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LETTERS TO THE EDITOR

A ROLE OF INITIAL CONDITIONS CHOICE ON THE RESULTS OBTAINED USING DIFFERENT PERTURBATION METHODS

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1. INTRODUCTION

In reference [1] (pp. 2900–2902) the following interesting problem has been stated. Consider the Duffing equation

$$\ddot{x} + \omega_0^2 x + \varepsilon x^3 = 0 \tag{1}$$

with the following initial conditions:

$$x(0) = A, \quad \dot{x}(0) = 0. \tag{2}$$

An application of the perturbation method in the form introduced by Linstedt–Poincaré [1, 2] results in a series approximation to the solution defined by the initial problem (1), (2) with regard to the powers of small parameter ε. However, there are different ways to fulfil the initial conditions (2). If in the zero approximation the following conditions are taken:

$$x_0(0) = A, \quad \dot{x}_0(0) = 0,$$

then the series of sought amplitude–frequency characteristics has the form [1]

$$\omega = \omega_0 + \frac{3A^2}{8\omega_0}\varepsilon - \frac{21A^4}{16^2\omega_0^3} + \frac{81A^6\varepsilon^3}{8{\cdot}16\omega_0^5} + \cdots . \tag{3}$$

Nayfeh [2] proposed another approach, where in the first approximation the amplitude α is unknown. Then one gets

$$\omega = \omega_0 + \frac{3\alpha^2\varepsilon}{8\omega_0} - 15\frac{\alpha^4\varepsilon^4}{16^2\omega_0^3} + \cdots \tag{4}$$

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and α is defined by the equation

$$A = \alpha + \frac{\alpha^3 \varepsilon}{32\omega_0^2} - \frac{5\alpha^2\varepsilon^2}{16^2\omega_0^4} + \cdots. \tag{5}$$

Both of the described approaches are equivalent from the point of view of asymptotic series. In fact, making an inverse of expression (5) and substituting

$$\alpha = A + \varepsilon\varphi_1(A) + \varepsilon^2\varphi_2(A) + \cdots$$

into equation (4) one obtains equation (3). However, asymptotical and real errors belong to qualitatively different aspects of the problem. Among others, we are going to show that the exactness of the solution of equation (1) depends on the form on the expansion of its solution, and that the exactness of the solution of equation (6) depends on the chosen variables (this question has been also discussed in reference [3]). We begin with a numerical experiment for $\omega_0 = \varepsilon = 1$. The results are shown in Figure 1, where the curves 1–3 correspond to the following solutions: governed by (3); exact solution which can be expressed via elliptic functions [2]; the solution defined by equations (4) and (5). In the last case, first α has been found numerically from equation (5), and then it has been substituted into equation (4). It is seen that the method proposed by Nayfeh leads to better results.

2. DISCUSSION OF THE SOLUTION STRUCTURE

It is known [4–8] that an asymptotical series better describes a sought solution if the analytical structure is better approximated by a particular structure of the problem. For instance, from a point of view of asymptotical approach two amplitude–frequency characteristics

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots,$$
$$\omega^2 = \omega_0^2 + \varepsilon\gamma_1^2 + \varepsilon^2\gamma_2^2 + \cdots$$

are essentially equivalent. However, from a point of view of numerical values the second results in better accuracy [7, 8].

We consider an example of pendulum oscillations governed by the equations

$$\ddot{\theta} + \sin\theta = 0, \quad \theta(0) = \theta_0, \quad \dot{\theta}(0) = 0. \tag{6, 7}$$

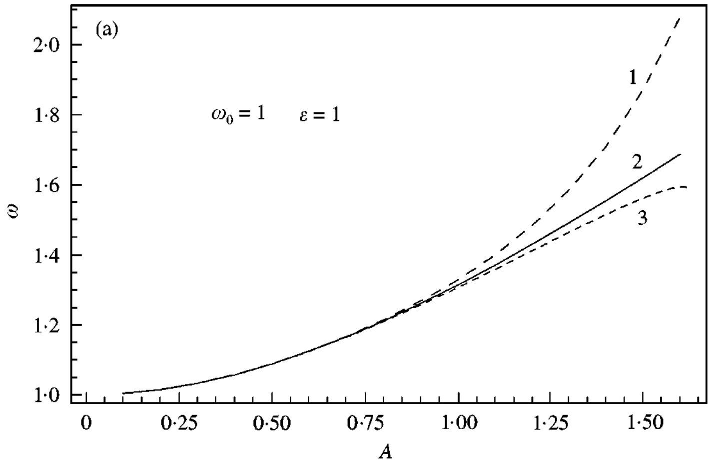
One of the typical approaches is to develop $\sin\theta$ into series of θ and then to apply the perturbation technique. However, it is more efficient to apply the transformation

$$\sin\theta = \varphi. \tag{8}$$

Then the initial Cauchy problem defined by equations (6) and (7) is substituted by the following one:

$$\ddot{\varphi} + \frac{\varphi\dot{\varphi}^2}{1-\varphi^2} + \varphi\sqrt{1-\varphi^2} = 0, \tag{9}$$

$$\varphi(0) = \varphi_0 \equiv \sin\theta_0, \quad \dot{\varphi}(0) = 0. \tag{10}$$



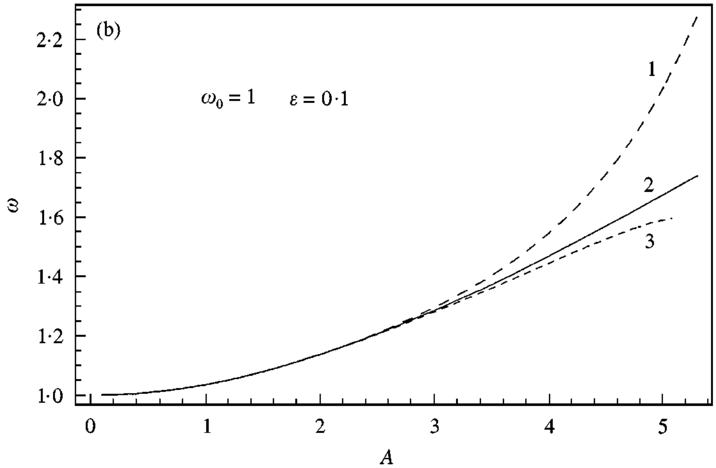


Figure 1. A comparison of results obtained using different analytical methods: (a) $\omega_0 = 1, \varepsilon = 0.1$; (b) $\omega_0 = \varepsilon = 1$.

The solutions to the problems (6), (7) and (9), (10) can be found using perturbation technique (8). In the first case, the following value of the period is obtained:

$$T = 2\pi(1 + \theta_0^2/16 + \cdots). \tag{11}$$

In the second case, we have

$$T = 2\pi[1 + 0{\cdot}25 \sin^2(0{\cdot}5\theta_0) + (9/64)\sin^4(0{\cdot}5\theta_0) + \cdots]. \tag{12}$$

The numerical results are shown in Figure 2. Curve 1 corresponds to the linear solution, whereas curves 2 and 3 correspond to solution (11) and (12) respectively. The curve 4 corresponds to exact solution which in this case can be expressed via elliptic functions [9]. It is obvious, that when an analytical structure of the problem is taken via transformation (8) then the obtained accuracy of the results increases. The same observation holds also for the problem defined in the introduction. From the point of view of numerical solution a more suitable approach is the one presented in reference [2], which more deeply exhibits key features of solution to the Cauchy problem. However, there is still an important

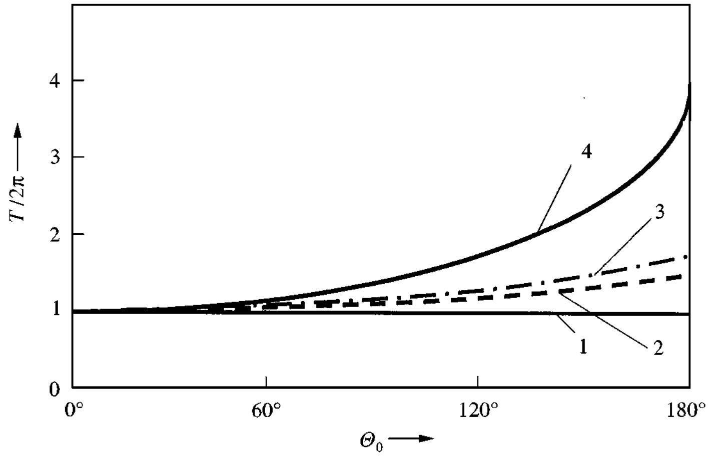


Figure 2. Period of oscillation of a mathematical pendulum against initial angle θ_0.

question not answered yet: what is the best choice of the parameters (ε, α) and the expansion of solution (this remark has been pointed out by a reviewer of this Letter).

3. CONCLUSIONS

Nowadays asymptotic methods play an important role during qualitative investigations of non-linear systems. The various numerical methods are used for a purpose of quantitative analysis. In order to extend an interval of asymptotic technique action one needs to use sometimes new but non-asymptotic ideas. Various methods devoted to series summation belong to the latter group (the Euler–Bellman method [4–6], Padé methods [6–8], quasirational approximations [8], the Laplace–Borel method [10], and others), as well as numerical asymptotical methods [11] and the methods taking into account the particular properties of a solution [4–6].

It should be emphasized that a question related to estimation of an asymptotical error belongs rather to purely mathematical problem, and there is no need to include this type of considerations to applicational oriented theory. The last one can be accepted even without obtaining asymptotical accuracy, and *vice versa*, and is non-acceptable when high asymptotical accuracy is achieved.

Researchers working in the fields of physics and/or mechanics are focused on achieving rather real accuracy than the asymptotical one. These two terms are not equivalent. In particular, with an increase of an asymptotical accuracy a real accuracy can decrease.

The last remark leads to the following conclusion. In a case of an asymptotical technique it is more efficient to focus rather on exhibiting a real analytical structure of the problem than to concentrate on achieving a high asymptotical accuracy.

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