



DAMPING EFFECT OF A RANDOMLY EXCITED AUTOPARAMETRIC SYSTEM

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An investigation into the modal interaction of an autoparametric system under a broadband random excitation is made. The specific system examined is an autoparametric vibration absorber with internal resonance, which is typical of many common structural configurations. By means of Gaussian closure scheme the dynamic moment equations explaining the random responses of the system are reduced to a system of autonomous ordinary differential equations of the first and second moments. In view of equilibrium solutions of this system and their stability we examine the system responses. We could not find the destabilizing effect of damping, which was observed by Ibrahim and Roberts (1977; *Zeitschrift für Angewandte Mathematik and Mechanik* 57, 643–649).

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1. INTRODUCTION

Modal interactions of harmonically excited non-linear systems with internal resonance have been studied extensively [1–12]. These systems have been known to exhibit complicated behaviors such as jump and saturation phenomenon, Hopf bifurcations and a sequence of period-doubling bifurcations leading to chaos [4–12]. In the meantime, Ibrahim and his colleagues [13–21] have studied influences of internal resonance on responses of randomly excited non-linear systems. For example, Ibrahim and Roberts [15, 19] and Roberts [20] included cubic non-linear terms in the analysis for systems with 1:2 internal resonance, and the destabilizing effect of a damping ratio was observed [19]. This destabilizing effect is a notable observation because the stabilizing effect of damping has been known as a generally accepted idea for resonance responses.

The motive of this study is to check the existence of destabilizing effect of damping. We selected an autoparametric vibration absorber [4] under a broadband random excitation as Ibrahim and Roberts [19] did. Obtaining moment equations from the Fokker–Planck equation corresponding to the equation of motion, we used Gaussian closure scheme to reduce a system of 14 autonomous ordinary differential equations for the first and second moments. We examined the equilibrium solution of this system and its stability.

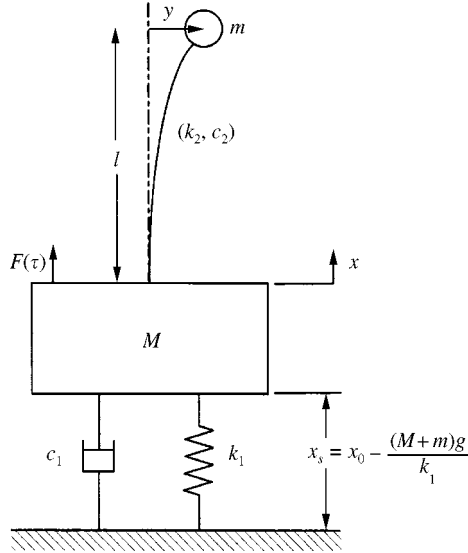


Figure 1. Schematic diagram of an autoparametric absorber system.

2. EQUATIONS OF MOTION

Figure 1 shows the autoparametric system under a broadband random excitation $F(t)$. The equations of motion of the system [4] are, for the main mass,

$$(M + m)\ddot{x} + c_1\dot{x} + k_1x - \left(\frac{6}{5}l\right)m(\dot{y}^2 + y\ddot{y}) = F(t) \quad (1)$$

and for the cantilever,

$$m\ddot{y} + c_2\dot{y} + \left\{k_2 - \left(\frac{6}{5}l\right)m\ddot{x}\right\}y + \left(\frac{36}{25}l^2\right)my(\dot{y}^2 + y\ddot{y}) = 0, \quad (2)$$

where x and y are normal co-ordinates corresponding to the linearized system. Introducing the notations

$$X = \frac{x}{x_s}, \quad Y = \frac{y}{l}, \quad \zeta_1 = \frac{c_1}{2(M + m)\omega_1}, \quad \zeta_2 = \frac{c_2}{2m\omega_2},$$

$$\tau = \omega_1 t, \quad r = \frac{\omega_2}{\omega_1}, \quad \varepsilon = \frac{l}{x_s},$$

$$R = \frac{m}{M + m}, \quad \omega_1^2 = \frac{k_1}{M + m}, \quad \omega_2^2 = \frac{k_2}{m}, \quad W(\tau) = \frac{F(\tau/\omega_1)}{(M + m)x_s\omega_1^2}, \quad \rho = \frac{6}{5}. \quad (3)$$

we have the non-dimensionalized equations as follows:

$$X'' + 2\zeta_1 X' + X - \rho\varepsilon R(Y'^2 + YY'') = W(\tau), \quad (4)$$

$$Y'' + 2\zeta_2 r Y' + \left(r^2 - \frac{\rho}{\varepsilon} X''\right)Y + \rho^2 Y(Y'^2 + YY'') = 0. \quad (5)$$

In the above equations, dot and prime denote differentiation with respect to t and τ respectively.

Eliminating the non-linear acceleration terms and neglecting the fourth and higher orders of non-linear terms we have

$$\begin{aligned} X'' + 2\zeta_1 X' + X + \rho\epsilon R(r^2 Y^2 + 2r\zeta_2 Y Y' - Y'^2) \\ + \rho^2 R(-W(\tau)Y^2 + XY^2 + 2\zeta_1 X'Y^2) = W(\tau), \end{aligned} \quad (6)$$

$$\begin{aligned} Y'' + 2\zeta_2 r Y' + r^2 Y + \frac{\rho}{\epsilon}(-YW(\tau) + XY + 2\zeta_1 Y X') \\ + \rho^2(1-R)(-r^2 Y^3 - 2\zeta_2 r Y^2 Y' + Y Y'^2) = 0. \end{aligned} \quad (7)$$

Random excitation $W(\tau)$ is assumed to be zero-mean white noise having the autocorrelation function

$$R_{WW}(\Delta\tau) = E[W(\tau)W(\tau + \Delta\tau)] = 2D \delta(\Delta\tau), \quad (8)$$

where $2D$ represents the spectral density when we express the frequency by $f(=\omega/2\pi)$, and $\delta(\Delta\tau)$ is the Dirac delta function.

3. MOMENT EQUATIONS BY GAUSSIAN CLOSURE SCHEME

Introducing the notations

$$\{X, Y, X', Y'\}^T = \{X_1, X_2, X_3, X_4\}^T = \mathbf{X}$$

and letting $W(\tau)$ be a formal derivative of a Brownian process, i.e., $W(\tau) = dB(\tau)/d\tau$, we can express equations (6) and (7) in the form of the Itô stochastic equation:

$$\begin{aligned} dX_1 &= X_3 d\tau, & dX_2 &= X_4 d\tau, \\ dX_3 &= \{-2\zeta_1 X_3 - X_1 + \rho\epsilon R(X_4^2 - 2r\zeta_2 X_2 X_4 - r^2 X_2^2) \\ &\quad + \rho^2 R(-2\zeta_1 X_2^2 X_3 - X_1 X_2^2)\} d\tau + (1 + \rho^2 R X_2^2) dB(\tau), \\ dX_4 &= \left\{-2\zeta_2 r X_4 - r^2 X_2 + \frac{\rho}{\epsilon}(-X_1 X_2 - 2\zeta_1 r X_2 X_3) \right. \\ &\quad \left. + \rho^2(1-R)(r^2 X_2^3 + 2\zeta_2 r X_2^2 X_4 - X_2 X_4^2)\right\} d\tau + \frac{\rho}{\epsilon} X_2 dB(\tau). \end{aligned} \quad (9)$$

The solution process of this equation is a Markov process and the Fokker-Planck equation may be applied for the Markov vector \mathbf{X} in the form

$$\frac{\partial}{\partial \tau} p(\mathbf{x}, \tau) = - \sum_{i=1}^4 \frac{\partial}{\partial x_i} [a_i(\mathbf{x}, \tau) p(\mathbf{x}, \tau)] + \frac{1}{2} \sum_{i=1}^4 \sum_{j=1}^4 \frac{\partial^2}{\partial x_i \partial x_j} [b_{ij}(\mathbf{x}, \tau) p(\mathbf{x}, \tau)], \quad (10)$$

where $p(\mathbf{x}, \tau)$ is the joint probability density function, and $a_i(\mathbf{x}, \tau)$ and $b_{ij}(\mathbf{x}, \tau)$ are the first and second incremental moments of the Markov process $\mathbf{X}(\tau)$. These are defined as follows:

$$a_i(\mathbf{x}, \tau) = \lim_{\delta\tau \rightarrow 0} \frac{1}{\delta\tau} E\{X_i(\tau + \delta\tau) - X_i(\tau) | \mathbf{X}(\tau) = \mathbf{x}\}, \quad (11)$$

$$b_{ij}(\mathbf{x}, \tau) = \lim_{\delta\tau \rightarrow 0} \frac{1}{\delta\tau} E\{[X_i(\tau + \delta\tau) - X_i(\tau)][X_j(\tau + \delta\tau) - X_j(\tau)] | \mathbf{X}(\tau) = \mathbf{x}\}. \quad (12)$$

From equations (9), a_i and b_{ij} are evaluated as follows:

$$\begin{aligned}
a_1 &= x_3, & a_2 &= x_4, \\
a_3 &= -2\zeta_1 x_3 - x_1 + \rho \varepsilon R (x_4^2 - 2r\zeta_2 x_2 x_4 - r^2 x_2^2) + \rho^2 r (-2\zeta_1 x_2^2 x_3 - x_1 x_2^2), \\
a_4 &= -2\zeta_2 r x_4 - r^2 x_2 + \frac{\rho}{\varepsilon} (-x_1 x_2 - 2\zeta_1 r x_2 x_3) \\
&\quad + \rho^2 (1 - R) (r^2 x_2^3 + 2\zeta_2 r x_2^2 x_4 - x_2 x_4^2), \\
b_{33} &= 2D + 4\rho^2 R x_2^2 D, & b_{34} &= b_{43} = 2\frac{\rho}{\varepsilon} x_2 D, & b_{44} &= 2\frac{\rho^2}{\varepsilon^2} x_2^2 D,
\end{aligned} \tag{13}$$

all other $b_{ij} = 0$.

Since it is impossible to obtain the exact solution $p(\mathbf{x}, \tau)$ to the Fokker–Planck equation, we are trying to examine the system responses by means of moment equations. First of all, introducing the following notations for the n th order moments of the system responses,

$$m'_{\alpha, \beta, \gamma, \eta}(\tau) = E[X_1^\alpha X_2^\beta X_3^\gamma X_4^\eta] = \iiint\int_{-\infty}^{\infty} x_1^\alpha x_2^\beta x_3^\gamma x_4^\eta p(\mathbf{x}, \tau) dx_1 dx_2 dx_3 dx_4$$

with $n = \alpha + \beta + \gamma + \eta$, we can derive a set of dynamic moment equations of any order by multiplying equation (10) by $x_1^\alpha x_2^\beta x_3^\gamma x_4^\eta$ and integrating by parts over the entire state space $-\infty < x_i < \infty$. This procedure results in the following general dynamic moment equation:

$$\begin{aligned}
m'_{\alpha, \beta, \gamma, \eta} &= \alpha m_{\alpha-1, \beta, \gamma+1, \eta} + \beta m_{\alpha, \beta-1, \gamma, \eta+1} - \gamma m_{\alpha+1, \beta, \gamma-1, \eta} - \varepsilon \rho R r^2 \gamma m_{\alpha, \beta+2, \gamma-1, \eta} \\
&\quad - \rho^2 R \gamma m_{\alpha+1, \beta+2, \gamma-1, \eta} + \varepsilon \rho R \gamma m_{\alpha, \beta, \gamma-1, \eta+2} + (\gamma-1) \gamma D m_{\alpha, \beta, \gamma-2, \eta} \\
&\quad + 2\rho^2 R (\gamma-1) \gamma D m_{\alpha, \beta+2, \gamma-2, \eta} - 2\zeta_1 \gamma m_{\alpha, \beta, \gamma, \eta} - 2\rho^2 R \zeta_1 m_{\alpha, \beta+2, \gamma, \eta} \\
&\quad - 2\varepsilon \rho R r \zeta_2 \gamma m_{\alpha, \beta+1, \gamma-1, \eta+1} - r^2 \eta m_{\alpha, \beta+1, \gamma, \eta-1} - \frac{\rho}{\varepsilon} \eta m_{\alpha+1, \beta+1, \gamma, \eta-1} \\
&\quad + \rho^2 (1 - R) r^2 \eta m_{\alpha, \beta+3, \gamma, \eta-1} - \rho^2 (1 - R) \eta m_{\alpha, \beta+1, \gamma, \eta+1} \\
&\quad + \frac{\rho^2}{\varepsilon^2} \eta (\eta-1) D m_{\alpha, \beta+2, \gamma, \eta-2} - 2\frac{\rho}{\varepsilon} \zeta_1 \eta m_{\alpha, \beta+1, \gamma+1, \eta-1} - 2r\zeta_2 \eta m_{\alpha, \beta, \gamma, \eta} \\
&\quad + 2\rho^2 (1 - R) r \eta m_{\alpha, \beta+2, \gamma, \eta} + 2\frac{\rho}{\varepsilon} \gamma \eta D m_{\alpha, \beta+1, \gamma-1, \eta-1}.
\end{aligned} \tag{14}$$

Equation (14) constitutes a set of infinite coupled equations. In other words, the differential equation of order n contains moment terms of order $n+1$ and $n+2$. The Gaussian closure is based on the assumption that the response process is nearly Gaussian and is carried out by setting third and fourth order cumulants to zero. The third and fourth order moments can be expressed in terms of lower order moments as follows [21, 22]:

$$\begin{aligned}
E[X_\alpha X_\beta X_\gamma] &= E[X_\alpha] E[X_\beta X_\gamma] + E[X_\beta] E[X_\alpha X_\gamma] + E[X_\gamma] E[X_\alpha X_\beta] \\
&\quad - 2E[X_\alpha] E[X_\beta] E[X_\gamma],
\end{aligned} \tag{15}$$

$$E[X_\alpha X_\beta X_\gamma X_\eta] = E[X_\alpha X_\beta]E[X_\gamma X_\eta] + E[X_\alpha X_\gamma]E[X_\beta X_\eta] \\ + E[X_\alpha X_\eta]E[X_\beta X_\gamma] - 2E[X_\alpha]E[X_\beta]E[X_\gamma]E[X_\eta]. \quad (16)$$

Substituting equations (15) and (16) into equations (14) we can obtain a system of 14 differential equations for 14 first and second order moments. For convenience, the system is expressed as follows:

$$\mathbf{m}' = f(\mathbf{m}), \quad (17)$$

where $\mathbf{m} = \{m_{1,0,0,0}, m_{0,1,0,0}, \dots, m_{0,0,1,1}\}^T$ is the moment vector consisting of 14 moments and $f(\mathbf{m}) = \{f_1(\mathbf{m}), f_2(\mathbf{m}), \dots, f_{14}(\mathbf{m})\}^T$ is the vector field of the system. We can obtain the equilibrium solution \mathbf{m}_0 from

$$f(\mathbf{m}_0) = \mathbf{0}. \quad (18)$$

In order to investigate the stability of the equilibrium solution, we let

$$\mathbf{m} = \mathbf{m}_0 + \delta\mathbf{m},$$

where $\delta\mathbf{m}$ is a small disturbance. The disturbance $\delta\mathbf{m}$ satisfies, to the first order

$$\delta\mathbf{m}' = \left. \frac{\partial f}{\partial \mathbf{m}} \right|_{\mathbf{m}=\mathbf{m}_0} \delta\mathbf{m}. \quad (19)$$

If real parts of all eigenvalues of the Jacobian matrix are negative, the solution \mathbf{m}_0 is considered asymptotically stable.

From equations (14) and (17) we can see that system (17) has the following equilibrium solution:

$$m_{2,0,0,0} \equiv E[X^2] = D/2\zeta_1, \quad m_{0,0,2,0} \equiv E[X'^2] = D/2\zeta_1 \quad (20)$$

and all other moments are zero. This equilibrium solution tells that the autoparametric vibration absorber undergoes the main system motion (X) with no cantilever motion ($Y = 0$); in other words the motion is unimodal. The mean square for X and X' are same and they are proportional to D and inversely proportional to ζ_1 .

4. NUMERICAL RESULTS

First of all, we solve numerically the algebraic equation (18) to ascertain that solution (20) is the only equilibrium solution. When the solution becomes unstable, we investigate the long-term behavior of the moments by integrating numerically the ordinary differential equation (17). Figure 2 shows how the mean-square values of the steady state motion depend on the frequency ratio $r = \omega_2/\omega_1$ when mass ratio, R is equal to 0.2 and 0.15. In Figures 2(a) and 2(b), two horizontal lines far from $r = 0.5$ imply that the corresponding response is a stationary process, because the mean-square values are independent of τ as well as r . The results showing the facts that the main system motion excited directly does not encourage the cantilever motion and the responses by a stationary excitation are stationary, coincide with the response characteristics of linear systems. According to the stability analysis, the equilibrium solution loses the stability at r_{b1} and r_{b2} by Hopf bifurcations which occur when the Jacobian matrix of equation (19) has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts. Therefore, in the region of

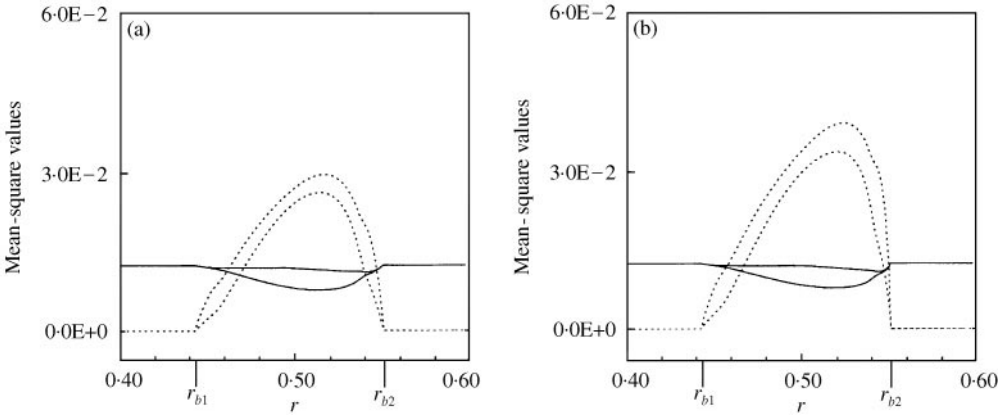


Figure 2. Limits of mean-square responses as functions of the frequency ratio r ($\zeta_1 = 0.01$, $\zeta_2 = 0.01$, $\varepsilon = 2$, $2D = 0.0005$). —, $E[X^2]$; ·····, $E[Y^2]$. (a) $R = 0.2$; (b) $R = 0.15$.

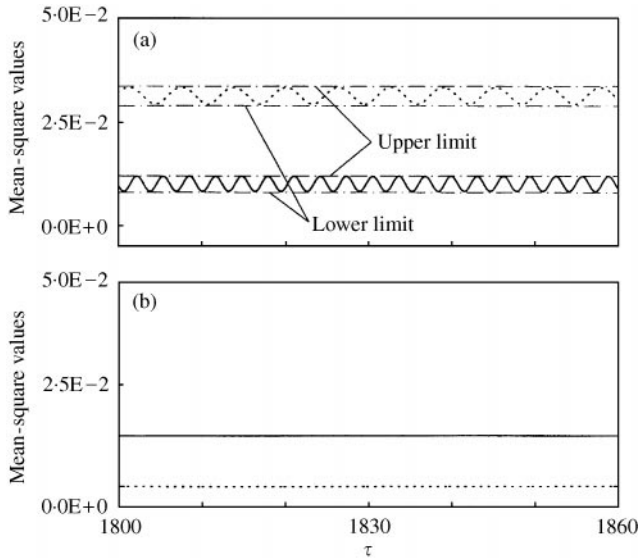


Figure 3. Time histories of mean-square responses ($\zeta_1 = 0.01$, $\zeta_2 = 0.01$, $\varepsilon = 2$, $R = 0.15$, $2D = 0.0005$). —, $E[X^2]$; ·····, $E[Y^2]$. (a) $r = 0.5$; (b) $r = 0.58$.

$r_{b1} < r < r_{b2}$, the moments can have the long-term behaviors such as periodic, quasi-periodic, and chaotic. In the figures the upper and lower limits of two moments are shown. These results show that the energy has been transferred from the main system motion excited directly to the cantilever motion not excited directly. Since the mean-square values of this motion vary between both the limits, the response is non-stationary. Due to the internal resonance condition ($r = 0.5$) strengthening the couplings between the non-linear terms, the system response shows the response characteristics of non-linear systems. Decreasing the mass ratio, R leads to an increase in the cantilever motion. However bifurcation points, r_{b1} and r_{b2} , are never changed according to varying mass ratio, R .

Figure 3 represents time histories of mean-square values of the system response at the steady state. Figures 3(a) and 3(b) show the non-stationary ($r = 0.5$) and stationary

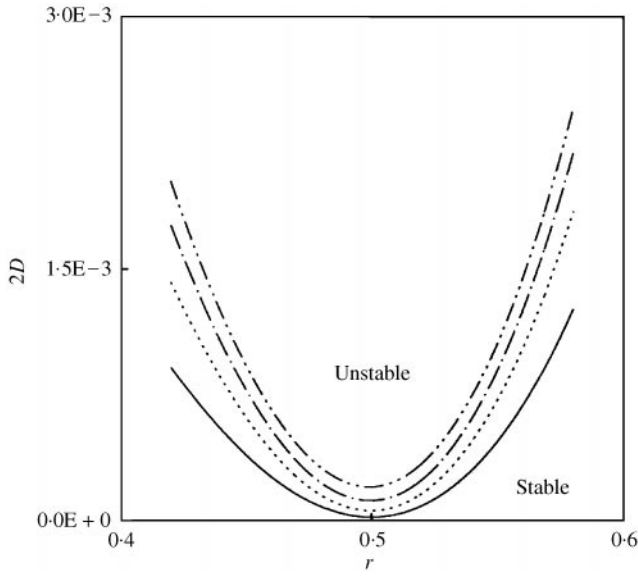


Figure 4. Influence of damping ζ_1 on the stability boundary ($\zeta_2 = 0.01, \varepsilon = 2$). —, $\zeta_1 = 0.01$; \cdots , $\zeta_1 = 0.02$; ---, $\zeta_1 = 0.03$; - · - · -, $\zeta_1 = 0.04$.

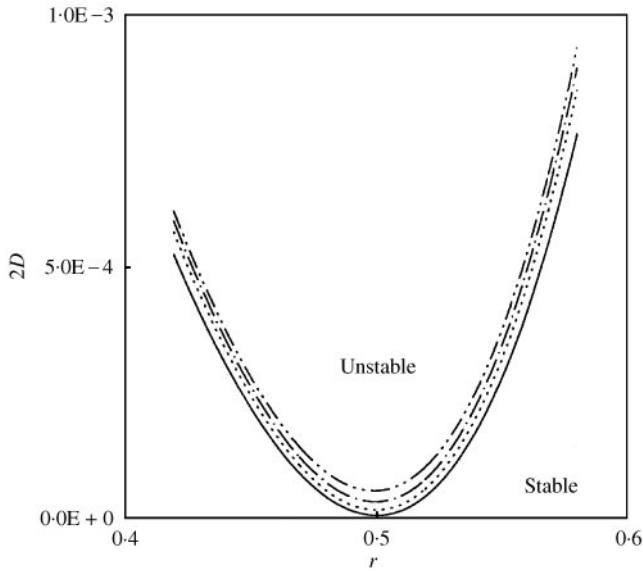


Figure 5. Influence of damping ζ_2 on the stability boundary ($\zeta_1 = 0.005, \varepsilon = 2$). —, $\zeta_2 = 0.01$; \cdots , $\zeta_2 = 0.02$; ---, $\zeta_2 = 0.03$; - · - · -, $\zeta_2 = 0.04$.

($r = 0.58$) processes respectively. Each of the mean-square values $E[X^2]$ and $E[Y^2]$ is oscillating with twice the corresponding mode natural frequency. This coincides with the fact that square of a harmonic function is oscillating with twice its frequency.

Figures 4 and 5 show how the system dampings affect Hopf bifurcation points in $2D-r$ planes, for $\zeta_2 = 0.01$ and $\zeta_1 = 0.005$ respectively. The figures show that stable regions expand as the system damping ratios increase. The result of Figure 4 contradicts Ibrahim

and Roberts' statement [19] saying "increase of the primary system damping ratio appears to have a destabilizing effect". Normalizing the mean-square displacement of the primary system, they could not avoid the constraint $D = 2\zeta_1$, which might cause them to misunderstand the influence of the damping ratio ζ_1 on the stability.

5. CONCLUSIONS

In order to investigate the influences of the internal resonance on the system responses of a two-degree-of-freedom system with a random excitation, we examined an autoparametric vibration absorber with a broadband random excitation to the main mass. The stability regions in the parameter space are expanded as the system dampings increase. This is a remarkable contrast with Ibrahim and Roberts' observation [19]. We believe that their misunderstanding is due to the normalization procedure of the analysis.

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