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A SIMPLE APPROXIMATE FORMULATION FOR THE FIRST TWO FREQUENCIES OF ASYMMETRIC WALL–FRAME MULTI-STOREY BUILDING STRUCTURES

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A simple approximate method is presented to estimate the first two coupled natural frequencies of uniform asymmetric wall–frame multi-storey buildings. Firstly, the fundamental uncoupled torsional-to-lateral frequency ratio is determined by using the maximum structural static deflection resulting from the gravity load acting horizontally. Secondly, the concept of equivalent eccentricity is proposed and determined by means of the frequency ratio of the second to the fundamental mode in the same direction of a building structure. Finally, a combination of the fundamental uncoupled torsional-to-lateral frequency ratio and the equivalent eccentricity is used to extend the simple formulae, which are limited to coupled natural frequencies of proportionate structural systems, to estimate the first two coupled natural frequencies of asymmetric wall–frame buildings. Numerical examples show that a good agreement between results from the proposed method and FEM has been achieved.

1. INTRODUCTION

The dynamic response analysis for a multi-storey building normally involves modal analysis. This is one of the most important and time-consuming steps in the response spectrum analysis procedure. The calculation of the mode frequencies and shapes of a tall building structure normally requires a full dynamic analysis by solving an eigenvalue problem, which can be a long and tedious process. In some circumstance the response in the fundamental mode may provide a good approximation for the total response; for instance in the initial stage of the design, a complex analysis is often not justified, and a more rapid approximate hand method is adequate. Therefore, a number of empirical formulae have been proposed to estimate the fundamental frequency of buildings. Most of these formulae given in some current building codes (e.g. the Uniform Building Code 1994 [1]) are based on the overall dimensions of a building, rather than the actual stiffness or mass properties. Moreover, Rayleigh's method, in which the static lateral deflection analysis of the building structure is inevitable, is given for more accurate estimation. However, no guidelines have yet been given to deal with the determination of the fundamental natural frequency of an asymmetric multi-storey building structure.

From Hejal and Chopra's study [2], the building response to earthquake ground motion can be determined by analyzing the response of the corresponding torsionally uncoupled system and a set of associated torsionally coupled, one-storey systems. The coupled natural frequencies of an asymmetric multi-storey building structure, due to asymmetric layout of structural components with respect to the floor plan, can be simply expressed in terms of uncoupled lateral frequency, uncoupled fundamental torsional-to-lateral frequency ratio and the eccentricity ratio. However, this relationship is valid only for proportionate structural systems, in which the lateral stiffness matrices of all resisting elements along one direction are proportional. As a result, the centres of rigidity of the floors of such buildings are uniquely defined and lie on a vertical line. For non-proportionate structural systems, such as wall–frame structures, the location of centres of rigidity can vary dramatically from floor to floor even for structures that are uniform over height, and also depend on the vertical distribution of the lateral applied loading.

The purpose of this paper is to extend the application of Hejal and Chopra's formulation to non-proportionate structural systems so as to estimate the first two coupled natural frequencies of uniform asymmetric wall-frame multi-storey buildings. The uncoupled fundamental lateral and torsional frequencies are obtained by means of the maximum static deflections of the building structure subjected to the gravity loading acting horizontally. The equivalent eccentricity is defined as a distance between the centres of mass and the fictitious centres of rigidity of floors. The fictitious centres of rigidity lie on a single vertical line and are also independent of lateral applied loading. They can be determined using the frequency ratio between the second and fundamental frequency in the direction in question. Although details presented herein are for floor plans having one axis of symmetry, the method as derived in this paper is also applicable to the general asymmetric case.

2. BASIC STRUCTURAL PARAMETERS

For the sake of clarity and comprehension, some basic structural parameter expressions presented by other researchers [3, 4] are repeated in this section.

Consider a uniform structure comprising a combination of structural walls and moment resisting frames having in-plane stiffness only and one axis of symmetry. A typical plan layout of the asymmetric wall–frame multi-storey building structure is shown in Figure 1. The distribution of mass over a floor is assumed to be identical for all floors, from which the centres of mass may be assumed to lie on a common vertical axis. This is chosen as axis z with the C_m at each floor level.

Due to the assumed uniformity of the structural properties of the members throughout the height of the building, the set of uniform walls can be represented by an equivalent

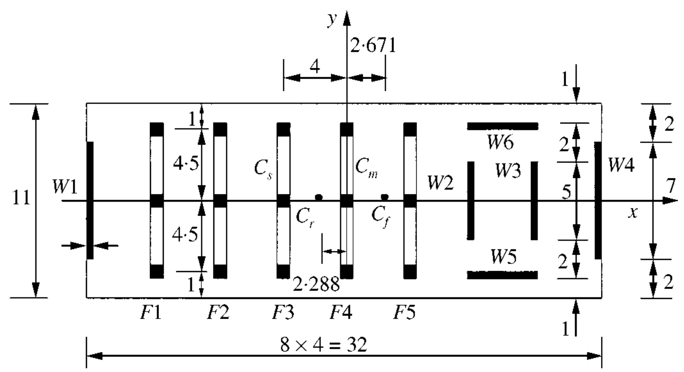

Figure 1. Floor plan of a 16-storey building supported by a dual system of frame and wall system. All dimensions in metres.

single flexural cantilever located at the centre of flexural rigidity, C_f, with a flexural rigidity of

$$EI = \sum_1^n (EI)_i \tag{1}$$

where $(EI)_i$ is the flexural rigidity of wall i.

The second moment of flexural rigidity, i.e., warping rigidity, of the wall system about the centre of flexural rigidity C_f, is

$$EI_{\omega f} = \sum_1^n (EIa^2)_i, \tag{2}$$

in which a_i is perpendicular distance of the plane of the wall i from C_f.

The warping rigidity of the equivalent flexural cantilever about the centre of mass C_m is then

$$EI_\omega = EI_{\omega f} + EIe_f^2 \tag{3}$$

in which e_f is distance of C_f from C_m.

Similarly, the set of moment resisting frames can be represented by an equivalent single shear cantilever at the centre of shear rigidity C_s, and having a shear rigidity of

$$GA = \sum_1^l (GA)_j, \tag{4}$$

where $(GA)_j$ is the racking shear rigidity of frame j, and as determined by

$$(GA)_j = \frac{12E}{h[1/\sum I_c/h + 1/\sum I_g/l]}, \tag{5}$$

in which $\sum(I_c/h)_j$ and $\sum(I_g/l)_j$ represent the sum of moments of inertia of the columns per unit height in one storey of frame j, and the sum of moments of inertia of each beam per unit span, across one floor of frame j respectively.

The second moment of racking shear rigidity, i.e., torsional rigidity, of the frame system about the centre of shear rigidity C_s is given by

$$GJ_s = \sum_1^l (GAb^2)_j, \tag{6}$$

in which b_j is the distance of frame j from C_s.

The torsional rigidity of the equivalent shear cantilever about the centre of mass C_m is given by

$$GJ = GJ_s + GAe_s^2, \tag{7}$$

in which e_s is the distance of C_s from C_m.

The additional non-dimensional parameters, k and p, can be written to account for the effect of the axial deformation of frame columns on the overall flexure of frames and the consequent effect on the lateral and torsional behaviours, respectively, of the structure. They are given by

$$k^2 = 1 + \frac{EI}{\sum(EAc^2)_j} = 1 + \frac{EI}{EAc^2} \quad \text{and} \quad p^2 = 1 + \frac{EI_\omega}{\sum(EAc^2)_j x_j^2} = 1 + \frac{EI_\omega}{EA_\omega}, \tag{8, 9}$$

in which EAc^2, the overall bending rigidity of the frames' column sectional areas, is equal to $\sum(EAc^2)_j$, and $(EAc^2)_j$ is the second moment of column sectional areas in frame j about their common centroid. EA_ω, the second moment of all the frames' column area bending rigidities about the centre of mass, is equal to $\sum(EAc^2)_j x_j^2$, and x_j is the distance of frame j from the centre of mass C_m.

The above basic rigidities may be used to form two more comprehensive non-dimensional structural parameters, $k\alpha H$ governing the lateral motion, and $p\beta H$ interpreting torsional behaviour. They are given as follows:

$$k\alpha H = k\sqrt{\frac{GA}{EI}}H \quad \text{and} \quad p\beta H = p\sqrt{\frac{GJ}{EI_\omega}}H. \tag{10, 11}$$

The relative behaviour of torsion to translation can then be expressed by a relative stiffness parameter, η [3], which is also non-dimensional:

$$\eta = \frac{p\beta H}{k\alpha H} = \frac{p}{k}\left(\frac{GJ}{EI_\omega}\right)^{1/2}\left(\frac{EI}{GA}\right)^{1/2} = \frac{p}{k}\left(\frac{GJ}{GA}\right)^{1/2}\left(\frac{EI}{EI_\omega}\right)^{1/2}. \tag{12}$$

Defining γ_f as the radius of gyration of the flexural rigidity of the wall system about C_m, and γ_s as the radius of gyration of the shear rigidity of the frame system about C_m:

$$\gamma_f^2 = \frac{EI_\omega}{EI} \quad \text{and} \quad \gamma_s^2 = \frac{GJ}{GA} \tag{13, 14}$$

and also introducing γ_a, the radius of gyration of column areas' overall flexural rigidities about the centre of mass C_m, as defined by the relationship

$$\gamma_a^2 = \frac{EA_\omega}{EAc^2} \tag{15}$$

the torsional-to-lateral relative stiffness parameter, η, can then be rewritten as

$$\eta = \frac{p}{k}\frac{\gamma_s}{\gamma_f}. \tag{16}$$

Equations (8), (9), (13) and (15) give

$$p^2 = \frac{\gamma_f^2}{\gamma_a^2}(k^2 - 1) + 1. \tag{17}$$

These basic structural parameters will be used to express the building dynamic behaviours.

3. FUNDAMENTAL UNCOUPLED LATERAL AND TORSIONAL FREQUENCIES

In an earlier paper [5], the writers proposed a simple method to predict the fundamental lateral frequency of symmetric multi-storey buildings. A uniform symmetric multi-storey building can be represented by a shear-flexural cantilever with the differential governing equation expressed as

$$\frac{d^4y}{dz^4} - (k\alpha)^2\frac{d^2y}{dz^2} = \frac{1}{EI}[w(z) - \alpha^2(k^2 - 1)M(z)], \tag{18}$$

where y is the lateral deflection, $w(z)$ is the intensity of the static lateral distributed loading, and $M(z)$ is the accumulated external moment at the co-ordinate z from the base of the structure.

The fundamental lateral frequency of a shear-flexural cantilever can be obtained from the maximum static deflection resulting from the gravity loading acting horizontally as

$$f_y = \frac{0{\cdot}5595}{H^2}\frac{1}{\sqrt{F(k, k\alpha H)}}\sqrt{\frac{EI}{m}}, \tag{19}$$

where

$$F(u, v) = 1 - \frac{1}{u^2}\left[1 - \frac{1}{v^2} + \frac{8}{v^4 \cosh v}(1 + v \sinh v - \cosh v)\right]. \tag{20}$$

The parameters $u = k$, $v = k\alpha H$ and EI were determined in section 2; H is the total height of buildings; and m is the building mass per unit height.

The maximum lateral deflection coefficient $F(k, k\alpha H)$ can be expressed graphically in terms of structural parameter k and $k\alpha H$, as shown in Figure 2.

The torsional behaviour of a symmetric wall–frame structure may be considered as a torsion-warping cantilever and the torsional motion can be approximated by the following governing differential equation [4]:

$$\frac{d^4\theta}{dz^4} - (p\beta)^2\frac{d^2\theta}{dz^2} = \frac{1}{EI_\omega}[t(z) - \beta^2(p^2 - 1)B(z)] \tag{21}$$

in which θ is the twist about the z-axis; $t(z)$ is the intensity of the static distributed horizontal torque; and $B(z)$ is the accumulated external bimoment at co-ordinate z from the base of the structure.

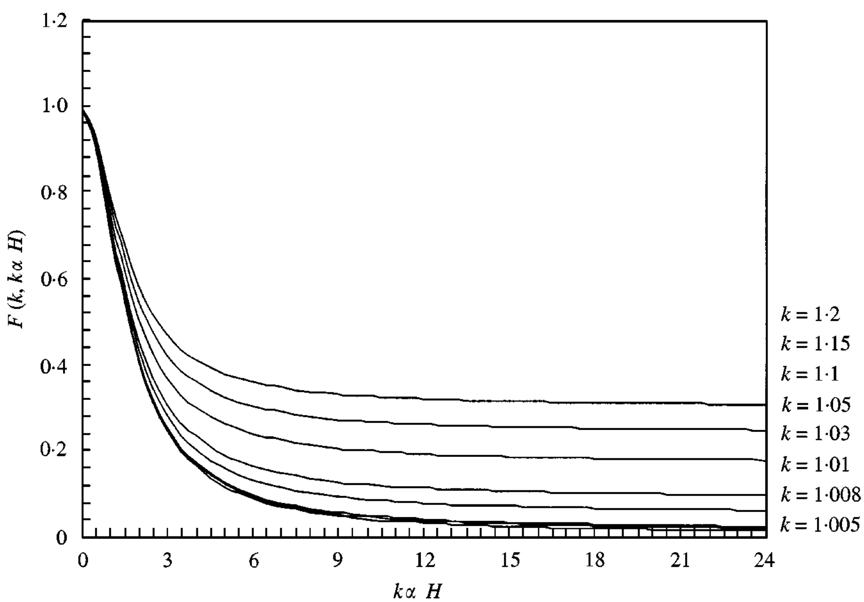

Figure 2. Top deflection coefficient of shear-flexural cantilevers.

Because of the similarity between equations (21) and (18), the fundamental torsional frequency can be obtained in a similar form as

$$f_\theta = \frac{0{\cdot}5595}{H^2}\frac{1}{\sqrt{F(p, p\beta H)}}\sqrt{\frac{EI_\omega}{m\gamma_m^2}} \tag{22}$$

in which γ_m is the mass radius of gyration about C_m; parameters p, $p\beta H$ and EI_ω are defined in section 2; $F(p, p\beta H)$, which governs the torsional behaviour of the building structure, is obtained by using the same expression as that for $F(k, k\alpha h)$ governing the lateral motion, equation (20) with $u = p$ and $v = p\beta H$, or from Figure 2.

For a uniform asymmetric wall–frame structural system, if the centre of flexural rigidity of the wall system and the centre of shear rigidity of the frame system are not located at the same point in floor plan, the centres of rigidity of all floors will not lie on a single vertical line [2]. Their locations are also dependent on the distribution of lateral applied loading. However if fictitious centres of rigidity at different floor levels are considered to be located on a single vertical axis and shifted to coincide with the centres of mass, while all other properties are kept identical to the actual asymmetric wall–frame system, an associated torsionally uncoupled wall–frame system can be obtained. Then equations (19) and (22) can be used to approximate the uncoupled fundamental lateral and torsional frequencies of uniform asymmetric wall–frame multi-storey buildings.

4. THE RATIO OF UNCOUPLED FUNDAMENTAL TORSIONAL TO LATERAL FREQUENCIES

From equations (22) and (19), an uncoupled fundamental torsional-to-lateral frequency ratio of a shear-flexure torsion-warping cantilever, Ω_0, can be obtained as

$$\Omega_0 = \frac{f_0}{f_y} = \left[\frac{F(k, k\alpha H)}{F(p, p\beta H)}\right]^{1/2}\frac{\gamma_f}{\gamma_m} = \lambda\frac{\gamma_f}{\gamma_m}, \tag{23}$$

where the frequency ratio coefficient is

$$\lambda = \left[\frac{F(k, k\alpha H)}{F(p, p\beta H)}\right]^{1/2} = \left[\frac{F(k, k\alpha H)}{F(p, \eta k\alpha H)}\right]^{1/2}. \tag{24}$$

4.1. UNCOUPLED FUNDAMENTAL FREQUENCY RATIO FOR TWO PARTICULAR TYPES OF STRUCTURES

4.1.1. *Structures with parameter $k = 1$*

For one category of structures having low height-to-width ratios, thereby negligible axial deformations of the vertical elements, i.e. $k = 1{\cdot}00$, and subsequently $p = 1{\cdot}00$, the frequency ratio coefficient becomes

$$\lambda = \left[\frac{F(\alpha H)}{F(\beta H)}\right]^{1/2} = \left[\frac{F(\alpha H)}{F(\eta\alpha H)}\right]^{1/2}, \tag{25}$$

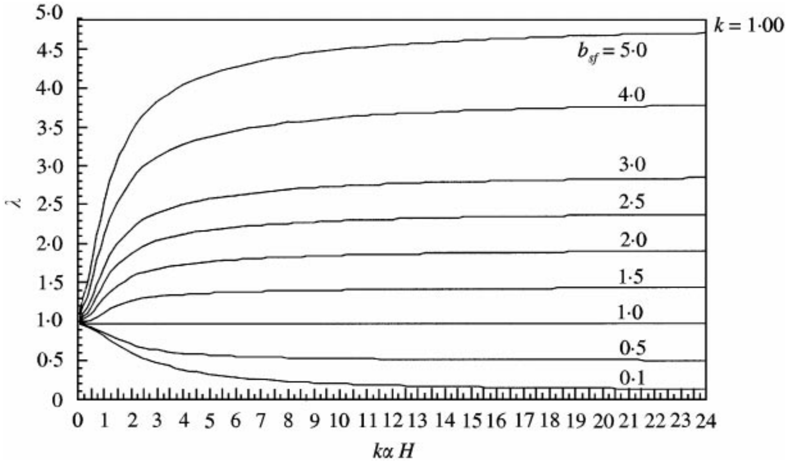

Figure 3. Uncoupled frequency ratio coefficient of the fundamental torsional-to-lateral modes when $k = 1{\cdot}00$.

where

$$F(v) = \frac{4}{v^2} - \frac{8}{v^4 \cos\mathrm{h}\, v}(1 + v \sinh v - \cosh v) \tag{26}$$

and the torsional/lateral relative stiffness parameter η becomes

$$\eta = \frac{\gamma_s}{\gamma_f} = b_{sf}. \tag{27}$$

The frequency ratio coefficient λ can be represented graphically in terms of the parameter $k\alpha H$ and torsional/lateral relative stiffness factor $\eta = b_{sf}$, as shown in Figure 3.

The curves show that the frequency ratio coefficient λ depends on the relative stiffness factor b_{sf}. It has a relatively small influence on λ at low values of $k\alpha H$, and has an increasing influence as $k\alpha H$ increases. However when $k\alpha H > 10$, the change of frequency ratio coefficient will be slight. This implies that for higher values of $k\alpha H$ increasing the coupling degree of connecting beams between columns or changing the building height will have less effect on the uncoupled frequency ratio than that on the frequency themselves, especially for those with the relative stiffness factor b_{sf} close to unity.

It is also noted that the frequency ratio coefficient λ for high values of $k\alpha H$ converges to the relative stiffness factor b_{sf}. The same observation was made by Yoon and Stafford–Smith [3] by using the decoupling eigenvalue approach. In other words, b_{sf} can be used to evaluate the value of λ or taken as an approximate of λ for structures with high values of $k\alpha H$.

For b_{sf} equal to unity, the frequency ratio coefficient λ remains as unity regardless of the value of $k\alpha H$. For b_{sf} lower than unity, the frequency ratio coefficient λ decreases as the value of $k\alpha H$ increases. The frequency ratio coefficient λ increases with increasing $k\alpha H$ and increases more rapidly for higher values of relative stiffness factor when b_{sf} is larger than unity.

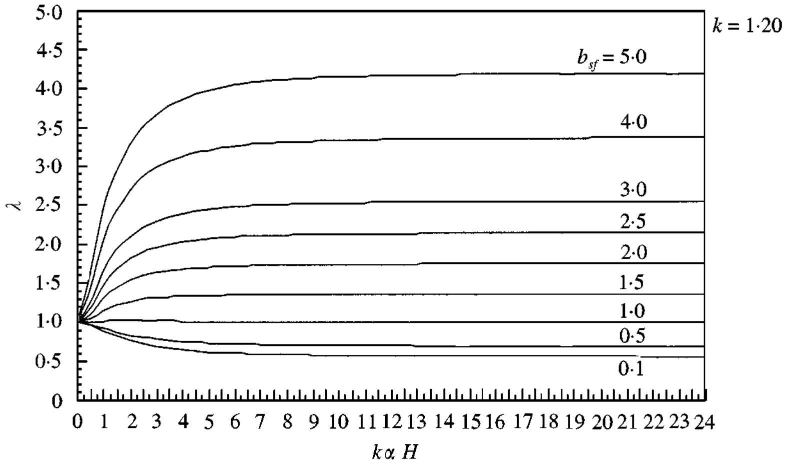

Figure 4. Uncoupled frequency ratio coefficient of the fundamental torsional-to-lateral modes when $k = 1{\cdot}20$.

4.1.2. *Structures with identical frames in frame system*

When all frames in different bents of a wall–frame building structure are identical, the ratio of (GA) to (EAc^2) for every frame is the same, and, consequently, the shear and axial radii of gyration about C_m of frame system are identical, i.e., $\gamma_s = \gamma_a$.

Using the expression of $b_{sf} = \gamma_s/\gamma_f$, equations (17) and (16) become

$$p^2 = \frac{1}{b_{sf}^2}(k^2 - 1) + 1 \tag{28}$$

and

$$\eta = \frac{1}{k}(k^2 - 1 + b_{sf}^2)^{1/2}. \tag{29}$$

The frequency ratio coefficient λ can also be represented graphically in terms of the parameter $k\alpha H$ and torsional/lateral relative stiffness factor b_{sf} for different values of k, as shown in Figure 4 for $k = 1{\cdot}20$.

As shown in Figures 3 and 4, the structural parameter k, which accounts for the axial deformations of vertical elements, has significant influences on the frequency ratio coefficient λ when the stiffness factor b_{sf} is far from unity. In other words, parameter k has small influences on λ when b_{sf} is close to unity. The λ keeps to unity regardless of the values of k and $k\alpha H$ when the relative stiffness factor b_{sf} equals unity. As k increases, it tends to shift the curves towards the unity line, as shown in Figures 3 and 4.

The fundamental uncoupled frequency ratio Ω_0 provides an approximate indicator for the degree of lateral-torsional coupling for an asymmetric building. If the ratio is unity, the maximum degree of lateral and torsional coupling is likely to occur, while if the ratio is between 0·8 and 1·2, the structure should be considered to be liable to severe increases in deflections and forces due to coupling.

A large value of the ratio Ω_0 for a building implies that the building is torsionally stiff with resisting elements near the perimeter of the building plan. On the other hand, a small value of the ratio Ω_0 indicates a torsionally flexible building with a stiff central core but a flexible perimeter. It has become customary to draw the line between flexible and stiff torsional systems at $\Omega_0 = 1$. For most buildings the ratio, Ω_0, range is normally between 0.8 and 1.5.

5. FIRST TWO COUPLED FREQUENCIES OF UNIFORM ASYMMETRIC WALL–FRAME BUILDINGS

5.1. APPROXIMATE FORMULATION OF THE FIRST TWO COUPLED FREQUENCIES OF ASYMMETRIC WALL–FRAME STRUCTURES

From studies of Hejal and Chopra [2], the modal and dynamic response analysis of a special class of building subjected to the component of the translational earthquake ground motion can be determined by analysing two smaller systems: (1) a corresponding torsionally uncoupled, N-story system with N degrees of freedom; and (2) a set of torsionally coupled, one-story systems with 2 degrees of freedom. This special class of building, called proportionate structural system, has been identified as that consisting of resisting elements, which are connected at each floor level by a rigid diaphragm, with the following characteristics: (1) the centres of mass of all floors lie on a vertical line; and (2) lateral stiffness matrices of all resisting elements along one direction are proportional. As a result, the centres of rigidity of floors of such buildings are uniquely defined and lie on a vertical line. Thus, the static eccentricity of each floor, which is defined as the distance between the centres of mass and rigidity of the floor, is the same for all floors.

For the floor plans of a building assumed to have one axis of symmetry, each floor has two degrees of freedom: y-translation and θ-rotation. The coupled natural frequencies and mode shapes can be determined by solving the eigenproblem of a set of one-storey, two-degree-of-freedom systems, in terms of the uncoupled natural frequencies f_{yj}, $f_{\theta j}$ and the eccentricity e, as follows:

$$\begin{bmatrix} 1-\left(\dfrac{f_n}{f_{yj}}\right)^2 & \dfrac{e}{\gamma_m} \\ \dfrac{e}{\gamma_m} & \left(\dfrac{e}{\gamma_m}\right)^2+\left(\dfrac{f_{\theta j}}{f_{yj}}\right)^2-\left(\dfrac{f_n}{f_{yj}}\right)^2 \end{bmatrix} \begin{Bmatrix} \varphi_y \\ \varphi_\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{30}$$

where $\varphi^{\mathrm{T}} = (\varphi_y, \varphi_\theta)$ is an eigenvector of equation (30) of a one-storey, two-degree-of freedom system; f_{yj} and $f_{\theta j}$ are the jth ($j = 1, 2, \ldots, N$) pair of uncoupled lateral and torsional frequencies, respectively, f_n is the nth ($n = 1, 2$) coupled natural frequency; e is eccentricity, which is the distance between the vertical axes through mass centres, C_m, and the centre of rigidity, C_r; and γ_m is the radius of gyration of the mass.

The solution of equation (30) yields the natural coupled frequencies for the first two modes of vibration

$$f_1 = f_y\left\{\frac{1}{2}\left[1+\left(\frac{e}{\gamma_m}\right)^2+\Omega_0^2\right]-\frac{1}{2}\sqrt{\left[1+\left(\frac{e}{\gamma_m}\right)^2-\Omega_0^2\right]^2+4\Omega_0^2\left(\frac{e}{\gamma_m}\right)^2}\right\}^{1/2} \tag{31a}$$

and

$$f_2 = f_y\left\{\frac{1}{2}\left[1+\left(\frac{e}{\gamma_m}\right)^2+\Omega_0^2\right]+\frac{1}{2}\sqrt{\left[1+\left(\frac{e}{\gamma_m}\right)^2-\Omega_0^2\right]^2+4\Omega_0^2\left(\frac{e}{\gamma_m}\right)^2}\right\}^{1/2}, \tag{31b}$$

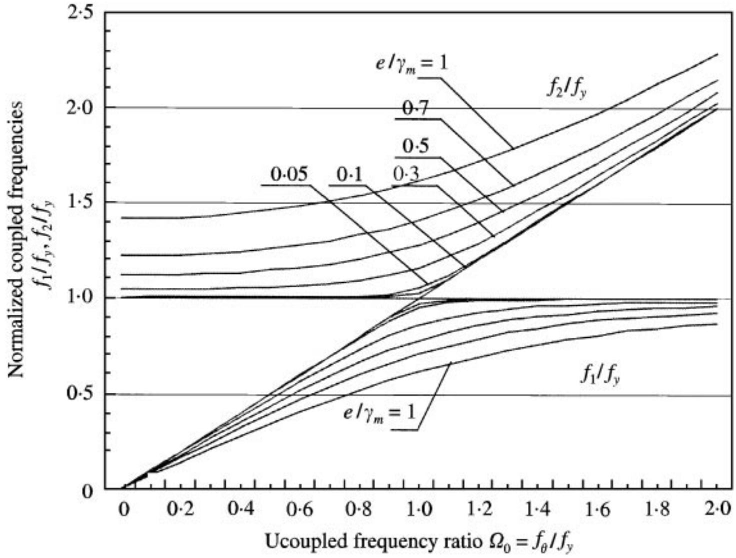

Figure 5. The first two normalized coupled vibration frequencies against frequency ratio Ω_0 and eccentricity ratio e/γ_m.

where f_y is the fundamental lateral frequency; and Ω_0 is the uncoupled fundamental torsional-to-lateral frequency ratio.

Equations (31a) and (31b) are applicable only to proportionate multi-storey buildings in which the centres of mass and the centres of rigidity are located along the two vertical axes.

The first two coupled natural frequencies $f_{1,2}$ normalized by the fundamental lateral frequency f_y depend only on the eccentricity ratio e/γ_m and the uncoupled torsional-to-lateral frequency ratio Ω_0. Normalized frequencies f_1/f_y, f_2/f_y are plotted is Figure 5 against Ω_0 for different values of e/γ_m. Also included for comparison are uncoupled frequencies f_y and f_θ, both normalized by the fundamental lateral frequency f_y, in order to identify effects of lateral–torsional coupling on the natural vibration frequencies. It is demonstrated from Figure 5 that the uncoupled fundamental lateral and torsional frequencies f_y and f_θ are upper and lower bounds of the coupled frequencies: as e/γ_m increases, f_1 decreases below f_y and f_θ, and f_2 increases above f_y and f_θ. Obviously, the coupled frequencies are the closest to the uncoupled ones for systems with the smallest e/γ_m values. For torsionally flexible systems (i.e., $\Omega_0 < 1$), f_θ is the upper bound of f_1, while f_y is the lower bound for f_2. On the other hand, for torsionally stiff systems (i.e., $\Omega_\theta > 1$), f_y is the upper bound for f_1, while f_θ is the lower bound for f_2. For systems with closely spaced uncoupled fundamental frequencies (i.e., Ω_0 is around unity), the first two coupled frequencies are close one another for systems with smaller values of e/γ_m.

For non-proportionate structural systems, such as asymmetric wall-frame multi-storey buildings, the Hejal's formulations of equations (31a) and (31b) are not applicable because the centres of rigidity of floors inevitably do not lie on a common vertical axis throughout the height of the building structure. However, the essentially non-proportionate structural system will be perceived, in the method proposed here, as an equivalent proportionate one, in which the centres of rigidity of floors are considered to be located along a fictitious single vertical axis with an eccentricity e^*, defined as the distance between the centres of mass C_m and fictitious centres of rigidity C_r. Thus, the first two coupled vibration frequencies, equations (31a) and (31b), can then be expressed in terms of the equivalent eccentricity ratio e^*/γ_m and uncoupled fundamental frequency ratio Ω_0, as follows:

$$f_1 = f_y \left\{ \frac{1}{2}\left[1 + \left(\frac{e^*}{\gamma_m}\right)^2 + \Omega_0^2\right] - \frac{1}{2}\sqrt{\left[1 + \left(\frac{e^*}{\gamma_m}\right)^2 - \Omega_0^2\right]^2 + 4\Omega_0^2\left(\frac{e^*}{\gamma_m}\right)^2}\right\}^{1/2} \tag{32a}$$

and

$$f_2 = f_y \left\{ \frac{1}{2}\left[1 + \left(\frac{e^*}{\gamma_m}\right)^2 + \Omega_0^2\right] - \frac{1}{2}\sqrt{\left[1 + \left(\frac{e^*}{\gamma_m}\right)^2 - \Omega_0^2\right]^2 + 4\Omega_0^2\left(\frac{e^*}{\gamma_m}\right)^2}\right\}^{1/2}. \tag{32b}$$

So the key question is how to determine fictitious centres of rigidity of floors for a non-proportionate asymmetric wall–frame multi-storey structure.

5.2. EQUIVALENT ECCENTRICITY e^* FOR NON-PROPORTIONATE STRUCTURAL SYSTEMS

The accuracy of the first two coupled natural frequencies of a non-proportionate wall–frame building structure will depend on the accuracy of the equivalent eccentricity e^* when using equations (32a) and (32b). In deciding how to estimate the equivalent eccentricity of asymmetrical wall–frame multi-storey building structures, it is reasonable to approach it in terms of significant centres of reference, such as the centres of rigidity and shear centres.

It has been demonstrated by Hejal and Chopra [2] that for proportionate multi-storey building structures the centres of rigidity and shear centres of floors of buildings are coincident. Locations of these centres are independent of the distribution of the applied lateral loading and lie on a vertical line. However, for non-proportionate multi-storey building structures these centres do not coincide. Their locations not only depend on geometric and stiffness characteristics of the building, but also on the distribution of the applied lateral loading. It is apparent that the concept and definition of the centres of rigidity in non-proportionate multi-storey building structures are still active subjects of discussion [6–12].

For prismatic structures consisting of only proportionate structural walls, i.e., walls whose flexural rigidities are constant in proportion throughout the height, the centre of rigidity and shear centre are at the centre of flexural rigidities, C_f, and are constant in plan location throughout the height of the building. Similarly, for prismatic structures containing only proportionate frames, i.e., frames whose racking shear rigidities are constant in proportion throughout the height, the centre of rigidity and shear centre are at the centre of shear rigidity, C_s, and are constant in plan location throughout the height. Although the majority of building structures comprises either non-proportionate frames, non-proportionate walls, or a combination of frames, walls, and other types of lateral load-resisting systems, most uniform wall–frame multi-storey buildings structures can be approximately treated as a combination of a proportionate frame system and a proportionate wall system.

For a uniform wall–frame multi-storey building having one axis of symmetry, it can be expressed as a combination of a proportionate frame system and a proportionate wall system. If this non-proportionate structure is treated as an equivalent proportionate structural system, the centres of rigidity of all floors can be considered to be located along a fictitious single vertical line. The equivalent eccentricity e^* is defined as the distance between the centres of mass C_m and these fictitious centres of rigidity C_r. When the wall system is predominant in the lateral behaviour of a building structure, the centres of rigidity will be close to the centre of flexural rigidity of the wall system. Otherwise, the centres of rigidity will be close to the centre of the shear rigidity of the frame system when the frame system is predominant in the lateral motion. In the case where neither the wall system nor the frame system is predominant in resisting the lateral loading, fictitious centres of rigidity will be located between the centre of flexural rigidity of the wall system and the centre of shear rigidity of the frame system.

As demonstrated in the previous study [13], the coupling degree of connecting beams between columns regarding the lateral behaviour in a bent of the building structure can be measured by the ratio of the second to fundamental frequency in the same direction. In the same way, whether the wall system or the frame system is more pronounced in lateral behaviour can also be measured by this frequency ratio. Theoretically, the frequency ratio for a pure flexure cantilever is 6·27 and 3 for a pure shear cantilever. Since a frame system of a regular shape basically has shear-type lateral deformation, it is expected that its lateral modal frequency ratio of the second of fundamental mode should be close to the value of 3. On the other hand, a slender wall system is expected to have frequency ratio close to the value of 6·27. It implies that if the frequency ratio of a uniform multi-storey wall–frame building in lateral motion is close to 6·27, the wall system in the building will be predominant in the lateral behaviour and the centres of rigidity will be close to the centre of flexural rigidity of the wall system. In contrast, the frame system in a building will be predominant in the lateral behaviour and the centres of rigidity will be close to the centre of shear rigidity of the frame system if the frequency ratio approaches the value of 3. Therefore, fictitious centres of rigidity and, consequently, the equivalent eccentricity e^* of a uniform wall–frame building may be determined by the frequency ratio of lateral vibration modes and using linear interpolation technique.

The frequency ratio of the second to fundamental mode in the lateral direction is given in [13] as

$$R_y = \frac{f_{y2}}{f_{y1}} = 0{\cdot}2845\sqrt{F(k, k\alpha H)}\,(\lambda H)^2_{y2}, \tag{33}$$

where $F(k, k\alpha H)$ is given by equation (20) or from Figure 2, and

$$(\lambda H)^2_{y2} = \frac{22{\cdot}03k}{\left[k^2 - 1 + \dfrac{1}{1 + \dfrac{(k\alpha H)^2}{22{\cdot}03}}\right]^{1/2}}. \tag{34}$$

The frequency ratio may be also expressed graphically in terms of parameters k and $k\alpha H$, as shown in Figure 6.

As can been seen in Figure 6, the lateral stiffness characteristic parameter, $k\alpha H$, may be used to measure the lateral relative stiffness of shear rigidity to flexural rigidity of wall–frame systems. Small values of $k\alpha H$ account for the combination of greater relative rigidity of the wall system and less height, while larger values of $k\alpha H$ normally indicate greater relative rigidity of the frame system and greater height. However, for larger values of parameter k, for example $k > 1{\cdot}1$, wall–frame building structural systems will behave more like a flexural beam when parameter $k\alpha H$ becomes relative larger, say $k\alpha H > 9$ for example.

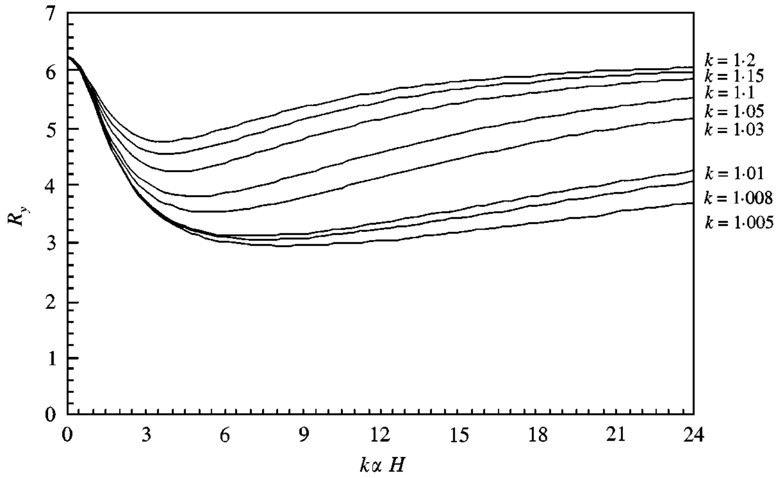

Figure 6. Frequency ratio, R_y, of the second to fundamental frequency.

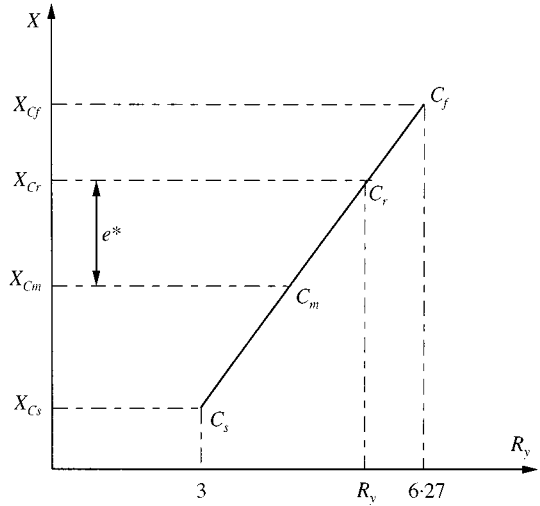

Figure 7. Linear interpolation method to determine the location of the centre of rigidity in wall–frame structures.

The equivalent eccentricity may be obtained by the following linear interpolation, and also from Figure 7.

$$e^* = |X_{Cr} - X_{Cm}| = \left| X_{Cs} + \frac{(R_y - 3)(X_{Cf} - X_{Cs})}{3{\cdot}27} - X_{Cm} \right| \tag{35}$$

in which, R_y is the lateral frequency ratio of the wall–frame building structure in the y direction (as shown in Figure 1); X_{Cm} is the co-ordinate of centres of mass; X_{Cf} and X_{Cs} are co-ordinates of the centre of flexural rigidity of the wall system and the centre of shear rigidity of the frame system respectively; and X_{Cr} is the co-ordinate of fictitious centre of rigidity of the wall–frame building structure.

6. SUMMARY OF PROCEDURE

The procedure for estimating the first two coupled natural frequencies of a uniform asymmetric wall–frame multi-storey building is given as follows:

- Determine locations of the centre of flexural rigidity of wall system C_f, the centre of shear rigidity of frame system C_s, the centre of mass of floor plan C_m, and radii of gyration γ_m, γ_f and γ_s about C_m.
- Calculate the lateral stiffness parameter k and $k\alpha H$, and torsional stiffness parameter p and $p\beta H$.
- Calculate uncoupled frequency ratio, Ω_0, of the fundamental torsional-to-lateral frequencies using equation (20) or Figure 2, and equation (23); obtain the fundamental lateral frequency, f_y, from equation (19).
- Use equations (34) and (33) or Figure 6 to estimate the frequency ratio, R_y, of the second to fundamental frequency in the (same) direction in question, and then determine the equivalent eccentricity e^* from equation (35).
- Predict, finally, the first two coupled natural frequencies by using equation (32).

7. NUMERICAL EXAMPLE

To assess the validity of the proposed method and to illustrate its application to estimation of the first two coupled natural frequencies of uniform asymmetric wall–frame multi-storey buildings, a 16-storey uniform concrete wall–frame building is analyzed using the presented simple method. The structural layout is the same as that in Gluck's study [12], as shown in Figure 1, with total height of 48 m. The storey mass is assumed as $3E + 5$ kg and elastic modulus E as $3{\cdot}3E + 10$ N/m^2. All member properties and co-ordinates are listed in Table 1.

7.1. LATERAL STIFFNESS PARAMETERS k, α AND $k\alpha H$

The flexural rigidity of wall system in the y direction is given by

$$EI = \sum (EI)_j = (2 \times 5{\cdot}7167 + 2 \times 2{\cdot}0833)E = 15{\cdot}6E$$

TABLE 1

Member properties for structure in Figure 1

Wall	Principal moment of inertia, m^4	Position co-ordinates x_j	Position co-ordinates y_j	Frame	Moment of inertia, m$^4 \times 10^{-4}$: Exterior column $0{\cdot}75 \times 0{\cdot}75$	Centre column $0{\cdot}9 \times 0{\cdot}75$	Beam $0{\cdot}8 \times 0{\cdot}4$	Position co-ordinates x_i
1	5·7167	− 16	0	1	2·6367	4·5563	1·7067	− 12
2	2·0833	8	0	2	2·6367	4·5563	1·7067	− 8
3	2·0833	12	0	3	2·6367	4·5563	1·7067	− 4
4	5·7167	16	0	4	2·6367	4·5563	1·7067	0
5	1·0667	10	− 4·5	5	2·6367	4·5563	1·7067	4
6	1·0667	10	4·5					

and the shear rigidity of the frame system by

$$GA = 5 \times \frac{12E}{h\left[\dfrac{1}{\sum \dfrac{I_c}{h}} + \dfrac{1}{\sum \dfrac{I_g}{l}}\right]} = 5 \times \frac{12E}{3 \times \left[\dfrac{1 \times 10^4}{\dfrac{2 \times 2{\cdot}6367 + 4{\cdot}5563}{3}} + \dfrac{1 \times 10^4}{\dfrac{2 \times 1{\cdot}7067}{4{\cdot}5}}\right]} = 0{\cdot}1232E.$$

Therefore

$$\alpha = \left(\frac{GA}{EI}\right)^{1/2} = \left(\frac{0{\cdot}1232E}{15{\cdot}6E}\right)^{1/2} = 0{\cdot}08887/\mathrm{m}$$

and

$$k = \left\{1 + \frac{EI}{\sum(EAc^2)}\right\}^{1/2} = \left\{1 + \frac{15{\cdot}6E}{(10 \times 0{\cdot}75 \times 0{\cdot}75 \times 4{\cdot}5^2 + 2 \times 0{\cdot}2 \times 4 \times 4{\cdot}5^2)E}\right\}^{1/2} = 1{\cdot}052.$$

Then the structural parameter for governing lateral behaviour is given by

$$k\alpha H = 1{\cdot}052 \times 0{\cdot}08887 \times 48 = 4{\cdot}488.$$

7.2. RADII OF GYRATION γ_m, γ_m AND γ_s ABOUT C_m

The mass radius of gyration is given by

$$\gamma_m = \left(\frac{32^2 + 11^2}{12}\right)^{1/2} = 9{\cdot}768.$$

The warping rigidity of the equivalent flexural cantilever about the centre of mass C_m is

$$EI_\omega = \sum_j (EI_j x_j^2 + EI_j y_j^2) = 2 \times 5{\cdot}7167 \times 16^2 + 2 \times 1{\cdot}0667 \times 4{\cdot}5^2$$
$$+ 2{\cdot}0833 \times 8^2 + 2{\cdot}0833 \times 12^2 = 3403{\cdot}48E.$$

Then the radius of gyration of the wall system's flexural rigidity about C_m is obtained from equation (13) as

$$\gamma_f = \left(\frac{EI_\omega}{EI}\right)^{1/2} = \left(\frac{3403{\cdot}48}{15{\cdot}6}\right)^{1/2} = 14{\cdot}771\ \mathrm{m}.$$

The torsional rigidity of the frame system about the centre of shear rigidity C_s is given by

$$GJ_s = \sum_j (GA)_j x_j^2 = 2 \times \frac{0{\cdot}1232E}{5} \times 8^2 + 2 \times \frac{0{\cdot}1232E}{5} \times 4^2 = 3{\cdot}9424E.$$

Thus, the radius of gyration of the frame system's shear rigidity about C_m can be obtained from equation (14) as

$$\gamma_s = \left(\frac{GJ_s}{GA} + 4^2\right)^{1/2} = \left(\frac{3{\cdot}9424E}{0{\cdot}1232E} + 4^2\right)^{1/2} = 6{\cdot}928\ \mathrm{m}.$$

7.3. TORSIONAL STIFFNESS PARAMETERS p, η AND $\eta k\alpha H$

The relative stiffness factor

$$b_{sf} = \frac{\gamma_s}{\gamma_f} = \frac{6{\cdot}928}{14{\cdot}771} = 0{\cdot}469.$$

From equations (28) and (29),

$$p = \left[\frac{1}{b_{sf}^2}(k^2 - 1) + 1\right]^{1/2} = \left[\frac{1}{0{\cdot}469^2}(1{\cdot}052^2 - 1) + 1\right]^{1/2} = 1{\cdot}219$$

and

$$\eta = \frac{1}{k}(k^2 - 1 + b_{sf}^2)^{1/2} = \frac{1}{1{\cdot}052}(1{\cdot}052^2 - 1 + 0{\cdot}469^2)^{1/2} = 0{\cdot}543.$$

Then the structural parameter for governing the torsional behaviour is given by

$$\eta k\alpha H = 0{\cdot}543 \times 4{\cdot}488 = 2{\cdot}437.$$

7.4. UNCOUPLED LATERAL FUNDAMENTAL FREQUENCY f_y AND FREQUENCY RATIO Ω_0

The frequency ratio of the fundamental torsional-to-lateral frequency can be obtained from equations (20), (24) and (23) with

$$F(k, k\alpha H) = 1 - \frac{1}{k^2}\left[1 - \frac{4}{(k\alpha H)^2} + \frac{8}{(k\alpha H)^4 \cosh k\alpha H}(1 + k\alpha H \sinh k\alpha H - \cosh k\alpha H)\right.$$

$$= 1 - \frac{1}{1{\cdot}052^2}\left[1 - \frac{4}{4{\cdot}488^2} + \frac{8}{4{\cdot}488^4 \cosh 4{\cdot}488}\right.$$

$$\times (1 + 4{\cdot}488 \sinh 4{\cdot}488 - \cosh 4{\cdot}488) = 0{\cdot}2133$$

and

$$F(p, \eta k\alpha H) = 1 - \frac{1}{p^2}\left[1 - \frac{4}{(\eta k\alpha H)^2} + \frac{8}{(\eta k\alpha H)^4 \cosh \eta k\alpha H}\right.$$

$$\times (1 + \eta k\alpha H \sinh \eta k\alpha H - \cosh \eta k\alpha H)$$

$$= 1 - \frac{1}{1{\cdot}219^4}\left[1 - \frac{4}{2{\cdot}437^2} + \frac{8}{2{\cdot}437^4 \cosh 2{\cdot}437}\right.$$

$$\times (1 + 2{\cdot}437 \sinh 2{\cdot}437 - \cosh 2{\cdot}437) = 0{\cdot}5401.$$

Therefore

$$\lambda = \left[\frac{F(k, k\alpha H)}{F(p, \eta k\alpha H)}\right]^{1/2} = \left[\frac{0{\cdot}2133}{0{\cdot}5401}\right]^{1/2} = 0{\cdot}6285$$

and

$$\Omega_0 = \lambda \frac{\gamma_f}{\gamma_m} = 0{\cdot}6285 \times \frac{14{\cdot}771}{9{\cdot}768} = 0{\cdot}950.$$

The uncoupled fundamental lateral frequency can be obtained by equation (19) as

$$f_y = \frac{0{\cdot}5595}{H^2} \frac{1}{\sqrt{F(k, k\alpha H)}} \sqrt{\frac{EI}{m}} = \frac{0{\cdot}5595}{48^2} \frac{1}{\sqrt{0{\cdot}2133}} \sqrt{\frac{15{\cdot}6 \times 3{\cdot}3 \times 10^{10}}{3 \times 10^5/3}} = 1{\cdot}193 \text{ Hz}.$$

7.5. EQUIVALENT ECCENTRICITY e^*

The frequency ratio of the second to fundamental lateral frequency is obtained from equations (34) and (33)

$$(\lambda H)_{y2}^2 = \frac{22{\cdot}03k}{\left[k^2 - 1 + \dfrac{1}{1 + \dfrac{(k\alpha H)^2}{22{\cdot}03}}\right]^{1/2}} = \frac{22{\cdot}03 \times 1{\cdot}052}{\left[1{\cdot}052^2 - 1 + \dfrac{1}{1 + \dfrac{4{\cdot}488^2}{22{\cdot}03}}\right]^{1/2}} = 29{\cdot}219$$

and

$$R_y = \frac{f_{y2}}{f_{yt}} = 0{\cdot}2845 \sqrt{F(k, k\alpha H)}\, (\lambda H)_{y2}^2 = 0{\cdot}2845 \times \sqrt{0{\cdot}2133} \times 29{\cdot}219 = 3{\cdot}839.$$

Then the equivalent eccentricity e^* can be determined as

$$e^* = |X_{Cr} - X_{Cm}| = \left| X_{Cs} + \frac{(R_y - 3)(X_{Cf} - X_{Cs})}{3{\cdot}27} - X_{Cm} \right|$$

$$= \left| -4 + \frac{(3{\cdot}839 - 3)(2{\cdot}671 + 4)}{3{\cdot}27} - 0 \right| = 2{\cdot}288 \text{ m}.$$

7.6. THE FIRST TWO COUPLED FREQUENCIES f_1 AND f_2

Finally, the first two coupled natural frequencies of this asymmetric uniform wall–frame building can be obtained as follows:

$$f_1 = f_y \left\{ \frac{1}{2}\left[1 + \left(\frac{e^*}{\gamma_m}\right) - \Omega_0^2\right] - \frac{1}{2}\sqrt{\left[1 + \left(\frac{e^*}{\gamma_m}\right)^2 - \Omega_0^2\right]^2 + 4\Omega_0^2 \left(\frac{e^*}{\gamma_m}\right)^2} \right\}^{1/2}$$

$$= 1{\cdot}193 \times \left\{ \frac{1}{2}\left[1 + \left(\frac{2{\cdot}288}{9{\cdot}768}\right)^2 + 0{\cdot}95^2\right] \right.$$

$$\left. - \frac{1}{2}\sqrt{\left[1 + \left(\frac{2{\cdot}288}{9{\cdot}768}\right)^2 - 0{\cdot}95^2\right]^2 + 4 \times 0{\cdot}95^2 \times \left(\frac{2{\cdot}288}{9{\cdot}768}\right)^2} \right\}^{1/2}$$

$$= 1{\cdot}029 \text{ Hz}.$$

TABLE 2

Comparison between results from proposed method and those from ANSYS FEM for the first two lowest coupled natural frequencies of 16-storey uniform building structures

Coupled natural frequencies (Hz)	Proposed method	ANSYS FEM
Mode 1	1·029	1·035
Mode 2	1·314	1·290

$$f_2 = f_y \left\{ \frac{1}{2}\left[1 + \left(\frac{e^*}{\gamma_m}\right)^2 - \Omega_0^2\right] + \frac{1}{2}\sqrt{\left[1 + \left(\frac{e^*}{\gamma_m}\right)^2 - \Omega_0^2\right]^2 + 4\Omega_0^2\left(\frac{e^*}{\gamma_m}\right)^2} \right\}^{1/2}$$

$$= 1{\cdot}193 \times \left\{ \frac{1}{2}\left[1 + \left(\frac{2{\cdot}288}{9{\cdot}768}\right)^2 + 0{\cdot}95^2\right] \right.$$

$$\left. + \frac{1}{2}\sqrt{\left[1 + \left(\frac{2{\cdot}288}{9{\cdot}768}\right)^2 - 0{\cdot}95^2\right]^2 + 4 \times 0{\cdot}95^2 \times \left(\frac{2{\cdot}288}{9{\cdot}768}\right)^2} \right\}^{1/2}$$

$$= 1{\cdot}314 \text{ Hz}.$$

In assessing the accuracy of the proposed method, the first two coupled natural frequencies of this 16-storey building structure are also determined by using a 3D finite element model in the ANSYS 5·4 software package. The results are listed in Table 2.

Furthermore, other structures with different plan arrangements and different lateral relative stiffness of the shear rigidity of the frame system to the flexural rigidity of the wall system are analyzed [13]. The maximum frequency discrepancy between the proposed simple method and the finite element method is 7·5%, while discrepancies, in most of the cases, are well within 6%. The differences are believed to arise mainly from the representation of the non-proportionate structural system of wall–frame buildings as an equivalent proportionate one, in which the centres of rigidity of floors are assumed to be located along a fictitious single vertical axis. The equivalent eccentricity e^*, defined as the distance between the centres of mass C_m and fictitious centres of rigidity C_r, is used for equation (32) to determine the coupled natural frequencies.

8. CONCLUSIONS

A simple approximate method to estimate the first two coupled natural frequencies of uniform asymmetric wall–frame multi-storey buildings is presented. The method first involves the determination of the uncoupled torsional-to-lateral frequency ratio, which is then used in combination with a fictitious equivalent eccentricity to determine the first two coupled natural frequencies.

The uncoupled fundamental torsional-to-lateral frequency ratio can be used to indicate the degree of lateral and torsional coupling of asymmetric wall–frame multi-storey structures. If the ratio is unity, the maximum degree of lateral and torsional coupling is likely to occur, and the coupled fundamental natural frequency will be quite different from the uncoupled one. The ratio Ω_0 can also serve to indicate whether the fundamental coupled mode of a wall–frame structure will be laterally predominant ($\Omega_0 > 1$) or torsionally predominant ($\Omega_0 < 1$).

For small eccentricities, the fundamental uncoupled natural frequency is a good approximation of the coupled one.

For non-proportionate structural systems with centres of rigidity varying from floor to floor over the building height, it is assumed that the centres of rigidity are located along a single fictitious vertical axis, at a distance called equivalent eccentricity from the vertical axis of mass centres. As a result, the simple formulae, used to determine the coupled natural frequencies of proportionate structural systems, have been extended to approximate to the coupled natural frequencies of wall–frame buildings.

The frequency ratio of the second to fundamental lateral frequency in the direction in question can be used to indicate whether the wall system or the frame system is more dominant in structural lateral behaviour, and also to locate fictitious centres of rigidity of wall–frame multi-storey buildings. When the frequency ratio is close to 6·27, the lateral motion of a wall–frame multi-storey structure is dominated by its wall system; in contrast, the frame structural behaviour will be predominant in lateral motion while the frequency ratio approaches 3.

The proposed method provides a useful design tool for practising engineers to estimate the fundamental natural frequency of a uniform asymmetric wall–frame multi-storey building at an early stage in design without having to perform a time-consuming modal analysis. By simply requiring the preliminary design information, including dimensions and material properties, the method can also be used to indicate the degree of lateral–torsional coupling, and the extent of effects on the lateral and torsional behaviour caused by the wall system and the frame system in a wall–frame multi-storey building.

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