Journal of Sound and Vibration (2000) **235**(5), 801–812
doi:10.1006/jsvi.2000.2951, available online at http://www.idealibrary.com on IDEAL®

ALMOST-SURE STABILITY OF A GYROPENDULUM SUBJECTED TO WHITE-NOISE RANDOM SUPPORT MOTION

S. F. Asokanthan

Department of Mechanical Engineering, The University of Queensland, Brisbane, Qld 4072, Australia

AND

S. T. Ariaratnam

Solid Mechanics Division, Faculty of Engineering, University of Waterloo, Waterloo, Ont., Canada N2L 3G1

(*Received* 26 *June* 1996, *and in final form* 7 *March* 2000)

Instability behaviour of a gyropendulum subjected to white noise vertical support motion is examined. Conditions for almost-sure asymptotic stability are obtained explicitly. A stochastic averaging procedure is employed to evaluate the maximal Lyapunov exponent. The sign of this exponent determines the instability behaviour of this system. Closed-form expressions for the instability conditions obtained in this study are employed to predict the minimum level of damping required to ensure almost-sure asymptotic stability.

1. INTRODUCTION

A gyropendulum forms an essential component of navigational instruments for providing accurate vertical reference or artificial horizon. Other devices such as a simple pendulum and bubble levels become totally unsuitable for this purpose since they suffer from effects due to support motion. In the case of a gyropendulum, however, a considerable amount of directional stability is provided by a spinning rotor, hence resulting in sufficiently accurate measurements. It has been shown that examination of stability and resonance behaviour plays a significant role in the design of vibration isolators for supporting the gyropendulum and the design of the gyropendulum itself [1, 2].

Motion of a gyropendulum due to steady support acceleration as well as horizontal harmonic excitation is well understood and has been well documented by Arnold and Maunder [1], and Krishnan and Maunder [3]. Experimental investigation to confirm this behaviour has been performed by Krishnan and Maunder [3]. Resonant behaviour at well-defined frequencies of excitation which correspond to theoretically predicted precession as well as nutational frequencies has been examined. Subsequently, Krishnan [2] considered a vertical harmonic support excitation on a gyropendulum in V-configuration and established conditions for instability in a closed form. He further showed that primary parametric instabilities were associated with frequencies close to a nutational frequency of the system. In the above studies, the vertical support motion was considered to be sinusoidal. In reality, however, some random fluctuation over a wide band of frequencies will be present (see, e.g., reference [4]). Investigation of the effect of this form of fluctuation on the instability behaviour of a gyropendulum is the prime intent of the present study.

In order to examine the instability behaviour of such a system, methods based on a stochastic framework is essential. Namachchivaya [4] established explicit instability conditions in the *mean square sense* using a stochastic averaging procedure. In this study, the amount of damping required to ensure stability of the response amplitude variance was found to be dependent on the excitation spectrum near certain critical frequencies. Also, as a special case, conditions for instability for the case of white-noise excitation was obtained. In general, moment stability conditions are found to be too conservative when compared with conditions for *sample* or *almost-sure* stability. The present paper, therefore, concentrates on establishing almost sure stability conditions which have not been obtained previously.

Equations which describe the motion of a gyropendulum in a double gimbal V-arrangement [2] subjected to randomly varying pendulous stiffness are considered in the present investigation. The excitation is considered to be represented by a white-noise process. The equations of motion represent the motion of a linear gyroscopic system with randomly varying stiffness. In the present investigation, a stochastic averaging procedure suggested by Khas'minskii [5] will be employed.

Stated briefly, the stochastic averaging procedure replaces the given set of equations by an approximate set of Itô equations valid under certain conditions. Since it is known that solutions of Itô equations are diffusive Markov processes, methods available in the theory of Markov processes can be employed to obtain approximate response statistics of the system. Explicit asymptotic expressions developed for the largest Lyapunov exponent are used in the determination of stability. It is known that Lyapunov exponents characterize the average exponential growth rates of the response of dynamical systems for large values of time (see, e.g., reference [6]). The trivial solution which corresponds to the equilibrium configuration of a dynamical system is stable or unstable with probability 1 (w.p.1) according to whether the largest Lyapunov exponent is negative or positive. Thus, the vanishing of the largest Lyapunov exponent gives the boundary of stochastic stability w.p.1.

Explicit stability conditions are developed when the excitation is a white noise. These conditions provide an estimate of the amount of damping necessary to ensure stability for a given level of excitation. Previously established conditions for mean square stability [4], when compared with the conditions for sample stability established in the present investigation, are found to be conservative. The conditions developed in the present study will, therefore, aid more efficient design of a gyropendulum and support mechanisms.

2. EQUATIONS OF MOTION

For the purpose of the present analysis, a gyropendulum in V-configuration as illustrated in Figure 1 is considered. In this double-gimbal arrangement, the spinning rotor is suspended from the inner gimbal. In order to study the stability behaviour of the gyropendulum about the equilibrium configuration, the equations representing the motion of the pendulum about this configuration are considered. This motion may be represented by the generalized co-ordinates θ_1,θ_2 which represent, respectively, the motion of the outer gimbal with respect to an inertial frame *OXYZ* and the motion of the inner gimbal about the y-axis fixed to the outer gimbal. When the support is excited by a random acceleration $f(t)$, the pendulum experiences an inertial force proportional to $f(t)$. In the case of practical instruments, angles θ_1, θ_2 will be small and the motion of the gyropendulum under this

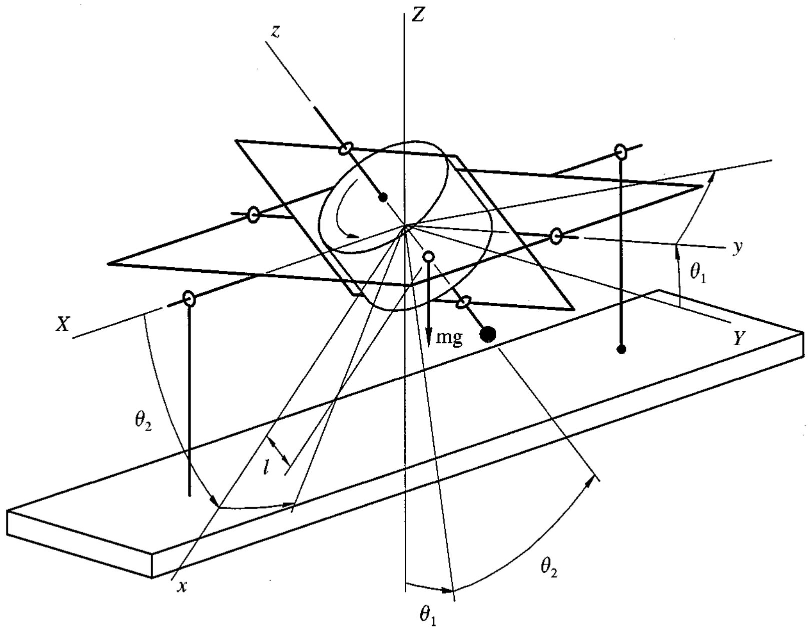

Figure 1. A gyropendulum in a V-configuration.

condition is governed by the following equation (see, e.g., references [1, 4]):

$$M\ddot{\mathbf{Q}} + G\dot{\mathbf{Q}} + D\dot{\mathbf{Q}} + [K + K_1 f(t)]\mathbf{Q} = \mathbf{0}, \tag{1}$$

where $\mathbf{Q}$ represents the vector of generalized co-ordinates $[\theta_2\ \theta_1]^{\mathrm{T}}$ and

$$M = \begin{bmatrix} A_0 & 0 \\ 0 & B_0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & -Cn \\ Cn & 0 \end{bmatrix}, \quad K = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}, \quad K_1 = \begin{bmatrix} k_1 & 0 \\ 0 & k_1 \end{bmatrix}, \quad D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}.$$

In equation (1), M, G represent the mass and the gyroscopic matrices, respectively, while D denotes the damping matrix. Matrices K, K_1 are the stiffness matrices associated with the system and the excitation respectively. The dissipative forces have been assumed to be of the viscous type in the present formulation. The rotor moments of inertia about the axis of rotation and about any axis perpendicular to Oz are C, A, respectively, while the equivalent inertias A_0, B_0 are composed of the gimbal inertias and rotor inertias to give $A_0 = A + B'$, $B_0 = A + A' + A''$. The quantities B' and A' represent the moments of inertia of the inner gimbal about the rotor principal axes Oy and Ox, respectively, while A'' stands for the moment of inertia of the outer gimbal about the outer axis OX. The quantity k stands for the pendulous stiffness mgl and n denotes the rotor spin speed while k_1 denotes the intensity of the stochastic excitation $f(t)$.

Upon letting $\mathbf{Q} = M^{-1/2}\mathbf{q}$ where $\mathbf{q}$ represents a vector with new generalized co-ordinates q_1, q_2, and premultiplying the resulting matrices by $M^{-1/2}$ the equations of motion become

$$\begin{aligned} &\ddot{q}_1 - 2\gamma\dot{q}_2 + d_{11}\dot{q}_1 + k_{11}^2[1 + \mu f(t)]q_1 = 0, \\ &\ddot{q}_2 + 2\gamma\dot{q}_1 + d_{22}\dot{q}_2 + \rho k_{11}^2[1 + \mu f(t)]q_2 = 0, \end{aligned} \tag{2}$$

where

$$\gamma = Cn/(2\sqrt{A_0B_0}), \quad k_{11}^2 = k/A_0, \quad k_{22}^2 = k/B_0, \quad \rho = k_{22}^2/k_{11}^2, \quad d_{11} = d_1/A_0, \quad d_{22} = d_2/B_0$$

and μ represents the amplitude of the stochastic stiffness fluctuation resulting from the random support motion of the gyropendulum. The dissipative forces have been assumed to be of the viscous type in the present formulation.

These equations represent the motion of a linear gyroscopic system with stochastically varying stiffness. It may be observed that the trivial solution of equation (2) corresponds to the equilibrium or operating state of the gyropendulum ($q_i = 0$, $\dot{q}_i = 0$, $i = 1, 2$). Also, for such a system, if the excitation $f(t)$ is periodic, instability is known to occur when the excitation frequency is in the neighbourhood of the frequencies $2\omega_1/r$, $2\omega_2/r$ and $(\omega_1 + \omega_2)/s$ where $r, s = 1, 2, \ldots$. The present analysis deals with the case when the excitation is a random function of time. In order to seek an approximate analytical solution, the spectral density of the stochastic fluctuation $f(t)$ is assumed to be of the same order of smallness as the damping coefficients d_1, d_2. In this study, the excitation will be assumed to be Gaussian white noise with zero mean. Under these conditions, a method of stochastic averaging [7] will be employed to study the *almost-sure* asymptotic stability of the above system represented by equation (2).

3. AMPLITUDE-PHASE EQUATIONS

In order to apply the method of stochastic averaging, it is first necessary to transform the generalized co-ordinates and velocities to amplitude and phase variables a, ϕ via the transformation

$$q_1 = a_1(t)s_1 + a_2(t)s_2, \quad q_2 = \alpha_1 a_1(t)c_1 + \alpha_2 a_2(t)c_2,$$
$$\dot{q}_1 = \omega_1 a_1(t)c_1 + \omega_2 a_2(t)c_2, \quad \dot{q}_2 = -\alpha_1\omega_1 a_1(t)s_1 - \alpha_2\omega_2 a_2(t)s_2, \tag{3}$$

where $c_r = \cos\mathbf{\Phi}_r$, $s_r = \sin\mathbf{\Phi}_r$, $\mathbf{\Phi}_r = \omega_r t + \phi_r(t)$, $\alpha_r = \alpha(\omega_r)$, $r = 1, 2$ and

$$\alpha(\omega_r) = \frac{\omega_r^2 - k_{11}^2}{2\gamma\omega_r} = \frac{2\gamma\omega_r}{\omega_r^2 - k_{22}^2}, \quad r = 1, 2. \tag{4}$$

In equations (3) and (4) the undamped natural frequencies ω_1 and ω_2 of the unperturbed system are given by the expressions

$$\omega_{1,2}^2 = \frac{1}{2}[(k_{11}^2 + k_{22}^2 + 4\gamma^2) \pm \sqrt{(k_{11}^2 + k_{22}^2 + 4\gamma^2)^2 - 4k_{11}^2 k_{22}^2}]. \tag{5}$$

Use of transformation (3) and the method of variation of parameters yields the following equations in the amplitude and phase variables:

$$\dot{a}_i = \chi L_i + \hat{\mu}\chi f(t) P_i,$$
$$\dot{\mathbf{\Phi}}_i = \omega_i + \chi M_i + \hat{\mu}\chi f(t) Q_i, \quad i = 1, 2, \tag{6}$$

where $\chi = 2\gamma/(\omega_1^2 - \omega_2^2)$, $\hat{\mu} = \mu k_{11}^2$ and L_i, M_i, P_i, Q_i, $i = 1, 2$ are functions of the amplitude and phase variables and are evaluated to be

$$L_i = (-1)^i \left[\frac{d_{11}\omega_i a_i c_i^2}{\alpha_i} + d_{11}\omega_j a_j c_i c_j + d_{22}\alpha_i \omega_i a_i s_i^2 + d_{22}\alpha_j \omega_j a_j s_i s_j \right],$$

$$M_i = (-1)^j \frac{d_{11}\omega_i s_i c_i}{\alpha_i} + (-1)^j d_{11}\omega_j a_j c_j s_i + (-1)^i d_{22}\alpha_i \omega_i s_i c_i + (-1)^i d_{22}\omega_j \frac{a_j}{a_i} s_j c_i,$$

$$P_i = (-1)^j \left[\left(\rho\alpha_i - \frac{1}{\alpha_i} \right) a_i s_i c_i + \rho\alpha_j a_j c_j s_i - \frac{a_j c_i s_j}{\alpha_i} \right],$$

$$Q_i = (-1)^j \left[\frac{s_i^2}{\alpha_i} + \frac{a_j}{a_i}\frac{s_i s_j}{\alpha_i} + \rho\alpha_i c_i^2 + \rho\alpha_j \frac{a_j}{a_i} c_j c_i \right], \quad i, j = 1, 2 \quad i \neq j. \tag{7}$$

It may be noted from equations (6) that the amplitude and phase equations for the two modes are coupled. In the sequel, since the right-hand sides of equation (6) are homogeneous in a_1, a_2 and of degree one, to study the stability behaviour of solutions of equation (6), a procedure developed by Khas'minskii [8] may be employed to derive an expression for the largest Lyapunov exponent of the amplitude process [9]. It is known that Lyapunov exponents characterize the average exponential growth rates of the response of dynamical systems for large values of the time t (see, e.g., reference [6]). The trivial solution which corresponds to the equilibrium configuration of a dynamical system is stable or unstable with probability 1 (w.p.1) according to whether the largest Lyapunov exponent is negative or positive. Thus, the vanishing of the largest Lyapunov exponent gives the boundary of stochastic stability w.p.1.

To this end, a further logarithmic polar transformation of the form

$$\rho = \frac{1}{2}\log|\beta_1 a_1^2 + \beta_2 a_2^2|, \quad \theta = \tan^{-1}\left[\sqrt{\frac{\beta_2}{\beta_1}} \frac{a_2}{a_1} \right], \quad 0 \leqslant \theta \leqslant \frac{\pi}{2} \tag{8}$$

is applied, where $\beta_1, \beta_2 > 0$ are to be chosen suitably. It may be pointed out that in order to apply the procedure developed by Khas'minskii [5], stochastic differential equations must be obtained in the Itô form. For this purpose, appropriate stochastic differential equations are first obtained in the Stratonovich form using the rules of ordinary calculus and these equations are then transformed to the Itô form. The resulting stochastic differential equations in the Stratonovich form are

$$\mathrm{d}v = \hat{m}_v \,\mathrm{d}t + \hat{\sigma}_v(\theta_1, \phi_1, \phi_2)\,\mathrm{d}W, \tag{9}$$

where W denotes the standard Wiener process which is the integral of a white-noise process $f(t)$ and v stands for the processes ρ, θ, ϕ_1 and ϕ_2. The drift terms $\hat{m}_\rho$, $\hat{m}_\theta$, $\hat{m}_{\phi_1}$, $\hat{m}_{\phi_2}$ and the diffusion terms $\hat{\sigma}_\rho$, $\hat{\sigma}_\theta$, $\hat{\sigma}_{\phi_1}$, $\hat{\sigma}_{\phi_2}$, that are associated with the above Stratonovich form are

$$\hat{m}_\rho = \frac{2\gamma}{\omega_1^2 - \omega_2^2}\left\{ \frac{L_1}{a_1}\cos^2\theta + \frac{L_2}{a_2}\sin^2\theta \right\}, \quad \hat{\sigma}_\rho = \frac{2\hat{\mu}\gamma}{\omega_1^2 - \omega_2^2}\left\{ \frac{P_1}{a_1}\cos^2\theta + \frac{P_2}{a_2}\sin^2\theta \right\},$$

$$\hat{m}_\theta = \frac{2\gamma\cos^2\theta}{\omega_1^2-\omega_2^2}\sqrt{\frac{\beta_2}{\beta_1}}\left\{\frac{L_2}{a_1}-\frac{L_1 a_2}{a_1^2}\right\}, \quad \hat{\sigma}_\theta = \frac{2\hat{\mu}\gamma\cos^2\theta}{\omega_1^2-\omega_2^2}\sqrt{\frac{\beta_2}{\beta_1}}\left\{\frac{P_2}{a_1}-\frac{P_1 a_2}{a_1^2}\right\},$$

$$\hat{m}_{\phi_1} = \omega_1 + \frac{2\gamma M_1}{(\omega_1^2-\omega_2^2)}, \quad \hat{\sigma}_{\phi_1} = \frac{2\hat{\mu}\gamma Q_1}{(\omega_1^2-\omega_2^2)}, \tag{10}$$

$$\hat{m}_{\phi_2} = \omega_2 + \frac{2\gamma M_2}{(\omega_1^2-\omega_2^2)}, \quad \hat{\sigma}_{\phi_2} = \frac{2\hat{\mu}\gamma Q_2}{(\omega_1^2-\omega_2^2)}.$$

The drift terms $m_\rho, m_\theta, m_{\phi_1}, m_{\phi_2}$ and the diffusion terms $\sigma_\rho, \sigma_\theta, \sigma_{\phi_1}, \sigma_{\phi_2}$ which correspond to the associated Itô equations may be evaluated by adding the so-called Wong and Zakai correction terms to the previous drift terms as follows:

$$m_v = \left[\hat{m}_v + \frac{1}{2}\left\{\frac{\partial\hat{\sigma}_v}{\partial\theta}\hat{\sigma}_\theta + \frac{\partial\hat{\sigma}_v}{\partial\phi_1}\hat{\sigma}_{\phi_1} + \frac{\partial\hat{\sigma}_v}{\partial\phi_2}\hat{\sigma}_{\phi_2}\right\}\right], \quad \sigma_v = \hat{\sigma}_v(\theta_1, \phi_1, \phi_2). \tag{11}$$

It may be observed that all of the above coefficients are functions of θ, ϕ_1, ϕ_2 only and do not depend on the ρ process. Since the solutions of Itô stochastic differential equations are Markov diffusion process (see, e.g., reference [7]), the processes $\theta(t)$, $\phi_1(t)$, $\phi_2(t)$ which are independent of the process ρ are governed by a joint stationary probability density function $p_\varepsilon(\theta, \phi_1, \phi_2)$ which satisfies the Fokker–Planck equation

$$-\sum_{v=\theta,\phi_1,\phi_2}\frac{\partial}{\partial v}[m_v p_\varepsilon] + \frac{1}{2}\sum_{v=\theta,\phi_1,\phi_2}\frac{\partial}{\partial v^2}[\sigma_v^2 p_\varepsilon] + \frac{\partial^2}{\partial\theta\,\partial\phi_1}[\sigma_\theta\sigma_{\phi_1}p_\varepsilon]$$
$$+\frac{\partial^2}{\partial\theta\,\partial\phi_2}[\sigma_\theta\sigma_{\phi_2}p_\varepsilon] + \frac{\partial^2}{\partial\phi_1\partial\phi_2}[\sigma_{\phi_1}\sigma_{\phi_2}p_\varepsilon] = 0. \tag{12}$$

4. METHOD OF STOCHASTIC AVERAGING

The solution of the above partial differential equation in the probability density function p_ε is not easily obtainable in a closed form in general. A solution in a closed form is essential for formulating the maximum Lyapunov exponent which characterizes the sample stability of the system under investigation. To this end, an approximate analysis based on the method of averaging will be employed. In order to get approximate solutions it is necessary to make the assumption that the damping coefficients d_1, d_2 and the intensity of the excitation process $f(t)$ are small, such that $d_1, d_2 = O(\varepsilon)$ and $\hat{\mu} = O(\varepsilon^{1/2})$, $0 < \varepsilon < 1$, so that the coefficients of the Fokker–Planck equation are of the same order of smallness. A theorem due to Khas'minskii [5] may now be used to simplify equation (12) and obtain solution valid in the first approximation. Stochastic averaging is performed for coefficients of the Fokker–Planck equation rather than for original SDE. Khasminskii [5] showed that the final results are the same using the above two averaging approaches. In this procedure, the coefficients of equation (12) are averaged with respect to explicitly appearing time only. According to this rule, if the averaging operator

$$\underset{t}{M}\{.\} = \lim_{T\to\infty}\frac{1}{T}\int_0^T\{\}\,\mathrm{d}t$$

is employed in equation (12), the following time-averaged Fokker–Planck equation for $p(\theta, \phi_1, \phi_2)$:

$$
-\sum_{v=\theta,\phi_1,\phi_2} \frac{\partial}{\partial v}[\bar{m}_v p] + \frac{1}{2}\sum_{v=\theta,\phi_1,\phi_2} \frac{\partial}{\partial v^2}[\bar{\sigma}_v^2 p] + \frac{\partial^2}{\partial\theta\,\partial\phi_1}[\bar{\sigma}_\theta\bar{\sigma}_{\phi_1} p_\varepsilon]
+ \frac{\partial^2}{\partial\theta\,\partial\phi_2}[\bar{\sigma}_\theta\bar{\sigma}_{\phi_2} p] + \frac{\partial^2}{\partial\phi_1\partial\phi_2}[\bar{\sigma}_{\phi_1}\bar{\sigma}_{\phi_2} p] = 0. \tag{13}
$$

is obtained. The averaged coefficients $\bar{m}_\theta$, $\bar{m}_{\phi_1}$, $\bar{m}_{\phi_2}$, $\bar{\sigma}_\theta^2$, $\bar{\sigma}_{\phi_1}^2$, $\bar{\sigma}_{\phi_2}^2$, $\bar{\sigma}_\theta\bar{\sigma}_{\phi_1}$, $\bar{\sigma}_\theta\bar{\sigma}_{\phi_2}$, $\bar{\sigma}_{\phi_1}\bar{\sigma}_{\phi_2}$, for the present system, can be evaluated in terms of the system parameters using equations (10) and (11) and are expressed as

$$
\begin{aligned}
\bar{m}_\theta = {} & \frac{\gamma\sin 2\theta}{2(\omega_1^2-\omega_2^2)}\left[d_{11}\left(\frac{\omega_1}{\alpha_1}+\frac{\omega_2}{\alpha_2}\right)+d_{22}(\alpha_1\omega_1+\alpha_2\omega_2)\right] \\
& + \frac{\hat{\mu}^2\gamma^2 S}{4(\omega_1^2-\omega_2^2)^2}\left[\left(\rho\alpha_2-\frac{1}{\alpha_2}\right)^2-\left(\rho\alpha_1-\frac{1}{\alpha_1}\right)^2+\frac{\beta_1}{2\beta_2}D_2-\frac{\beta_2}{2\beta_1}D_1\right]\sin 2\theta \\
& + \frac{\hat{\mu}^2\gamma^2 S}{16(\omega_1^2-\omega_2^2)^2}\left[\left(\rho\alpha_2-\frac{1}{\alpha_2}\right)^2+\left(\rho\alpha_1-\frac{1}{\alpha_1}\right)^2-8\rho-\frac{\beta_1}{\beta_2}D_2-\frac{\beta_2}{\beta_1}D_1\right]\sin 4\theta \\
& + \frac{\hat{\mu}^2\gamma^2 S}{4(\omega_1^2-\omega_2^2)^2}\left[\frac{\beta_2}{\beta_1}D_1\frac{\cos^3\theta}{\sin\theta}-\frac{\beta_1}{\beta_2}D_2\frac{\sin^3\theta}{\cos\theta}\right],
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\bar{\sigma}_\theta^2 = {} & \frac{\hat{\mu}^2\gamma^2 S}{8(\omega_1^2-\omega_2^2)^2}\left[\left(\rho\alpha_2-\frac{1}{\alpha_2}\right)^2+\left(\rho\alpha_1-\frac{1}{\alpha_1}\right)^2-8\rho\right]\sin^2 2\theta \\
& + \frac{\hat{\mu}^2\gamma^2 S}{2(\omega_1^2-\omega_2^2)^2}\left[\frac{\beta_1}{\beta_2}D_2\sin^4\theta+\frac{\beta_2}{\beta_1}D_1\cos^4\theta\right],
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
& \bar{m}_{\phi_1} = \omega_1, \quad \bar{\sigma}_{\phi_1}^2 = \frac{\hat{\mu}^2\gamma^2 S}{2(\omega_1^2-\omega_2^2)^2}\left[2\left(\rho^2\alpha_1^2+\frac{1}{\alpha_2^2}\right)^2+2\rho+\frac{\beta_1}{\beta_2}D_2\tan^2\theta\right], \\
& \bar{m}_{\phi_2} = \omega_2, \quad \bar{\sigma}_{\phi_2}^2 = \frac{\hat{\mu}^2\gamma^2 S}{2(\omega_1^2-\omega_2^2)^2}\left[2\left(\rho^2\alpha_2^2+\frac{1}{\alpha_2^2}\right)^2+2\rho+\frac{\beta_2}{\beta_1}D_1\cot^2\theta\right],
\end{aligned} \tag{16}
$$

where

$$
D_1 = 2(\rho^2\alpha_1^2 + 1/\alpha_2^2), \quad D_2 = 2(\rho^2\alpha_2^2 + 1/\alpha_1^2).
$$

In obtaining equation (13), account has been taken of the fact that $\omega_1 \neq \omega_2$. It is known that for the present system, the natural frequencies are always unequal when the rotor is stationary. Further, it has been shown that the smaller frequency gets smaller and the larger gets larger as the rotor spin speed increases (see, e.g., papers by Krishnan [2, 3]). It may also be remarked that the convergence of the exact solution $p_\varepsilon(\theta, \phi_1, \phi_2)$ to the averaged solution is in the weak sense. The solution of the equation (13) satisfying the periodicity condition

$$
p(\theta, \phi_1 + 2\pi, \phi_2) = p(\theta, \phi_1, \phi_2 + 2\pi) = p(\theta, \phi_1, \phi_2)
$$

will be independent of ϕ_1 and ϕ_2 and hence of the form $p(\theta, \phi_1, \phi_2) = Cf(\theta)$ where $f(\theta)$ satisfies

$$-\frac{d}{d\theta}[\bar{m}_\theta f(\theta)] + \frac{1}{2}\frac{d^2}{d\theta^2}[\bar{\sigma}_\theta^2 f(\theta)] = 0, \quad 0 \leqslant \theta \leqslant \pi/2, \tag{17}$$

whose solution is

$$f(\theta) = \frac{C}{\bar{\sigma}_\theta^2 U(\theta)}, \tag{18}$$

where

$$U(\theta) = \exp\left[-2\int_0^\theta \bar{m}_\theta(\psi)\bar{\sigma}_\theta^{-2}(\psi)\,d\psi\right], \tag{19}$$

and C is a constant to be determined from the normalization condition [10]. Using equations (14), (15) and (18) and performing the integration in equation (19) an expression for the probability density function $p(\theta)$ can be obtained. Now, the maximal Lyapunov exponent of the present system can be evaluated from the *Itô* stochastic differential equation which governs the evolution of the amplitude process (see e.g., reference [10]). Further, for the purpose of a closed-form evaluation of the maximum Lyapunov exponent, the mean and the drift coefficients m_ρ, m_θ, $\bar{\sigma}_\theta^2$ may be expressed in the form

$$\begin{aligned}
\bar{m}_\rho(\theta) &= \tfrac{1}{2}(\lambda_1 + \lambda_2) + \tfrac{1}{2}(\lambda_1 - \lambda_2)\cos 2\theta + \bar{\sigma}_\theta^2(\theta),\\
\bar{m}_\theta(\theta) &= \tfrac{1}{2}(\lambda_1 - \lambda_2)\sin 2\theta + \bar{\sigma}_\theta^2(\theta)\cot 2\theta,\\
\bar{\sigma}_\theta^2(\theta) &= a - b\cos^2\theta,
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
\lambda_1 &= -\frac{\beta}{2}\frac{\omega_1}{\omega_1 + \omega_2} + \lambda_1', \quad \lambda_1' = \frac{\hat{\mu}^2\gamma^2}{2(\omega_1^2 - \omega_2^2)}\left(\rho\alpha_1 + \frac{1}{\alpha_1}\right)^2 S,\\
\lambda_2 &= -\frac{\beta}{2}\frac{\omega_2}{\omega_1 + \omega_2} + \lambda_2', \quad \lambda_2' = \frac{\hat{\mu}^2\gamma^2}{2(\omega_1^2 - \omega_2^2)}\left(\rho\alpha_2 + \frac{1}{\alpha_2}\right)^2 S,
\end{aligned} \tag{21}$$

$$a = \frac{\hat{\mu}^2\gamma^2}{4(\omega_1^2 - \omega_2^2)}(1 - 4\rho + D)S, \quad b = \frac{\hat{\mu}^2\gamma^2}{4(\omega_1^2 - \omega_2^2)}(1 - 4\rho - D)S, \quad D = \left|2\rho^2\alpha_1\alpha_2 + \frac{2}{\alpha_1\alpha_2}\right|,$$

and $\beta = (d_{11} + d_{22})/2$. In equation (21), the power-spectral density of the white-noise process $f(t)$ is represented by S. Also, in obtaining equation (20), as stated earlier, the arbitrary coefficients β_1, β_2 have been chosen to conform to $\sqrt{\beta_1/\beta_2} = D_1/D_2$ which ensures the above simplified form for equation (20). It is known that, for the *Itô* system of equations (13), the largest Lyapunov exponent is

$$\lambda = E[\bar{m}_\rho(\theta)]. \tag{22}$$

It has been shown by Ariaratnam and Xie [10] that the expression for the largest Lyapunov exponent may be evaluated, for the case $K > D$, as

$$\lambda = \frac{1}{2}(\lambda_1 + \lambda_2) + \frac{1}{2}(\lambda_1 - \lambda_2)\coth\left[\frac{\lambda_1 - \lambda_2}{(\Delta)^{1/2}}\alpha\right],$$

$$\alpha = \cosh^{-1}\frac{K}{D}, \quad \Delta = \frac{\gamma^4 S^2}{(\omega_1^2 - \omega_2^2)^4}(K^2 - D^2), \tag{23}$$

and for the case $K < D$ as

$$\lambda = \frac{1}{2}(\lambda_1 + \lambda_2) + \frac{1}{2}(\lambda_1 - \lambda_2)\coth\left[\frac{\lambda_1 - \lambda_2}{(-\Delta)^{1/2}}\alpha\right],$$

$$\alpha = \cos^{-1}\frac{K}{D}, \quad \Delta = \frac{\gamma^4 S^2}{(\omega_1^2 - \omega_2^2)^4}(K^2 - D^2), \tag{24}$$

where

$$K = \frac{1}{2}\left(\rho\alpha_1 - \frac{1}{\alpha_1}\right)^2 + \frac{1}{2}\left(\rho\alpha_1 - \frac{1}{\alpha_1}\right)^2 - 4\rho,$$

employing the expression for the invariant density $p(\theta)$ and equation (22). Expressions (23) and (24) can be further reduced using equations (21). The largest Lyapunov exponent may be shown to be, for the case $K > D$:

$$\lambda = -\frac{\beta}{2} + \frac{1}{2}\left[-\frac{\beta(\omega_1 - \omega_2)}{2(\omega_1 + \omega_2)} + \lambda_1' - \lambda_2'\right]\coth\left[\frac{(-\beta/2((\omega_1 - \omega_2)/(\omega_1 + \omega_2)) + \lambda_1' - \lambda_2')\alpha}{(\Delta)^{1/2}}\right],$$

$$\alpha = \cosh^{-1}\frac{K}{D} \tag{25}$$

and for the case $K < D$:

$$\lambda = -\frac{\beta}{2} + \frac{1}{2}\left[-\frac{\beta(\omega_1 - \omega_2)}{2(\omega_1 + \omega_2)} + \lambda_1' - \lambda_2'\right]\coth\left[\frac{(-\beta/2((\omega_1 - \omega_2)/(\omega_1 + \omega_2)) + \lambda_1' - \lambda_2')\alpha}{(-\Delta)^{1/2}}\right],$$

$$\alpha = \cos^{-1}\frac{K}{D}. \tag{26}$$

Expressions (25) and (26) which directly relate the top Lyapunov exponent to the system parameters in a closed form can be used to determine the *almost-sure* asymptotic stability of the present system. It is known that the system is asymptotically stable only if the top Lyapunov exponent λ is negative. For the system under consideration, these expressions are employed to examine the stability in the following section.

5. NUMERICAL RESULTS

In this section, the analytical findings presented in the previous sections are applied to a typical gyropendulum whose support is subjected to a vertical white-noise acceleration.

The dimensions and the speed selected conform to the experimental pendulum developed by Krishnan [2]. Also, in order to study the effects of the stiffness ratios on the stability behaviour, two other variations of this pendulum has been considered in the present study. The parameters of the experimental gyropendulum are

mass	$m = 0{\cdot}866\,\mathrm{kg}$,
length	$l = 70\,\mathrm{mm}$,
moment of inertia	$C = 0{\cdot}278 \times 10^{-3}\,\mathrm{kg\,m^2}$,
moment of inertia	$B_0 = 4{\cdot}760 \times 10^{-3}\,\mathrm{kg\,m^2}$,
moment of inertia	$A_0 = 5{\cdot}425 \times 10^{-3}\,\mathrm{kg\,m^2}$.

The natural frequencies ω_1 and ω_2 vary with the speed parameter γ in accordance with equation (5) and expressions (25) and (26) are used for evaluating the maximal Lyapunov exponent λ as a function of the rotor spin speed parameter γ. Figure 2 provides the graphical representation of conditions for instabilities which correspond to the condition $\lambda > 0$ where expressions (25) and (26) for the top Lyapunov exponent λ has been employed. The shaded regions shown in the figure indicate the instability regions.

In order to illustrate the applicability of the analytical results of the present analysis, three different inertia ratios have been considered. These values were chosen to study the stability behaviour for inertia ratios in the ranges close to unity, greater than unity and less than unity. The curve for the parameter values with $\rho = 1{\cdot}14$ correspond to the experimental pendulum. In all three cases, the curves are composed of two parts associated with the

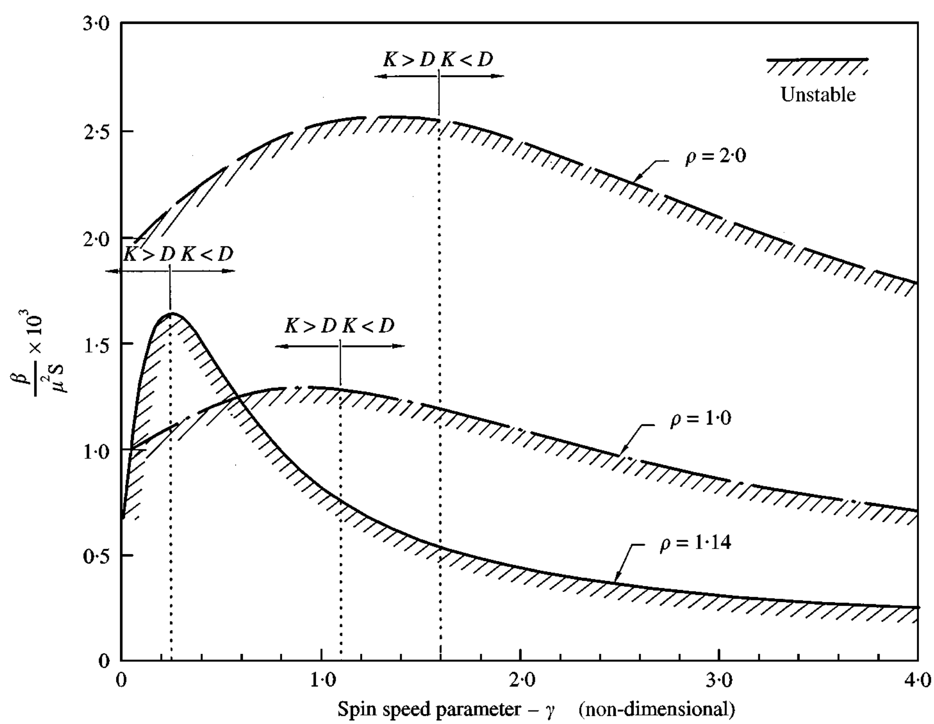

Figure 2. Instability regions.

conditions $K < D$ and $K > D$ as shown in Figure 2. All three curves illustrate that minimum values of damping to ensure asymptotic stability under white noise support excitation may be predicted using expressions (25) and (26). The curves also show that in each case there is a value of rotor speed parameter γ at which a maximum amount of damping is required to ensure stability. It may also be observed from this figure that for higher values of ρ, higher damping values are required to ensure asymptotic stability. The speed parameter that is associated with the maximum damping tends to be larger for stiffness ratios that are either larger or smaller than unity.

Explicit conditions for instability behaviour of the gyropendulum evaluated in this study as demonstrated above, can be employed to facilitate efficient designs for the gyropendulum and the associated support structure. The almost sure stability conditions obtained in this study would help attain more efficient designs, since it is known that the use of moment stability conditions which were obtained in previous studies would result in more conservative designs.

6. CONCLUSION

Instability behaviour of a gyropendulum under vertical white noise support acceleration is examined by explicitly evaluating the maximal Lyapunov exponent. Methods based on a stochastic averaging procedure are employed for this purpose. Closed-form expressions obtained for the maximal Lyapunov exponent provide means of ascertaining the almost sure asymptotic stability for any set of system parameters and excitation levels. They provide a minimum level of damping required to ensure stability. The conditions for almost sure asymptotic stability conditions obtained are more accurate than the mean square stability conditions obtained previously and hence will lead to more efficient design of a gyropendulum and the associated support structures.

ACKNOWLEDGEMENTS

This research was supported (in part) by grant No. ARCS0776G 95 from the Australian Research Council (ARC).

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