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A NOTE ON MODELLING MULTI-DEGREE-OF-FREEDOM VIBRO-IMPACT SYSTEMS USING COEFFICIENT OF RESTITUTION MODELS

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1. INTRODUCTION

In this letter, we consider the problem of modelling multi-degree of freedom (d.o.f.) impact oscillator systems using coefficient of restitution models. Impact oscillator systems have been studied extensively in recent years following the work of Shaw and Holmes [1]. Most of this work has been focused on models for single-d.o.f. systems [2], although inevitably experimental systems studied were often multi-modal (see, for example, references [3–6]). In the majority of both mathematical and experimental impact oscillator studies the impact process was modelled using a coefficient of restitution rule. This rule was both effective and simple in capturing the key properties of an impacting body. However, in experimental studies using an impacting beam system, Thompson *et al.* [7] noted that a low value of the coefficient of restitution must be used in numerical simulations of the system in order to match dynamical behaviour. Similar results were found by Weger *et al.* [8]. Thompson *et al.* [7] conjectured that this reduction in value for the coefficient of restitution was due to the transfer of energy into higher modes of vibration.

We consider here the dynamics of a multi-d.o.f. linear system, with proportional damping and a single point of impact. An energy balance approach for a periodic impacting motion is described. Then we derive a relationship between modal energy and the coefficient of restitution which exists for periodic impact orbits of multi-d.o.f. systems. For certain experimental systems this analysis can be used to estimate values for the coefficient of restitution [9]. Here we demonstrate the effects of increased modal behaviour by modelling a multi-modal beam system using models with one and four modes.

2. MATHEMATICAL MODEL

For a multi-modal system with N d.o.f. such as the lumped mass model shown in Figure 1, the equations of motion can be expressed in matrix form as

$$[\mathbf{M}]\ddot{\mathbf{x}} + [\mathbf{C}]\dot{\mathbf{x}} + [\mathbf{K}]\mathbf{x} = \mathbf{f}(\tau), \quad x_N < x_s, \tag{1}$$

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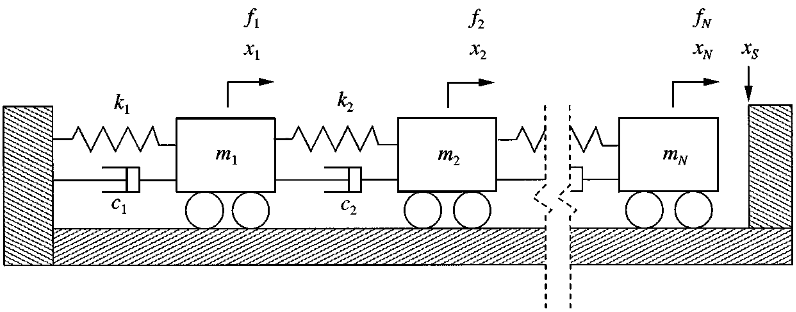

Figure 1. Schematic representation of an N-d.o.f. impact oscillator.

where $[\mathbf{M}], [\mathbf{C}], [\mathbf{K}]$ are the mass, damping and stiffness matrices, respectively, $\mathbf{x} = \{x_1, x_2, \dots, x_N\}^{\mathrm{T}}$ the displacement vector, x_s is the distance to the impact stop and $\mathbf{f}(\tau) = \{f_1, f_2, \dots, f_N\}^{\mathrm{T}}$ the external forcing vector. For a lumped mass system, the coupling between masses occurs via the matrices $[\mathbf{C}]$ and $[\mathbf{K}]$, which are non-diagonal. The mass matrix $[\mathbf{M}]$ is a diagonal matrix. For this system, we assume that only mass N can impact. When an impact occurs, we apply a coefficient of restitution rule, which in matrix form can be written as

$$\dot{\mathbf{x}}(\tau_+) = [\mathbf{R}]\dot{\mathbf{x}}(\tau_-), \quad x_N = x_s, \tag{2}$$

where $\tau-$ is the time just before impact, $\tau+$ is the time just after impact, $r \in [0, 1]$ is the coefficient of restitution and $[\mathbf{R}] = \mathrm{diag}\{1, 1, 1, \dots, 1, 1, -r\}$. Thus, the impacting mass, mass N, has its velocity reversed and reduced by r at impact. This process is assumed to be *instantaneous.*

As we are primarily interested in systems with uniformly distributed parameters, we consider the case where $m_j = m$, $c_j = c$, $k_j = k$ for $j = 1, 2, \dots, N$, such that c is proportional to k. Then we can write $[\mathbf{C}] = c[\mathbf{E}]$ and $[\mathbf{K}] = k[\mathbf{E}]$, where $[\mathbf{E}]$ is the coupling matrix [10]. Then, by considering the undamped, unforced, (non-impacting) system the natural frequencies of the system are given by $\omega_{nj} = \sqrt{\lambda_j k/m}$ for $j = 1, 2, \dots, N$ [11], where λ_j $j = 1, 2, \dots, N$ are the eigenvalues of $[\mathbf{E}]$. The eigenvectors $\boldsymbol{\xi}_j$ corresponding to each λ_j normalized such that $\| \boldsymbol{\xi}_j \| = 1$ define the corresponding mode shapes of the system. Using these eigenvectors we can construct a modal matrix $[\boldsymbol{\Psi}] = [\{\boldsymbol{\xi}_1\}, \{\boldsymbol{\xi}_2\}, \dots, \{\boldsymbol{\xi}_N\}]$. We can then define modal co-ordinates, using the linear transform $\mathbf{x} = [\boldsymbol{\Psi}]\mathbf{q}$ where $\mathbf{q} = \{q_1, q_2, \dots, q_N\}^{\mathrm{T}}$. Substituting this into equation (1) and premultiplying by $[\boldsymbol{\Psi}]^{\mathrm{T}}$ decouples the system ($[\boldsymbol{\Psi}]$ is orthogonal such that $[\boldsymbol{\Psi}]^{\mathrm{T}} = [\boldsymbol{\Psi}]^{-1}$) to give

$$[\mathbf{I}]\ddot{\mathbf{q}} + \frac{c}{m}[\boldsymbol{\Lambda}]\dot{\mathbf{q}} + \frac{k}{m}[\boldsymbol{\Lambda}]\mathbf{q} = \frac{1}{m}[\boldsymbol{\Psi}]^{\mathrm{T}}\,\mathbf{f}(\tau), \tag{3}$$

where $[\boldsymbol{\Lambda}]$ is the diagonal eigenvalue matrix. We will consider only harmonic forcing of the form $\mathbf{f}(\tau) = \mathbf{A}\cos(\Omega\tau)$, $\mathbf{A} = \{A_1, A_2, \dots, A_N\}^{\mathrm{T}}$. Thus, we can simplify equation (3) such that for each mode,

$$\ddot{q}_j + 2\zeta_j\omega_{nj}\dot{q}_j + \omega_{nj}^2 q = \frac{F_j}{m}\cos(\Omega\tau), \quad j = 1, 2, \dots, N, \tag{4}$$

where $\mathbf{F} = [\boldsymbol{\Psi}]^{\mathrm{T}}\mathbf{A}$, $\mathbf{F} = \{F_1, F_2, \ldots, F_N\}^{\mathrm{T}}$ and $\zeta_j = (c/2)\sqrt{\lambda_j/km}$ is the modal damping coefficient.

However, the motion of the system is constrained such that $x_N < x_s$ during excitation. If we define the vector $\boldsymbol{\psi} = \{\boldsymbol{\Psi}_{N1}, \boldsymbol{\Psi}_{N2}, \ldots, \boldsymbol{\Psi}_{NN}\}^{\mathrm{T}}$, then in terms of modal co-ordinates an impact occurs when $\boldsymbol{\Psi}^{\mathrm{T}}\mathbf{q} = x_s$. Hence, equation (3) is valid only for $\boldsymbol{\psi}^{\mathrm{T}}\mathbf{q} < x_s$ which is equivalent to the condition that $x_N < x_s$.

We are considering a linear coupled system in physical co-ordinates, $\mathbf{x}$, with a plane of discontinuity representing an impact law; the coefficient of restitution. This plane of discontinuity crosses only a single co-ordinate axis x_N, in phase space. Transforming this system into modal co-ordinates, using the linear transform defined by the modal matrix $[\boldsymbol{\Psi}]$ decouples the equations of motion. However, transforming the discontinuity into this modal space results in all modal co-ordinates becoming discontinuous. So now the system is coupled via the impact event.

3. MODAL ENERGY ANALYSIS

For a multi-d.o.f. system, energy loss occurs at impact via the coefficient of restitution rule, and during free flight due to viscous damping. Now we consider a modal energy analysis of a steady state vibro-impact motion of a multi-d.o.f. impact oscillator. From this analysis we derive a relation between energy lost during impact and free flight (modal energy).

In modal space the coefficient of restitution rule, equation (2) becomes

$$[\boldsymbol{\Psi}]\dot{\mathbf{q}}(\tau_+) = [\mathbf{R}][\boldsymbol{\Psi}]\dot{\mathbf{q}}(\tau_-), \qquad \boldsymbol{\psi}^{\mathrm{T}}\mathbf{q} = x_s. \tag{5}$$

This leads to the relation for the modal velocities after impact

$$\dot{\mathbf{q}}(\tau_+) = [\hat{\mathbf{R}}]\dot{\mathbf{q}}(\tau_-), \qquad \boldsymbol{\psi}^{\mathrm{T}}\mathbf{q} = x_s, \tag{6}$$

where $[\hat{\mathbf{R}}] = [\boldsymbol{\psi}]^{-1}[\mathbf{R}][\boldsymbol{\Psi}]$ is the matrix which represents a linear transform of modal velocities just before impact to modal velocities just after impact. This transformation also represents the discontinuous jump in velocities at impact.

Premultiplying the (modal) equation of motion for an N-d.o.f. system, equation (3) by $m\dot{\mathbf{q}}^{\mathrm{T}}$ and integrating with respect to τ gives an expression for the energy at time τ^* as

$$\frac{m}{2}(\dot{\mathbf{q}}^{\mathrm{T}}\dot{\mathbf{q}}(\tau^*) - \dot{\mathbf{q}}^{\mathrm{T}}\dot{\mathbf{q}}(\tau_i)) + \frac{k}{2}(\mathbf{q}^{\mathrm{T}}[\boldsymbol{\Lambda}]\mathbf{q}(\tau^*) - \mathbf{q}^{\mathrm{T}}[\boldsymbol{\Lambda}]\mathbf{q}(\tau_i))$$
$$= \int_{\tau_i}^{\tau^*} \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{F}\cos(\Omega\tau)\,\mathrm{d}\tau - c\int_{\tau_i}^{\tau^*} \dot{\mathbf{q}}^{\mathrm{T}}[\boldsymbol{\Lambda}]\,\dot{\mathbf{q}}\,\mathrm{d}\tau, \tag{7}$$

where $\tau_i < \tau^* < \tau_{i+1}$ is a time between two consecutive impacts, τ_i and τ_{i+1}. The terms on the right-hand side of this expression represent the modal forcing energy FE and modal damping energy DE respectively. For each mode, we refer to $FE_j - DE_j$ at the end of a period as the *residual* modal energy, where

$$FE_j = \int_{\tau_i}^{\tau^*} \dot{q}_j F_j \cos(\Omega\tau)\,\mathrm{d}\tau, \qquad DE_j = c\int_{\tau_i}^{\tau^*} \dot{q}_j\lambda_j\dot{q}_j\,\mathrm{d}\tau. \tag{8}$$

This is a measure of modal energy gain during free flight, over one period of motion for each mode, and the sum over all modes, with $FE = \sum_{j=1}^{N} FE_j$ and $DE = \sum_{j=1}^{N} DE_j$, represents the energy gain for the whole system.

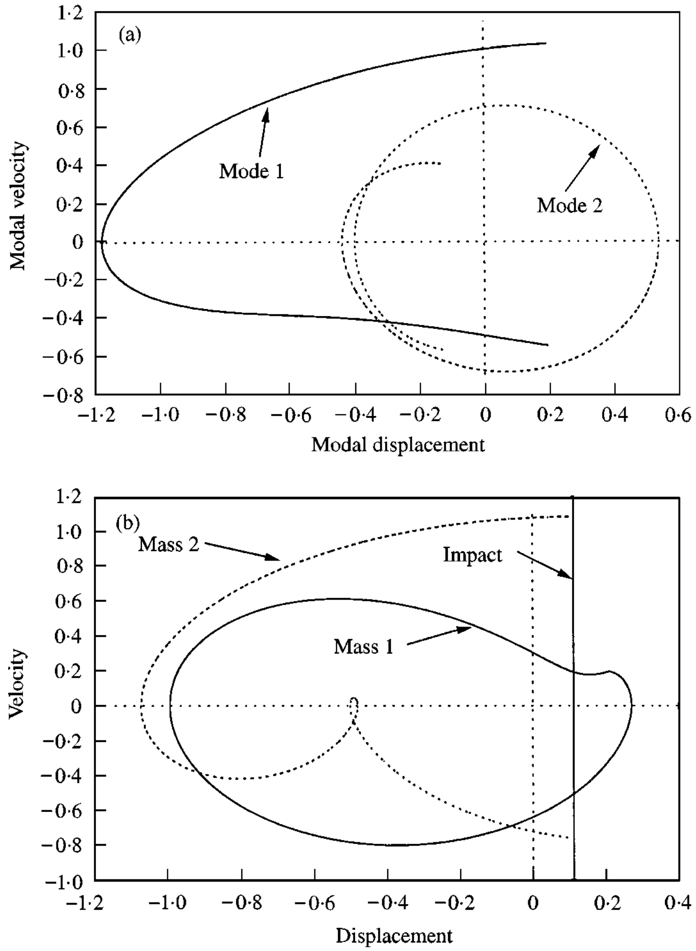

Figure 2. Two-d.o.f. impact oscillator numerically generated phase portraits. Parameter values $m_1 = m_2 = 1$, $k_1 = k_2 = 1$, $c_1 = c_2 = 0{\cdot}1$, $x_s = 0{\cdot}1$, $A_2 = 0{\cdot}0$, $A_1 = 0{\cdot}5$, $\Omega = 0{\cdot}9$ and $r = 0{\cdot}7$. (a) Modal co-ordinates: Solid line q_1, broken line q_2. (b) Physical co-ordinates: Solid line x_1, broken line x_2.

We evaluate this expression for a period one, one impact motion, denoted $\mathbf{P}(1, 1)$. A numerical simulation of such a $\mathbf{P}(1, 1)$ orbit is shown in Figure 2 for a two-d.of. system. The phase portrait is plotted in both modal (Figure 2(a)) and physical (Figure 2(b)) co-ordinates.

We examine the whole period of motion between two impacts τ_i and τ_{i+1} for such an orbit by setting $\tau^* = \tau_{i+1}$. As we are considering the system from one impact to the next, the modal displacement will be the same at τ_i and τ_{i+1}, and hence the potential energy term (second term in equation (7)) is zero. For each impact $\tau_i \equiv \tau_+$ and $\tau_{i+1} \equiv \tau_-$. The kinetic energy term (first term in equation (7)) can be evaluated using the relations; $\dot{\mathbf{q}}(\tau_-) = [\mathbf{\Psi}]^{-1}\dot{\mathbf{x}}(\tau_-)$, $\dot{\mathbf{q}}^{\mathrm{T}}(\tau_-) = \dot{\mathbf{x}}^{\mathrm{T}}(\tau_-)[\mathbf{\Psi}]$, $\dot{\mathbf{q}}(\tau_+) = [\mathbf{\Psi}]^{-1}[\mathbf{R}]\dot{\mathbf{x}}(\tau_-)$ and $\dot{\mathbf{q}}^{\mathrm{T}}(\tau_+) = \dot{\mathbf{x}}^{\mathrm{T}}(\tau_-)$ $[\mathbf{R}][\mathbf{\Psi}]$, to give

$$\frac{m}{2}(\dot{\mathbf{x}}^{\mathrm{T}}(\tau_-)\dot{\mathbf{x}}(\tau_-) - \dot{\mathbf{x}}^{\mathrm{T}}(\tau_-)[\mathbf{R}][\mathbf{R}]\dot{\mathbf{x}}(\tau_-)) \tag{9}$$

which reduces to

$$\frac{m}{2} v_N^2 (1 - r^2), \tag{10}$$

where v_N denotes the velocity of mass N at impact (τ_-). Thus, we can reduce equation (7) to an energy balance expression for a $\mathbf{P}(1, 1)$ orbit

$$\frac{m}{2} v_N^2 (1 - r^2) = \sum_{j=1}^{N} (FE_j - DE_j). \tag{11}$$

The energy lost at impact (left-hand side of equation (11)) is equal to the sum of the residual energy of all the modes in the system. In other words, the energy lost at impact, IE say, is equal to the energy input, FE, less the energy dissipated due to viscous damping, DE. So $IE = FE - DE$ or $FE = IE + DE$ for a $\mathbf{P}(1, 1)$ orbit.

By rearranging equation (11) we can obtain an expression for the coefficient of restitution

$$r = \sqrt{1 - \frac{2}{m v_N^2} \left\{ \sum_{j=1}^{N} (FE_j - DE_j) \right\}}. \tag{12}$$

Then we can reduce equation (12) to

$$r = \sqrt{1 - \frac{RE}{KE_i}}, \tag{13}$$

where $KE_i = m v_N^2/2$ is the kinetic energy at impact, and $RE = \sum_{j=1}^{N} (FE_j - DE_j)$ is the residual energy for all modes.

We know by definition that $r \in [0, 1]$, is a real positive quantity. The kinetic energy at impact $KE_i > 0$ is a strictly positive quantity (excluding zero velocity impacts, which do not occur in stable periodic motion—the case considered here). Thus we can see that for real values of r, $0 < RE < KE_i$. These bounds apply to $\mathbf{P}(1, 1)$ orbits in systems with an arbitrary N degrees of freedom. Then from equation (13), we can see that by increasing residual energy or decreasing kinetic energy at impact, the coefficient of restitution is reduced. This is shown in Figure 3(a) for $KE_i = 1{\cdot}0$, and in Figure 3(b) for $KE_i > RE$. Thus, from Figure 3, we see that for a multi-modal system, it is possible to have a complete range of r values $0 \leqslant r \leqslant 1$, dependent on the energy balance of the system.

4. MODELLING PHYSICAL VIBRO-IMPACT PROBLEMS

The problem of estimating a value for the coefficient of restitution arises when modelling a physical system such as an impacting beam [7, 8, 12]. In reference [7] a value for r was selected using the material type during impact, steel on steel, as the criteria. Using reference sources such as reference [13] one can estimate that for steel on steel impact a coefficient of restitution value in the range 0·85–0·95 should be chosen. As an alternative direct experimental measurements can be made of the velocity before and after impact to find the coefficient [14]. However, in multi-modal systems the influence of higher modes after impact may impede this process. In addition, it is clear from reference [13] and equation (12) that the coefficient of restitution is a non-linear function of impact velocity as

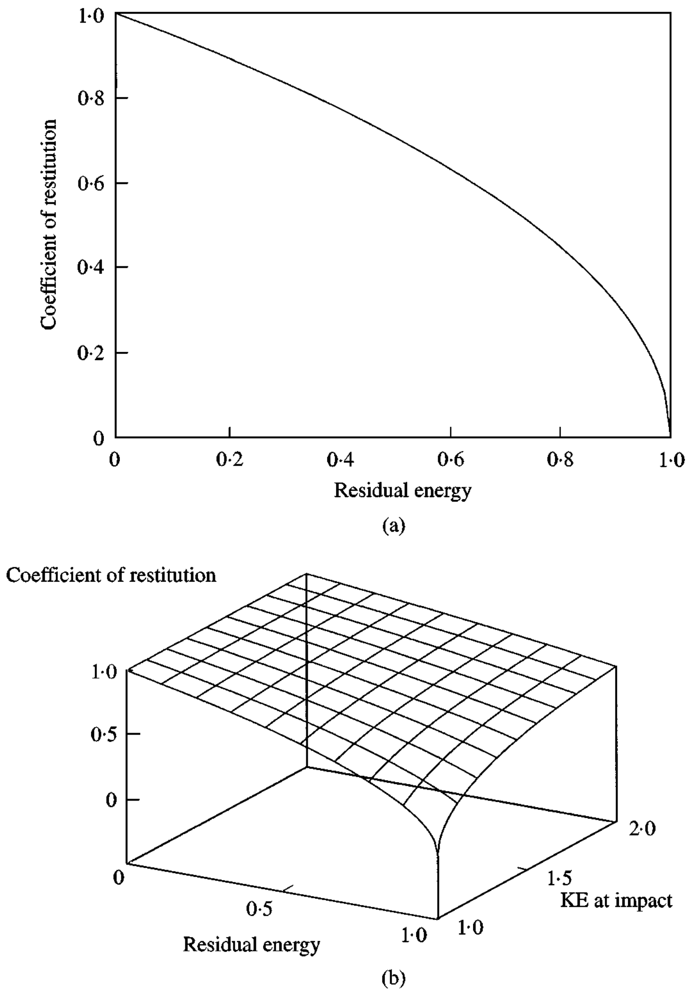

Figure 3. Coefficient of restitution as a function of energy for multi-d.o.f. impact oscillators. (a) r versus RE with $KE_i = 1{\cdot}0$. (b) r as a function of RE and KE_i, $KE_i > RE$.

well as material property. For a particular periodic impact motion the impact velocity is constant, so we can consider a value for the coefficient of restitution to be constant in this case only.

As an example, we consider the vibro-impact cantilever beam system described in detail by Thompson *et al.* [7]; see also references [12, 15]. A time series taken from a typical $\mathbf{P}(1, 1)$ motion is shown in Figure 4. When modelling the dynamics of this system Thompson *et al.* [7] found that a coefficient of restitution value of 0·2 was required in order to simulate the motion of the beam for a range of frequency values close to the first natural frequency of the beam. This simulation was carried out using a single-d.o.f. model. A simulated time series from this model is shown in Figure 5(a). We can see that this simulation captures the essential dynamics of the system, in terms of periodicity and maximum (positive) displacement amplitude.

In order to obtain a multi-modal simulation, we have used a model based on a Galerkin reduction of the Euler–Bernoulli equation for the vibro-impacting beam system [15]. This

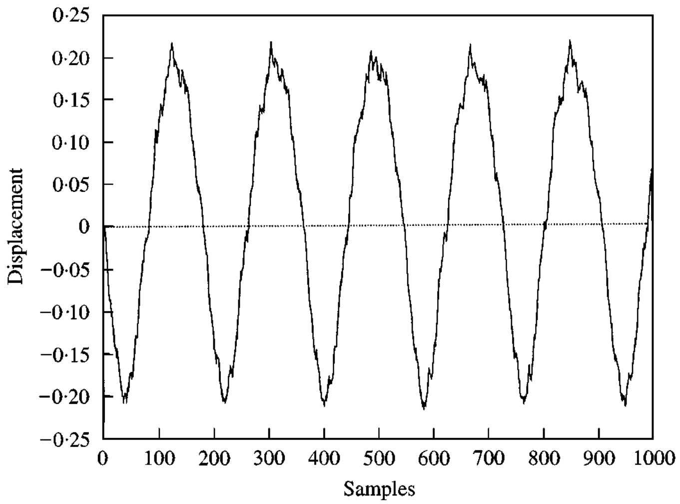

Figure 4. Experimental **P**(1, 1) beam data. Parameter values $A = 2{\cdot}5$ V, $x_s = -0{\cdot}2$ V and $\Omega = 138{\cdot}42$ Hz. Sampling rate: 4000 samples/s.

method incorporates an instantaneous impact rule, and assumes that the impact occurs at the free end of the beam. The result of simulating the motion of the free end of the beam using four modal co-ordinates and a coefficient of restitution value of 0·75 is shown in Figure 5(b). This value of the coefficient of restitution is much closer to the steel on steel material range 0·85–0·95, which suggests that the four-d.o.f. model is a closer representation of the physical system, in terms of both dynamical behaviour and energy transfer. The fact that the four-d.o.f. system models the dynamics of the beam more closely than the single-d.o.f. model is perhaps not surprising. However, the energy balance is distinctly different between the two models, and demonstrates the difficulties encountered when trying to model the behaviour of these complex systems.

As for many systems, the use of a low-dimensional model may be sufficient to model the key dynamics, the problem is that an appropriate reduced coefficient value needs to be estimated. This can be achieved by using equation (12) in conjunction with experimentally recorded data [9, 15].

5. CONCLUSIONS

When modelling physical phenomena such as an impacting beam, the main focus of attention is on simulating the dynamics of the physical system. Thompson *et al.* [7] show that for an impacting beam (forced close to its first natural frequency) this can be achieved with a simple (single-d.o.f.) numerical model. However, although the simple model captures the qualitative dynamics of the system, it fails to simulate the energy loss characteristics. These must be accounted for in the model by reducing the coefficient of restitution value to an appropriate level to account for the energy dissipated due to the excitation of higher modes at impact.

Here we have shown the relationship which exists between the coefficient of restitution and modal energy during periodic impacting motion of arbitrary N-d.o.f. system. Using simulations of experimental data from impacting beam experiments, we have

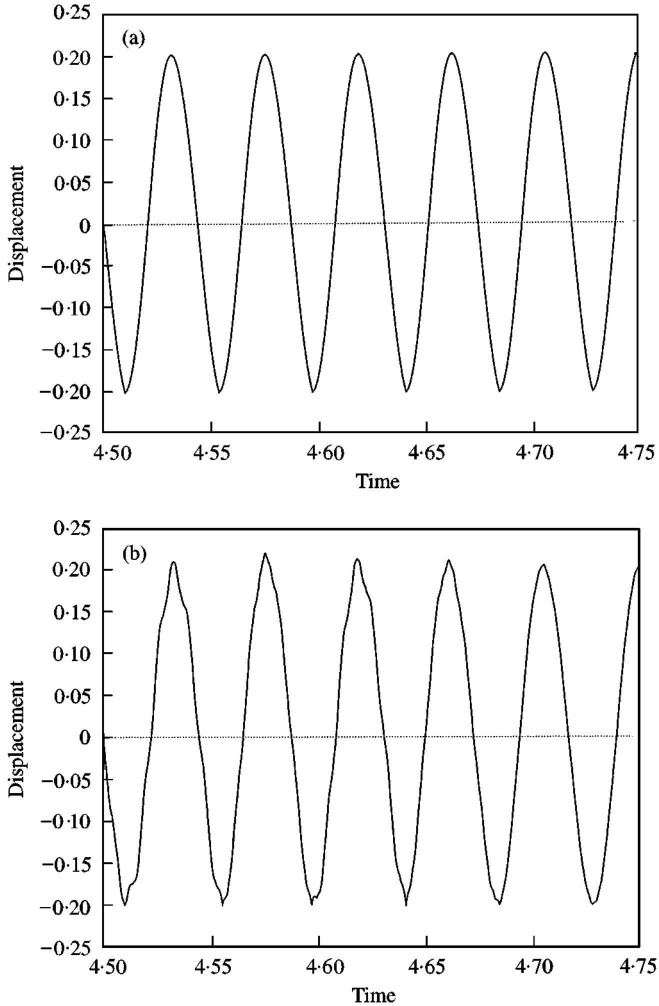

Figure 5. Numerical simulations of the vibro-impact beam data shown in Figure 4(a) Single-d.o.f. model $r = 0{\cdot}2$. (b) Four-d.o.f. model $r = 0{\cdot}75$.

demonstrated the effect of using multi-modal models on the value of the coefficient of restitution.

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