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TWO-WAY STATE-FLOW GRAPH MODEL FOR ANALYZING SOUND PROPAGATION IN STEPWISE TUBES

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This study presents a novel two-way state-flow (TWSF) graph model to graphically represent the acoustic wave in tubes in terms of the states flow, pressure and velocity. The proposed model is then used to derive analytical solutions of the acoustic pressure and velocity and acoustic properties of the stepwise tube, such as the input impedance and resonance frequency. Owing to the novel structure of the proposed model, the graph model of a steeped tube can be obtained by directly cascade connection of that of each section. Moreover, the exact and analytical results for the tubes can be directly calculated by topology methods according to the constructed graph model without solving multiple coupling differential equations.

1. INTRODUCTION

Propagation of plane waves in tubes is an important topic in acoustic applications on acoustic horns and filters, mufflers, and musical instruments. For single uniform tubes, exact solutions for the propagation and properties of sound in the tube with various boundary conditions have been proposed in many books [1, 2]. Wave propagation in the tubes of conical and exponential cross-sections have also investigated analytically [2, 3]. For the tubes of stepwise cross-section, it is convenient to approach it by a cascade of multiple uniform tubes. Typical technique for the analysis of this problem is using an electroacoustic analogue, by which the acoustic system is analogy to the electrical components and network [4, 5]. Then, multiple coupled differential equations can be formulated using the electrical theory. Several numerical methods have been proposed for solving these equations [3, 5–7]. Recently, an algebraic algorithm has been developed to obtain the analytical solution of the velocity ratio of the system based on transfer matrix method [8]. However, it is difficult to obtain the analytical and exact solutions of the acoustic states, velocity and pressure, inside the tube by the classical methods.

In this paper, analytical and exact solutions for plane wave propagation in a tube with stepwise cross-section are proposed. A two-way state-flow (TWSF) graph model [9, 10] for the representation of the acoustic behavior in a uniform acoustic tube is firstly developed. The graph model of the stepped tube is connecting each uniform segment in series. The acoustical pressure and velocity in the tube, and its input impedance and the resonance frequency of the tube are derived. Finally, some numerical examples are examined to illustrate the performance of the proposed method.

2. TWSF GRAPH MODEL OF UNIFORM TUBE

A stepwise tube consisting of N uniform sections with stationary medium is considered for the investigation. It is assumed that the plane acoustic wave (one-dimensional wave) is

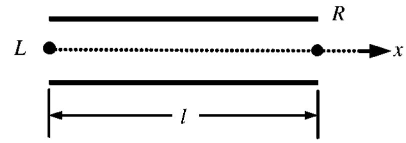

Figure 1. Schematic diagram of a uniform tube.

propagated through each uniform section. The description of wave propagation in a tube with uniform cross-section, as shown in Figure 1, is governed by the wave equation

$$\frac{\partial^2 p(x,t)}{\partial t^2} - c^2 \frac{\partial^2 p(x,t)}{\partial x^2} = 0, \tag{1}$$

where p is the acoustic pressure, and c is the phase speed of the wave propagation. The general solution for the acoustic pressure in the tube is given as

$$p(x,t) = f_1(ct - x) + f_2(ct + x), \tag{2}$$

where f_1 and f_2 are the arbitrary functional relationships of the parameters $ct - x$ and $ct + x$ respectively. This expression represents a pair of displacement waves propagating in the positive and negative directions along the axis of the tube. When the motion of the particles of the fluid as a function of harmonic wave is considered in this investigation, the solution can be expressed by complex form as

$$p(x,t) = (P^+ \mathrm{e}^{-\mathrm{j}kx} + P^- \mathrm{e}^{\mathrm{j}kx})\mathrm{e}^{-\mathrm{j}\omega t}, \tag{3}$$

where k is the wave number defined as $k = \omega/c$. P^+ is the complex pressure amplitude of a plane wave travelling in the positive direction, and P^- is the amplitude of the wave travelling in the negative direction. Both amplitudes are dependent on the length and the complex pressure amplitude at both terminals of the tube given by

$$P^+ = \frac{P_L \mathrm{e}^{\mathrm{j}kl} - P_R}{2\mathrm{j}\sin kl}, \qquad P^- = \frac{P_R - P_L \mathrm{e}^{-\mathrm{j}kl}}{2\mathrm{j}\sin kl}, \tag{4, 5}$$

where l is the length of the tube, P_L and P_R are the complex pressure amplitude at the terminals of $x = 0$ and $x = l$ respectively.

Moreover, the relationships between the acoustic volume velocity u and pressure p at any location x should satisfy the dynamical equilibrium equation

$$\frac{\partial u(x,t)}{\partial t} + \frac{A}{\rho_0}\frac{\partial p(x,t)}{\partial x} = 0, \tag{6}$$

where A is the cross-sectional area of the tube. Substituting equation (3) into equation (6), the volume velocity in the tube can be expressed as

$$u(x,t) = \frac{A}{c\rho_0}(P^+ \mathrm{e}^{-\mathrm{j}kx} - P^- \mathrm{e}^{\mathrm{j}kx})\mathrm{e}^{\mathrm{j}\omega t}. \tag{7}$$

According to equations (3) and (7), we see that the complex amplitudes of the pressure and volume velocity at both ends of the tube are dependent. If the volume velocity at the left end

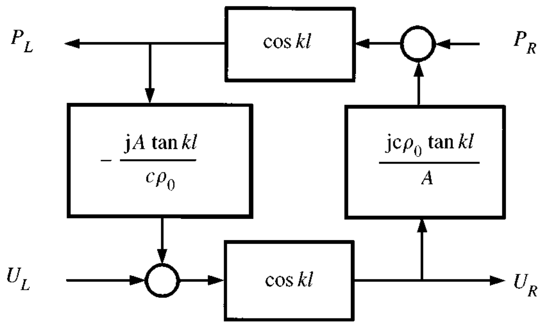

Figure 2. TWSF graph model of the uniform tube.

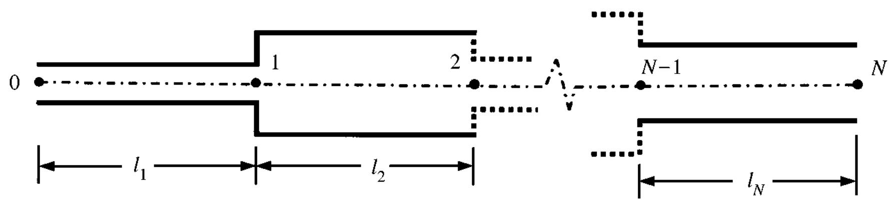

Figure 3. Schematic diagram of an N stepped tube.

where $x = 0$, U_L, and the acoustic pressure P_R at the other end where $x = l$ are chosen as input states, the other two states, volume velocity U_R and acoustic pressure P_L, should be a function of these two selected input states as follows:

$$U_R = \cos kl\left(-\frac{jA \tan kl}{c\rho_0}P_L + U_L\right), \tag{8}$$

$$P_L = \cos kl\left(\frac{jc\rho_0 \tan kl}{A}U_R + P_R\right). \tag{9}$$

Based on equations (8) and (9), the relationships between each complex amplitude of pressure and volume velocity at both ends of the tube can be expressed as a TWSF graph model as shown in Figure 2.

A stepwise tube consisting of N uniform sections with different cross-sections is considered as shown in Figure 3, in which l_i and A_i are the length and the cross-sectional area of the section i respectively. P_i and U_i are the complex pressure amplitude of the acoustic pressure and volume velocity at the junction of the sections $i - 1$ and i respectively. If the cross-sectional area of each section is sufficiently small within the limit of pure plane-wave propagation, the acoustic pressure and the volume velocity remain the same through each junction plane. Thus, the TWSF graph models for two connected sections, $i - 1$ and i, are compatible and can be directly connected in series as a ladder type as shown in Figure 4.

If the acoustic impedance, defined by the ratio of acoustic pressure in a medium to the associated volume velocity [2], at the end of the tube looking outward from the tube Z_a is considered, the relationships between the acoustic impedance and the states at the end P_N and U_N is

$$Z_a = \frac{P_N}{U_N}. \tag{10}$$

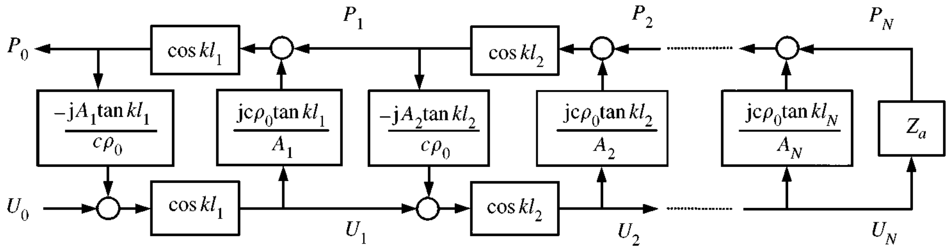

Figure 4. TWSF graph model of the N stepped tube.

Thus, TWSF graph model of the stepped tube including the effect of the output can be obtained by connecting the output impedance to the states P_N and U_N of the tube as shown in Figure 4.

3. ACOUSTIC PROPERTIES OF STEPWISE TUBES

For the graph model of the N stepped tube, there are $N(N+1)/2$ closed loops in the model if the output acoustic impedance Z_a is combined into the gain $jc\rho_0 \tan kl_N/A_N$ is considered. The structure of the TWSF graph model of the stepped tubes is very similar to that of the stepped rod except the most right vertical path. Thus, the analysis scheme presented in reference [9–11] can be applied in this problem.

The loop gain of each closed loop of the graph model passing through the ith and jth vertical path in the graph model L_{ij} is expressed as

$$L_{i,j} = A_i \tan kl_i \left(\frac{\tan kl_j}{A_j} - \frac{j\delta_{j,N} Z_a}{c\rho_0} \right) \prod_{k=i}^{j} \cos^2 kl_k \quad \text{for } j \geqslant i, \tag{11}$$

where $\delta_{j,N}$ is the Kronecker delta notation defined by

$$\delta_{i,j} = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases} \tag{12}$$

The complex frequency response of the volume velocity at the right end of section i can be obtained by calculating the transfer function from U_0 to U_i. According to the graph model, we see that there is only one forward path from U_0 to U_i as shown in Figure 4. The gain of the forward path is given by

$$F_1 = \prod_{j=1}^{i} \cos kl_j. \tag{13}$$

The cofactor of this forward path, which is formed by the part of the graph model on the right side of U_i, can be represented as [10]

$$D_i = \sum_{k=0}^{N-i} E_{i+1,N,k}, \tag{14}$$

where

$$E_{i,N,k} = \begin{cases} \displaystyle\sum_{i_{2k}=i+k-1}^{N} \sum_{i_{2k-1}=i+k-1}^{i_{2k}} \cdots \sum_{i_2=i}^{i_3-1} \sum_{i_1=i}^{i_2} \prod_{j=1}^{k} (-L_{i_{2j-1},i_{2j}}) & \text{for } k \geqslant 1, \\ 1 & \text{for } k = 0. \end{cases} \tag{15}$$

Based on equations (13) and (14), the complex frequency response U_i can be calculated and represented as

$$U_i = \frac{\prod_{j=1}^{i} \cos kl_j \sum_{k=0}^{N-i} E_{i+1,N,k}}{\sum_{k=0}^{N} E_{1,N,k}} U_0. \tag{16}$$

When the complex frequency response of the volume pressure at the right end of the ith section is considered, there are $N - i$ forward paths from U_0 to P_i. The cofactor of each forward path passing through the gain of $(\mathrm{j}c\rho_0 \sin kl_n/A_n)\prod_{m=i+1}^{n-1} \cos kl_m \prod_{j=1}^{n} \cos kl_j$ is formed by the part on the right of state U_n. Then, the complex frequency response P_i leads to

$$P_i = \begin{cases} \dfrac{\sum_{n=i+1}^{N} (\mathrm{j}c\rho_0 \sin kl_n/A_n + \delta_{n,N} Z_a \cos kl_N) \prod_{m=i+1}^{n-1} \cos kl_m \prod_{j=1}^{n} \cos kl_j \sum_{p=0}^{N-n} E_{n+1,N,p}}{\sum_{k=0}^{N} E_{1,N,k}} U_0 & \text{for } i \neq N, \\ \dfrac{Z_a \prod_{j=1}^{n} \cos kl_j}{\sum_{k=0}^{N} E_{1,N,k}} U_0 & \text{for } i = N. \end{cases} \tag{17}$$

According to the results obtained by equation (17), the pressure and velocity response at any location x_i of section i can be calculated by equations (3) and (7), in which the coefficients P^+ and P^- are expressed as the complex pressure amplitude by substituting equation (17) into equations (4) and (5).

The acoustic impedance at the input of tube seen toward the tube Z_{in} is defined as P_0/U_0, which can be calculated from equation (17) for $i = 0$, given as

$$Z_{in} = \frac{\sum_{n=1}^{N} (\mathrm{j}(c\rho_0/A_n) \sin kl_n + \delta_{n,N} Z_a \cos kl_N) \prod_{m=1}^{n-1} \cos kl_m \prod_{j=1}^{n} \cos kl_j \sum_{p=0}^{N-n} E_{n+1,N,p}}{\sum_{k=0}^{N} E_{1,N,k}}. \tag{18}$$

4. APPLICATION IN SIMPLIFIED CASES

4.1. SECTIONS WITH EQUIVALENT LENGTH

If each section of the stepped tube has the same length, the terms of kl_i in the graph model will be identical and denoted as kl. Thus, each loop gain of the graph model can be simplified as

$$L_{i,j} = A_i \tan kl \left(\frac{\tan kl}{A_j} - \frac{j\delta_{j,N} Z_a}{c\rho_0} \right) \cos^{2(j-i+1)} kl. \tag{19}$$

Since the forward path gain from U_0 to U_i becomes $\cos^i kl_j$, the complex frequency response U_i can be reduced to

$$U_i = \frac{\cos^i kl \sum_{k=0}^{N-i} E_{i+1,N,k}}{\sum_{k=0}^{N} E_{1,N,k}} U_0. \tag{20}$$

In the same way, the complex frequency response P_i and the input impedance may be rewritten as

$$P_i = \begin{cases} \dfrac{\sum_{n=i+1}^{N} (\mathrm{j}(c\rho_0/A_n)\sin kl + \delta_{n,N} Z_a \cos kl)\cos^{2n-i-1} kl \sum_{p=0}^{N-n} E_{n+1,N,p}}{\sum_{k=0}^{N} E_{1,N,k}} U_0 & \text{for } i \neq N, \\ \dfrac{Z_a \cos^N kl}{\sum_{k=0}^{N} E_{1,N,k}} U_0 & \text{for } i = N, \end{cases} \tag{21}$$

$$Z_{in} = \frac{\sum_{n=1}^{N} (\mathrm{j}(c\rho_0/A_n)\sin kl + \delta_{n,N} Z_a \cos kl)\cos^{2n-1} kl \sum_{p=0}^{N-n} E_{n+1,N,p}}{\sum_{k=0}^{N} E_{1,N,k}}. \tag{22}$$

4.2. FOR LOW-FREQUENCY RANGE

When the length of each section is small enough to reach $kl_i \ll 1$ in the low acoustic frequency range, each loop gain of the graph model can be approximated as

$$L_{i,j} = A_i k l_i \left(\frac{k l_j}{A_j} - \frac{\mathrm{j}\delta_{j,N} Z_a}{c\rho_0} \right). \tag{23}$$

Thus, the volume velocity, pressure, and input impedance can be approximated by substituting equation (23) into equations (16)–(18) is given by

$$U_i = \frac{\sum_{k=0}^{N-i} E_{i+1,N,k}}{\sum_{k=0}^{N} E_{1,N,k}} U_0, \tag{24}$$

$$P_i = \begin{cases} \dfrac{\sum_{n=i+1}^{N} (\mathrm{j}c\rho_0 k l_n/A_n + \delta_{n,N} Z_a) \sum_{p=0}^{N-n} E_{n+1,N,p}}{\sum_{k=0}^{N} E_{1,N,k}} U_0 & \text{for } i \neq N, \\ \dfrac{Z_a}{\sum_{k=0}^{N} E_{1,N,k}} U_0 & \text{for } i = N, \end{cases} \tag{25}$$

$$Z_{in} = \frac{\sum_{n=1}^{N} (\mathrm{j}c\rho_0 k l_n/A_n + \delta_{n,N} Z_a) \sum_{p=0}^{N-n} E_{n+1,N,p}}{\sum_{k=0}^{N} E_{1,N,k}}. \tag{26}$$

5. EXAMPLES

Some examples are investigated to illustrate the feasibility and performance of the presented method. Since a tube of uniform cross-section is the simplest type which has been discussed in many texts, calculation of the acoustic properties and wave propagation of a uniform tube with various termination conditions by the presented method is first examined.

5.1. SINGLE UNIFORM TUBE

A tube with length l and uniform area of cross-section A driven by a vibrating piston located at the left end where $x = 0$ and that terminated with an acoustic impedance Z_a at the right end where $x = l$ is considered. The acoustic velocity, pressure at the end 1 and the acoustic input impedance can be calculated by equations (16)–(18) for $N = 1$ and $i = 1$ is given by

$$U_1 = \frac{1}{\cos kl + (\mathrm{j}A/c\rho_0)Z_a \sin kl} U_0, \qquad P_1 = Z_a U_1, \tag{27, 28}$$

$$Z_{in} = \frac{(\mathrm{j}c\rho_0/A)\sin kl + Z_a \cos kl}{\cos kl + (\mathrm{j}A/c\rho_0)\, Z_a \sin kl}. \tag{29}$$

The resonance frequency of the tube is defined as that at which the input impedance is a minimum. The determinate equation for the resonant frequency at the tube leads to

$$\tan^2 kl + b \tan kl - 1 = 0, \tag{30}$$

where

$$b = \frac{R_a^2}{(c\rho_0/A)X_a} + \frac{X_a}{(c\rho_0/A)} - \frac{(c\rho_0/A)}{X_a}. \tag{31}$$

R_a is the acoustic resistance and X_a is the reactance of the acoustic output impedance Z_a. Thus, the resonance frequency of the tube is the root of the determinate equation given as

$$\omega = \frac{c}{l}\left(\tan^{-1}\left(\frac{-b + \sqrt{b^2 + 4}}{2}\right) + n\pi\right) \quad \text{for } n = 0, 1, 2, \ldots. \tag{32}$$

If the tube is closed at the end $x = l$, the velocity at the end will be zero. The acoustic output impedance Z_a at the right end where $x = l$, becomes infinity. According to equation (29), the input impedance at the point 0 seen toward the tube Z_{in} is given as

$$Z_{in} = -j(c\rho_0/A)\cot kl. \tag{33}$$

Thus, the resonance frequency of the tube becomes

$$\omega = \frac{c}{l}\left(\frac{(2n+1)\pi}{2}\right) \quad \text{for } n = 0, 1, 2, \ldots. \tag{34}$$

The results of equations (27)–(34), calculated from the derived formula based on the presented method, are identical to the ones in texts [2, 3].

5.2. THREE-SECTION TUBE

A tube with three uniform steps, having the same length l for each section as shown in Figure 5 is considered. The area of cross-section for sections 1, 2 and 3 of the tube are A, $2A$ and $3A$, respectively, and the acoustic output impedance of the tube at the right end is Z_a. If the acoustic output impedance Z_a is combined into the gain $\mathrm{j}c\rho_0 \tan kl_3/A_3$ in the graph model, the number of the closed loop of the graph model will be reduced to six. Thus, the

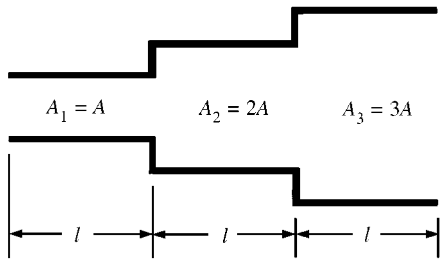

Figure 5. Acoustic tube with stepwise cross-sections.

loop gain of each closed loop is

$$L_{1,1} = L_{2,2} = \sin^2 kl, \quad L_{1,2} = \tfrac{1}{2}\cos^2 kl \sin^2 kl, \quad L_{1,3} = \tfrac{1}{3}\cos^4 kl \sin^2 kl - (\mathrm{j}A/c\rho_0)Z_a \sin kl \cos^5 kl,$$

$$L_{2,3} = \tfrac{2}{3}\cos^2 kl \sin^2 kl - (2\mathrm{j}A/c\rho_0)Z_a \sin kl \cos^3 kl, \quad L_{2,3} = \sin^2 kl - (3\mathrm{j}A/c\rho_0)Z_a \sin kl \cos kl. \tag{35}$$

From the derived formula, equations (20)–(22), the complex amplitudes of the volume velocity at node 3 and the input impedance of the tube are

$$U_3 = \frac{2}{\cos^2 kl(5\cos^2 kl - 3) + (3\mathrm{j}A/c\rho_0)Z_a \sin kl(-5\cos^2 kl + 1)} U_0, \tag{36}$$

$$Z_{in} = \frac{(4\mathrm{j}c\rho_0/15A)\sin kl(24\cos^2 kl - 5) + Z_a \cos^2 kl(9\cos^2 kl - 7)}{\cos^2 kl(5\cos^2 kl - 3) + (3\mathrm{j}A/c\rho_0)Z_a \sin kl(-5\cos^2 kl + 1)}. \tag{37}$$

If the cross-sectional area of each section is replaced by the same area A, equations (36) and (37) will become

$$U_3 = \frac{1}{(4\cos^2 kl - 3)\cos kl + (\mathrm{j}A/c\rho_0)Z_a(4\cos^2 kl - 1)\sin kl} U_0, \tag{38}$$

$$Z_{in} = \frac{(\mathrm{j}A/c\rho_0)(4\cos^2 kl - 1)\sin kl + Z_a(4\cos^2 kl - 3)\cos kl}{(4\cos^2 kl - 3)\cos kl + (\mathrm{j}A/c\rho_0)Z_a(4\cos^2 kl - 1)\sin kl}. \tag{39}$$

The acoustical response and property at the output terminal calculated using equations (38) and (39) are equal to those of using equations (27) and (29) by replacing kl to $3kl$. The volume velocity at junctions 1 and 2 are

$$U_1 = \frac{2\cos^2 kl - 2 + (2\mathrm{j}A/c\rho_0)Z_a \cos kl \sin kl}{(4\cos^2 kl - 3)\cos kl + (\mathrm{j}A/c\rho_0)Z_a(4\cos^2 kl - 1)\sin kl} U_0, \tag{40}$$

$$U_2 = \frac{\cos kl + (\mathrm{j}A/c\rho_0)Z_a \sin kl}{(4\cos^2 kl - 3)\cos kl + (\mathrm{j}A/c\rho_0)Z_a(4\cos^2 kl - 1)\sin kl} U_0. \tag{41}$$

The volume velocities obtained from equations (40) and (41) are equal to those calculated using the equations for single uniform tube, equations (7), (4) and (5), at the location of $x_1 = l$, $x_1 = 2l$ of tube length $3l$.

6. CONCLUSIONS

In this work, TWSF graph models for the single uniform tube and the multiple stepped tubes have been developed. Based on the proposed graph model, the acoustic pressure and velocity in the tube, and the acoustic input impedance and the resonance frequency of the tube are directly calculated.

Compared to the classical methods, the graph model can offer a clearer picture representing the interaction of acoustic states, pressure and velocity. Due to the special configuration of the graph model, the models for two cascade sections can be directly connected in a series. Another advantage is that the acoustic stepwise tube can be analyzed directly based on the graph model without solving multiple sets of differential equations. Moreover, the results of the analysis can be expressed as analytical and closed forms.

The developed graph model has been applied for the analysis of the tube with stepwise cross-section. This method has the potential to be extended for the analysis of more complex acoustical systems.

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