatrix} \{\mathbf{A}_j\} \\ \{\mathbf{B}_j\} \end{Bmatrix}_k - \begin{Bmatrix} \{\mathbf{A}_j\} \\ \{\mathbf{B}_j\} \end{Bmatrix}_{k-1} \right\|}{\left\| \begin{Bmatrix} \{\mathbf{A}_j\} \\ \{\mathbf{B}_j\} \end{Bmatrix} \right\|_{k-1}}. \tag{18}$$

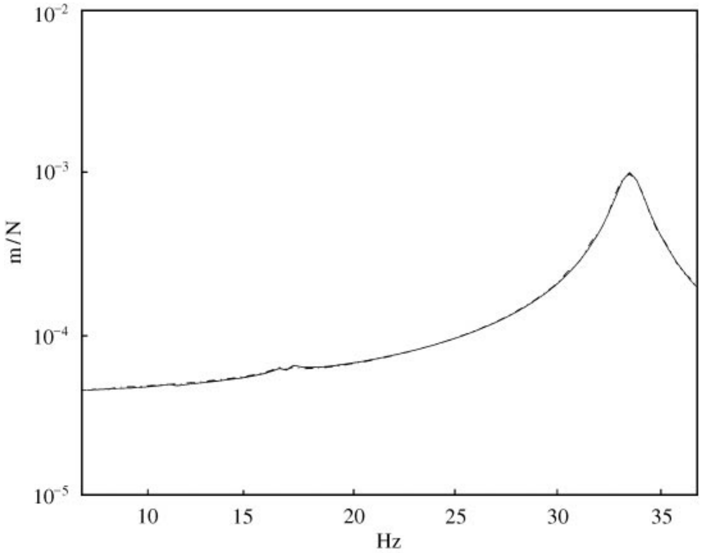

Figure 9. Comparison of results for step-sine simulation obtained with numerical integration and with harmonic balance for case no 1: ——, harmonic balance; –·–·–, numerical integration.

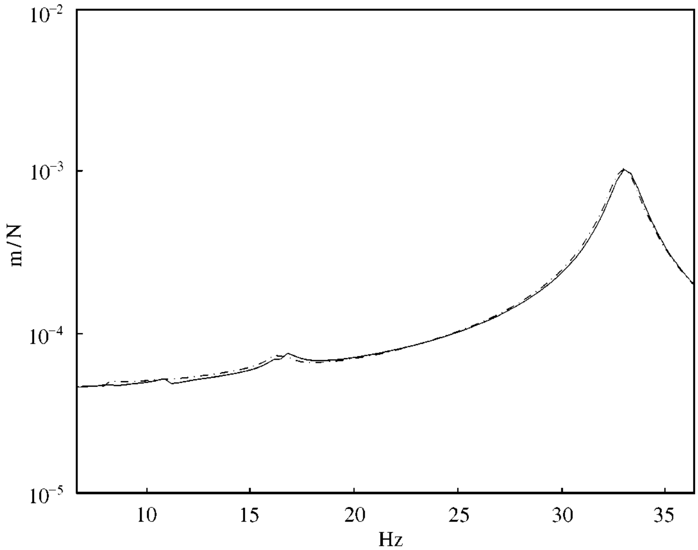

Figure 10. Comparison of results for step-sine simulation obtained with numerical integration and with harmonic balance for case no 2: ——, harmonic balance; –·–·–, numerical integration.

In order to illustrate the good agreement between results obtained through numerical integration and the new procedure proposed, over the excitation frequency range $[\omega_1 \ldots \omega_P]$, a step-sine test was simulated: for each of the test cases, a simulation was performed to determine the harmonic component of the steady dynamic response at the free-end of the beam at each excitation frequency $\omega \in [\omega_1 \ldots \omega_P]$ and calculating the

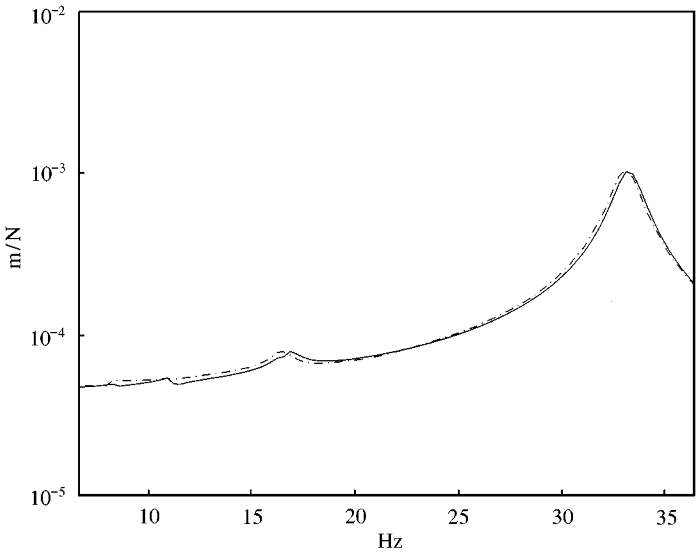

Figure 11. Comparison of results for step-sine simulation obtained with numerical integration and with harmonic balance for case no 3: ——, harmonic balance; –·–·–, numerical integration.

response envelope function defined as ratio between the maximum steady-state levels attained by the displacement response and excitation force signals.

Results of simulated step-sine are shown in Figure 9–11 for all the considered cases, demonstrating that the procedure gives good results over an extended range of excitation frequencies.

4. CONCLUSIONS

This article presents a technique aimed to evaluate the dynamic response of a beam with multiple breathing cracks to an applied sinusoidal force. The approach used makes it possible to reach a closely approximate solution for the beam equation of motion while a significant reduction (by approximately 100 times) of the computation times is obtained in comparison with direct numerical integration.

The obtained results clearly demonstrate that the presence of breathing cracks in a beam results in non-linear dynamic behaviour which gives rise to superharmonics in the spectrum of the response signals, the amplitude of which depends on the number, location and depth of any cracks present.

As a consequence, the fast evaluation of the dynamic response of the structure under analysis permitted by the proposed method opens new possibilities in the development of an appropriate damage detection method of level 2 and/or 3 (according to Rytter's classification [2]).

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