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INTRODUCTION OF MODIFIED COMPARISON FUNCTIONS FOR VIBRATION ANALYSIS OF A RECTANGULAR CRACKED PLATE

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In this paper, new functions named "modified comparison functions" are introduced and used for vibration analysis of a simply supported rectangular cracked plate. It is assumed that the crack having an arbitrary length, depth and location is parallel to one side of the plate. Elastic behavior of the plate at crack location is considered as a line spring with a varying stiffness along the crack. Because there is no exact solution for this problem, one has to use some approximate methods. Although among the functions which are used for vibration analysis of a cracked plate, the comparison functions are more accurate, obtaining these functions is very difficult. In spite of this difficulty, a method for obtaining the comparison functions of the above cracked plate satisfying all the geometric and natural boundary conditions as well as the inner boundary conditions at crack location is introduced. The main purpose of this paper is to improve the accuracy of these comparison functions which only satisfy all the boundary conditions and the inner boundary conditions at the crack location, but their accuracy is questionable at a distance away from the boundaries. In order to increase the accuracy of the comparison functions, it is assumed that the crack affects the mode shape functions in its neighborhood, and its maximum influence is at the crack location, and the influence will vanish at a sufficient distance from the crack. The comparison functions obtained in this way are called the "modified comparison functions" and they are more accurate than the comparison functions. Using the Rayliegh–Ritz method, the "modified comparison functions" are used to obtain the natural frequencies of the cracked plate mentioned above. The results are presented by appropriate curves showing the variations of the natural frequencies of the cracked plate in terms of the crack depth, length and location.

1. INTRODUCTION

Energy methods are usually used for investigating the vibrational behavior of plates having different types of boundary conditions as well as damaged plates. The Ritz method is one of the usual methods by which the natural frequencies of thin plates may be obtained approximately.

The accuracy of the method will be mainly influenced by the functions which are used for the vibrational mode shape functions [1]. Admissible functions are mainly used for estimating the natural frequencies of damaged circular and rectangular plates. Leissa *et al.* [2] investigated the vibration of circular plates with V-notches, using the Ritz method and considering admissible functions. In another investigation, Lee [3] used the Rayleigh–Ritz method and investigated admissible functions for the vibration of annular circular plates having a cut out between the internal and external radii. Finite element methods are also used for vibration analysis of damaged rectangular plates. For instance, Guan-Liang and

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others [4] investigated the vibration analysis of a rectangular plates having a through-thickness crack being parallel to one side of the plate, using the finite element method. In another investigation, Prabhakara and Datta [5] used the finite element method to examine the statical stability and vibrations of damaged rectangular plates. Considering the rotary inertia and shear deformation effects, Lee and Lim [6] used a numerical approach based on the Rayleigh–Ritz method to predict the natural frequencies of rectangular plates with a centrally located crack.

Having in mind that small damages have a minor effect on the natural frequencies of the plate, it is impossible to accurately predict such variations using admissible functions. On the other hand, the process of obtaining the eigenfunctions in such problems is quite complicated and in some cases it is even impossible. So one needs to use functions which are more accurate than admissible functions, i.e., so-called comparison functions.

In this investigation among the class of comparison functions, for the first time new functions which are called “modified comparison functions” and satisfy natural and geometric boundary conditions are introduced. The “modified comparison functions” are more accurate than comparison functions. In this paper, the “modified comparison functions” are suggested for the vibrational analysis of a damaged rectangular plate. The crack having an arbitrary length, depth, and location is parallel to one side of the plate. It is assumed that the rectangular plate is simply supported at all edges. Using the Rayliegh–Ritz method the “modified comparison functions” are used to obtain the natural frequencies of cracked plates.

2. MODELLING ELASTIC BEHAVIOR OF A CRACK HAVING A FINITE LENGTH

To model a crack with a finite length in a cracked rectangular plate, a rectangular plate may be considered as shown in Figure 1; the crack is 2C in length and runs parallel with one side of the plate. The co-ordinates of the crack center are represented by x_0 and y_0. Using non-dimensional parameters as

$$\begin{aligned} &2c = \frac{2C}{a} \quad \text{crack relative length,} \\ &\zeta_0 = \frac{x_0}{a}, \qquad \eta_0 = \frac{y_0}{b} \quad \text{crack center-co-ordinates,} \\ &\xi = \frac{h_0}{H} \quad \text{crack relative depth at its center,} \end{aligned} \tag{1}$$

a, b and H represent dimensions of the plate in x, y and z directions, respectively, and h_0 represents the crack depth at its center.

A hypothetical boundary along the crack direction which divides the plate into two regions is considered. Rice and Levy [7] obtained the ratio of the stress-intensity factor of a finite-length crack at a plate, k, to the stress-intensity factor of an all-over crack at another plate, k_∞, (same as stress-intensity factor in an edge-cracked strip in plane strain state and under bending moments), for which both plates are subjected to the same bending moments. By increasing the ratio of the crack length to the plate thickness, $2C/H$, the ratio of k/k_∞ for all values of relative depth of the crack, i.e., h_0/H, approaches unity. The smaller the relative depth of the crack, the more the value of k/k_∞ approaches unity in quite small values of $2C/H$. On the other hand, since k is proportional to the stress, so decrease in k of a crack with small length, is followed by decreasing stress at the region of the crack with small

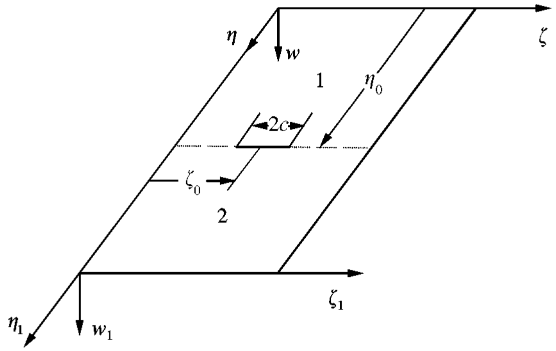

Figure 1. Rectangular plate with a part-through finite-length crack.

length and a given depth in comparison with the lengthy crack with the same depth. Rice and Levy [7] derived an approximate formula for nominal bending stress at the location of the crack with a finite length. When the plate is only subjected to the bending moments, the formula becomes

$$\sigma_b^0 = \frac{\sigma_{b_\infty}}{1 + [3(3+\nu)(1-\nu)\alpha_{bb}^0/(2C/H)]} \tag{2}$$

where σ_b^0 is the nominal bending stress at the crack location and on the surface of the plate, σ_{b_∞} is the nominal bending stress at the location of the crack with an infinite length and on the surface of the plate, α_{bb}^0 is the non-dimensional bending compliance coefficient at the crack center, and ν is the Poisson ratio.

As a result, the discontinuity of the slope at both sides of the crack with a finite length is less than the one for the lengthy crack with the same depth and bending load. Now it is necessary to find a function to model the shape of the crack in regard to some parameters characterizing the crack (relative depth, length and co-ordinates of the center). If the shape of the crack is considered as a semi-ellipse, in Cartesian co-ordinate system, then the function representing the shape of the crack will be

$$h(x) = \begin{cases} h_0\left[1 - \left(\dfrac{x - x_0}{C}\right)^2\right]^{1/2}, & \text{for } x_0 - C < x < x_0 + C \\ 0, & \text{for } \begin{cases} 0 < x < x_0 - C \\ x_0 + C < x < a \end{cases} \end{cases} \tag{3}$$

For vibration analysis of the plate having a crack with a finite length, relation (3) can be expanded as a sum of sine and cosine functions in the domain $0 \leqslant x \leqslant a$ by Fourier series. However, the application of this method may be inefficient due to time consuming and intensive computational efforts. Therefore, by using the following equation [7]

$$\alpha_{bb} = \frac{1}{H}\int_0^h g_b^2 \, \mathrm{d}h, \tag{4}$$

where g_b is the dimensionless function of the relative crack depth ($\xi = h/H$) in range of $0 < \xi < 0{\cdot}7$ and is defined as [8]

$$g_b = \xi^{1/2}(1{\cdot}99 - 2{\cdot}47\xi + 12{\cdot}97\xi^2 - 23{\cdot}117\xi^3 + 24{\cdot}80\xi^4), \tag{5}$$

a function representing dimensionless bending compliance coefficient is directly suggested as a function of dimensionless co-ordinate, ζ, which is free of the above-mentioned difficulties, as follows [9]

$$\alpha_{bb}(\zeta) = \alpha_{bb}^0 \mathrm{e}^{-[(\zeta - \zeta_0)e]^2/2c^2} \tag{6}$$

where α_{bb}^0 is the dimensionless bending compliance coefficient at crack center, e is the base of natural logarithm, and ζ is the dimensionless coordinate ($0 \leqslant \zeta \leqslant 1$).

The value of the above function is very small for the range of $0 \leqslant \zeta \leqslant \zeta_0 - c$ and $\zeta_0 + c \leqslant \zeta \leqslant 1$, and by keeping a distance from both ends of the crack, this value really vanishes. By growing the length of the crack, the value of $\alpha_{bb}(\zeta)$ according to equation (6) approaches the fixed value of α_{bb}^0 as it is for an all-over crack. The variation of $\alpha_{bb}(\zeta)$ for crack with $\zeta_0 = 0{\cdot}5$, a relative length of $2c = 0{\cdot}015$, and $\alpha_{bb}^0 = 2{\cdot}5$ as well as the curve for the variation of the relative depth of the crack based on equation (6) are shown in Figure 2.

As already mentioned, for the crack of a finite length the value of the nominal bending stress is less than this value for the center of an all-over crack. The value of the stress for the range of $0 \leqslant \zeta \leqslant \zeta_0 - c$ and $\zeta_0 + c \leqslant \zeta \leqslant 1$, where the plate is free of a crack, has a normal value and reaches its minimum value at the crack zone when approaching the center of the crack from both ends.

Considering the variation of $\alpha_{bb}(\zeta)$ on the hypothetical boundary in accordance with equation (6), and α_{bb}^0 as a dimensionless bending compliance coefficient at the crack center, one may suggest the variation of the nominal bending stress on the hypothetical boundary as the following new function [9]

$$\sigma_b(\zeta) = \sigma_{b_\infty} - (\sigma_{b_\infty} - \sigma_b^0) f(\zeta), \tag{7}$$

where $f(\zeta)$ is the "crack shape function" and is defined as [9]

$$f(\zeta) = \mathrm{e}^{-[(\zeta - \zeta_0)e]^2/2c^2}. \tag{8}$$

On the other hand, the slope discontinuity at both sides of the crack location due to bending moments is proportional to bending compliance of the crack and nominal bending stress,

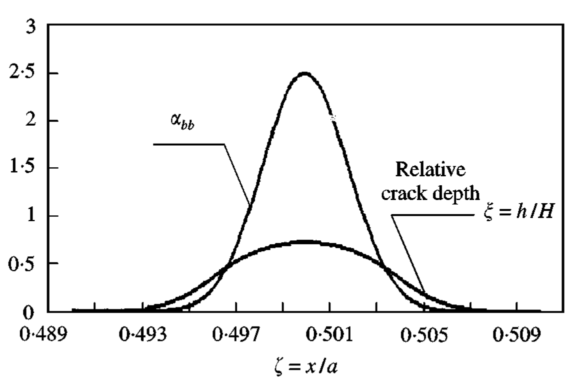

Figure 2. Variation of $\alpha_{bb}(\zeta)$ for a crack with $\zeta_0 = 0{\cdot}5$, $2c = 0{\cdot}015$, and $\alpha_{bb}^0 = 2{\cdot}5$ based on equation (6).

and is given by [7]

$$\theta = \frac{12(1-\nu^2)}{E}\,\sigma_b \alpha_{bb} \tag{9}$$

and σ_{b_∞} is given as [10]

$$\sigma_{b_\infty} = \frac{-EH}{2(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right). \tag{10}$$

By substituting equations (6) and (7), respectively, for α_{bb} and σ_b into equation (9), and by expressing equation (10) in the dimensionless form, one may obtain an expression for the slope discontinuity at both sides of the hypothetical boundary ($\eta = \eta_0$) as [9]

$$\theta(\zeta)\Big|_{\eta=\eta_0} = \frac{-6H}{b}\left(\frac{\partial^2 w}{\partial \eta^2} + \nu\phi^2 \frac{\partial^2 w}{\partial \zeta^2}\right)\alpha_{bb}^0 E(\zeta) f(\zeta)\Big|_{\eta=\eta_0}, \tag{11}$$

where

$$E(\zeta) = \frac{2C/H + 3(\nu+3)(1-\nu)\alpha_{bb}^0[1-f(\zeta)]}{2C/H + 3(\nu+3)(1-\nu)\alpha_{bb}^0} \tag{12}$$

Equation (11) gives the slope discontinuity at both sides of the hypothetical boundary along the crack in terms of the characteristics of the crack, the plate elastic behavior and the plate curvature at crack location

3. VIBRATION ANALYSIS OF A PLATE HAVING A CRACK OF FINITE LENGTH

3.1. HOW TO OBTAIN VIBRATIONAL MODE SHAPE FUNCTIONS

The governing equation for the free vibration of a rectangular plate is given by [11, 12]

$$-D_E \nabla^4 w = M \frac{\partial^2 w}{\partial t^2}, \tag{13}$$

where $w = w(x, y, t)$, ∇^4 is the biharmonic operator, M is the mass per unit area of the plate and D_E is the plate flexural rigidity.

Using the separation of variables technique, one may obtain

$$\frac{\mathrm{d}^2 T(t)}{\mathrm{d}t^2} + \omega^2 T(t) = 0, \qquad \nabla^4 W - \frac{\omega^2 M}{D_E} W = 0, \tag{14a, b}$$

where $w(x, y, t) = W(x, y)T(t)$, and $T(t)$ is a harmonic function. Equation (14b) may be written in terms of dimensionless co-ordinates ζ and η as follows:

$$\frac{\partial^4 W(\zeta,\eta)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 W(\zeta,\eta)}{\partial \eta^2 \partial \zeta^2} + \phi^4 \frac{\partial^4 W(\zeta,\eta)}{\partial \zeta^4} - \phi^4 \lambda^4 W(\zeta,\eta) = 0 \tag{15}$$

where $\phi = b/a$ is the plate aspect ratio, and $\lambda^2 = \omega a^2 \sqrt{M/D_E}$. Note that all dependent and independent variables of equation (15) are dimensionless.

For a plate with simple supports along the edges $\zeta = 0$ and 1, and an arbitrary edge condition at $\eta = 0$ and 1, the solution of equation (15) may be expressed in the form [13]

$$W(\zeta, \eta) = \sum_{m=1}^{\infty} Y_m(\eta) \sin(m\pi\zeta). \tag{16}$$

By substituting equation (16) into equation (15), one obtains

$$\frac{\mathrm{d}^4 Y_m(\eta)}{\mathrm{d}\eta^4} - 2\phi^2 (m\pi)^2 \frac{\mathrm{d}^2 Y_m(\eta)}{\mathrm{d}\eta^2} + \phi^4[(m\pi)^4 - \lambda^4]\, Y_m(\eta) = 0. \tag{17}$$

By applying equation (17) to two regions of the cracked plate shown in Figure 1, one may require eight boundary conditions. The boundary conditions are applied to regions (1) and (2), respectively, at $\eta = 0$ and 1, and to a hypothetical boundary separating the two regions. Because of the form of the plate supports at all four edges (simple support), the boundary conditions at $\zeta = 0$ and 1 for two regions are satisfied by equation (16). The boundary conditions at $\eta = 0$ and 1, for regions (1) and (2), respectively, are as follows:

$$Y_{1m}(\eta)\Big|_{\eta=0} = \frac{\mathrm{d}^2 Y_{1m}(\eta)}{\mathrm{d}\eta^2}\Big|_{\eta=0} = 0, \qquad Y_{2m}(\eta)\Big|_{\eta=1} = \frac{\mathrm{d}^2 Y_{2m}(\eta)}{\mathrm{d}\eta^2}\Big|_{\eta=1} = 0. \tag{18a, b}$$

On the other hand, one may obtain solution of equation (17) for $\lambda^2 > (m\pi)^2$ and $\lambda^2 < (m\pi)^2$ as

$$Y_m(\eta) = A_m \cosh\beta_m\eta + B_m \sinh\beta_m\eta + C_m \sin\gamma_m\eta + D_m \cos\gamma_m\eta, \quad \lambda^2 > (m\pi)^2, \tag{19a}$$

$$Y_m(\eta) = A_m \cosh\beta_m\eta + B_m \sinh\beta_m\eta + C_m \sinh\gamma_m\eta + D_m \cosh\gamma_m\eta, \quad \lambda^2 < (m\pi)^2, \tag{19b}$$

where $\beta_m = \phi\sqrt{\lambda^2 + (m\pi)^2}$ and $\gamma_m = \phi\sqrt{\lambda^2 - (m\pi)^2}$ or $\phi\sqrt{(m\pi)^2 - \lambda^2}$ whichever is real, and A_m through D_m are constants.

By applying the boundary conditions (18a) and (18b) to equations (19a) and (19b), one may obtain two solutions for two mentioned ranges of λ^2, where, for the range of $\lambda^2 > (m\pi)^2$ it will be as

$$W_1(\zeta, \eta) = (B_{1m} \sinh\beta_m\eta + C_{1m} \sin\gamma_m\eta) \sin m\pi\zeta, \quad 0 \leqslant \eta \leqslant \eta_0, \tag{20a}$$

$$W_2(\zeta, \eta) = [B_{2m} \sinh\beta_m(\eta - 1) + C_{1m} \sin\gamma_m(\eta - 1)] \sin m\pi\zeta, \quad \eta_0 \leqslant \eta \leqslant 1. \tag{20b}$$

The boundary conditions along the crack at $\eta = \eta_0$ are

$$W_1 = W_2, \qquad M_{1\eta} = M_{2\eta}, \qquad V_{1\eta} = V_{2\eta} \tag{21a–c}$$

and

$$\left[\frac{\partial w_1}{\partial \eta} - \theta(\zeta) - \frac{\partial w_2}{\partial \eta}\right]_{\eta=\eta_0} = 0. \tag{21d}$$

The first three relations show the equality of deflections, bending moments and shear forces on the two sides of the hypothetical boundary, respectively. The fourth equation shows the relation between the slope of two sides of the crack and the slope discontinuity at crack location.

By application of the aforementioned boundary conditions (21) to equations (20a) and (20b), one gets a set of homogeneous equations. Setting the determinant of the coefficient matrix equal to zero in order to obtain non-trivial solutions for B_{1m}, C_{1m}, B_{2m} and C_{2m}, one may reach a determinant as [9]

$$\begin{vmatrix} \sinh\beta_m\eta_0 & \sin\gamma_m\eta_0 & -\sinh\beta_m(\eta_0-1) & -\sin\gamma_m(\eta_0-1) \\ \lambda_+^2\sinh\beta_m\eta_0 & \lambda_-^2\sin\gamma_m\eta_0 & -\lambda_+^2\sinh\beta_m(\eta_0-1) & -\lambda_-^2\sin\gamma_m(\eta_0-1) \\ \beta_m\lambda_-^2\cosh\beta_m\eta_0 & -\gamma_m\lambda_+^2\cos\gamma_m\eta_0 & -\beta_m\lambda_-^2\cosh\beta_m(\eta_0-1) & \gamma_m\lambda_+^2\cos\gamma_m(\eta_0-1) \\ \begin{array}{c}\{\beta_m\cosh\beta_m\gamma_0+(6H/b)\alpha_{bb}^0 \\ \times[\lambda_+^2E(\zeta)f(\zeta)\sin\gamma_m\eta_0]\}\end{array} & \begin{array}{c}\{\gamma_m\cos\gamma_m\gamma_0-(6H/b)\alpha_{bb}^0 \\ \times[\lambda_-^2E(\zeta)f(\zeta)\sin\gamma_m\eta_0]\}\end{array} & -\beta_m\cosh\beta_m(\eta_0-1) & -\gamma_m\cos\gamma_m(\eta_0-1) \end{vmatrix}$$
$$=0, \tag{22}$$

where for the sake of brevity, the following definitions are used

$$\lambda_+^2 \equiv \phi^2[\lambda^2+(m\pi)^2(1-\nu)], \qquad \lambda_-^2 \equiv \phi^2[\lambda^2-(m\pi)^2(1-\nu)].$$

Equation (22) has been written in terms of variable ζ and it should be equal to zero for all ζ's for the range of $0 \leqslant \zeta \leqslant 1$. Since the problem is an eigenvalue problem, the vibrational mode shape functions for two regions (1) and (2) are obtained as [9]

$$W_{1m} = B_{1m}\left(\sinh\beta_m\eta + \frac{e_3+e_4E(\zeta)f(\zeta)}{e_5f(\zeta)}\sin\gamma_m\eta\right)\sin m\pi\zeta = Y_{1m}(\eta)\sin m\pi\zeta, \tag{23a}$$

$$W_{2m} = B_{1m}\left(e_1\sinh\beta_m(\eta-1) + e_2\frac{e_3+e_4E(\zeta)f(\zeta)}{e_5f(\zeta)}\sin\gamma_m(\eta-1)\right)\sin m\pi\zeta = Y_{2m}(\eta)\sin m\pi\zeta, \tag{23b}$$

where parameters e_1–e_5 are defined as

$$e_1 = \sinh\beta_m\eta_0/\sinh\beta_m(\eta_0-1), \qquad e_2 = \sin\gamma_m\eta_0/\sin\gamma_m(\eta_0-1), \tag{24a, b}$$

$$e_3 = 2\beta_m\phi^2\lambda^2[\cosh\beta_m\eta_0 - \sinh\beta_m\eta_0\cosh\beta_m(\eta_0-1)/\sinh\beta_m(\eta_0-1)], \tag{24c}$$

$$e_4 = \frac{6H}{b}\alpha_{bb}^0(\lambda_+^2)^2\sinh\beta_m\eta_0, \qquad e_5 = \frac{6H}{b}\alpha_{bb}^0\lambda_-^2\lambda_+^2\sin\gamma_m\eta_0. \tag{24d, e}$$

As already mentioned, the presence of the crack causes reduction of local flexibility at crack location. Therefore, for a crack with finite length, the value of the nominal bending moment reduces and the bending moment function will be continuous at the crack location. In other words, despite the presence of the part-through crack, the bending moment is equal at two sides of the crack. By keeping a distance from the crack along a line normal to the crack, the effect of the crack on bending moment function decreases. This effect will vanish quite far from the crack. The length and depth of the crack are important parameters in determining the above-mentioned distance. The more the crack length, the more is the distance in which the effect of the crack on bending moments vanishes. For an all-over crack, the distance covers the length of the plate throughout. The effect of the crack depth appears mainly as variations of bending moment function at crack zone. By assuming that the effect of crack having a length of $2C$ on the bending moment function vanishes at distance $D = y_0 \pm \frac{b}{a}(2C)$, where y_0 is the co-ordinate of the crack center along a line normal to the crack, and a, and b are the plate dimensions, the distance in dimensionless co-ordinates, becomes

$$\frac{D}{b} = \eta_0 \pm \frac{2b}{ab} C \Rightarrow d = \eta_0 \pm 2C, \quad c = \frac{C}{a} \tag{25}$$

considering that the value of B_{1m} is arbitrary in equations (23a) and (23b), one may determine B_{1m} by assuming $Y_{1m}(\eta_0 - 2c) = 1$ or $Y_{1m}(\eta_0 + 2c) = 1$ for $\eta_0 < 2c$ as

$$B_{1m} = \frac{1}{\{\sinh \beta_m(\eta_0 - 2c) + [(e_3 + e_4 E(\zeta) f(\zeta))/(e_5 E(\zeta) f(\zeta))] \sin \gamma_m(\eta_0 - 2c)\}} \tag{26}$$

upon substituting equations (26) into equations (23a) and (23b) and arranging terms, one gets

$$\begin{aligned} W_{1m} &= F_1(\zeta) \sinh \beta_m \eta + F_2(\zeta) \sin \gamma_m \eta, \\ W_{2m} &= e_1 F_1(\zeta) \sinh \beta_m(\eta - 1) + e_2 F_2(\zeta) \sin \gamma_m(\eta - 1), \end{aligned} \tag{27a, b}$$

where $F_1(\zeta)$ and $F_2(\zeta)$ are defined as

$$F_1(\zeta) \equiv \frac{e_5 E(\zeta) f(\zeta) \sin m\pi\zeta}{(e_8 E(\zeta) f(\zeta) + e_9)}, \qquad F_2(\zeta) \equiv \frac{[e_3 + e_4 E(\zeta) f(\zeta)] \sin m\pi\zeta}{(e_8 E(\zeta) f(\zeta) + e_9)} \tag{28a, b}$$

and

$$e_6 \equiv \sinh \beta_m(\eta_0 - 2c), \qquad e_7 \equiv \sin \gamma_m(\eta_0 - 2c), \tag{29a, b}$$

$$e_8 \equiv e_5 e_6 + e_4 e_7, \qquad e_9 \equiv e_3 e_7 \tag{29c, d}$$

by applying the aforementioned method for the range of $\lambda^2 < (m\pi)^2$, one may obtain similar equations.

3.2. DISCUSSION ON VIBRATIONAL MODE SHAPE FUNCTIONS: "MODIFIED COMPARISON FUNCTIONS"

The functions which are obtained for determining the vibrational mode shape of a cracked plate with a finite length crack, according to equations (27a) and (27b), result in an exact solution within the range of $0 \leqslant \zeta \leqslant \zeta_0 - c$ and $\zeta_0 + c \leqslant \zeta \leqslant 1$, but their accuracy is questionable within the range of $\zeta_0 - c \leqslant \zeta \leqslant \zeta_0 + c$. In fact, the aforementioned functions will act as comparison functions in the above range. Here, the aim is to obtain more accurate results and push the above comparison functions toward the eigenfunctions of the system in the range of $\zeta_0 - c \leqslant \zeta \leqslant \zeta_0 + c$.

By applying the crack shape function, defined as a continuous function in the range of $0 \leqslant \zeta \leqslant 1$, to equation (22) and by considering the physical aspect of the problem, it is observed that the values obtained for λ are constant in the uncracked zones ($0 \leqslant \zeta \leqslant \zeta_0 - c$ and $\zeta_0 + c \leqslant \zeta \leqslant 1$). However, the values of λ vary at the cracked zone ($\zeta_0 - c \leqslant \zeta \leqslant \zeta_0 + c$). At the cracked zone, λ begins from its maximum value ($\lambda_{max} = \lambda_I$) at two ends of the crack and reaches its minimum value at the center of crack. The value of λ at the range of $0 \leqslant \zeta \leqslant 1$ can be obtained by solving equation (22) (for the case of $\lambda^2 < (m\pi)^2$ a similar method is used). However, computer calculations for obtaining λ in accordance with the aforementioned method are time-consuming. As an alternative technique, a function which, in addition to adequate accuracy, decreases the calculation time considerably may be suggested for λ. In order to increase the accuracy of the vibrational mode shape functions of the cracked plate, the λ function may be considered as one which will be influenced by the crack in the crack zone and remain constant in the other zones, quite far from the crack. For this purpose, λ may be expressed as a two-variable function in terms of ζ and η. If the function for λ is expressed, at a direction normal to the crack, as an exponential function in such a way that its maximum difference from λ_I happens at crack center ($\eta = \eta_0$), and its difference from λ_I reduces to zero through distance $\eta = \eta_0 \pm 2c$, the proposed aims will be satisfied [9]:

$$\lambda(\zeta, \eta) = \lambda(\zeta) + (\lambda_I - \lambda(\zeta))[1 - \mathrm{e}^{-(\eta - \eta_0)^2/c^2}], \tag{30}$$

where $\lambda(\zeta)$ is defined as

$$\lambda(\zeta) = \lambda_I - (\lambda_I - \lambda_d)\mathrm{e}^{-[(\zeta - \zeta_0)^2/c^2]CF}, \tag{31}$$

where CF is a correction factor.

In Figure 3, for the range of $0 \leqslant \zeta \leqslant 1$ and at $\eta = \eta_0$, the curves for the variations of λ obtained by exactly solving equation (22) are compared with the values of λ obtained directly through the suggested equation (30). These curves are plotted for a typical case in which $2c = 0{\cdot}2$, $\eta_0 = 0{\cdot}4$, $\xi_0 = 0{\cdot}5$, $\lambda_I = 3{\cdot}776$ and $\lambda_d = 3{\cdot}671$.

This figure shows, how the desired accuracy is reachable by choosing $CF = (\pi/2)^2$.

Using equation (30) results in more accurate vibrational mode shape functions. Obviously, the influence of quite small cracks on the natural frequencies is minor. In order to evaluate such small variations in natural frequencies, it is necessary to consider

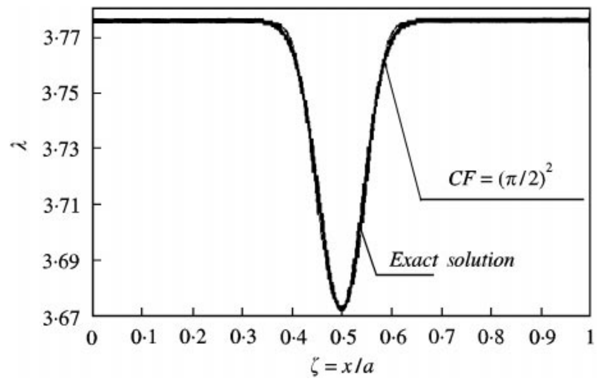

Figure 3. Variation of λ against x/a obtained from the exact solution of equation (22) and values of λ obtained through equation (30) for the case of $\zeta_0 = 0{\cdot}5$, $\eta_0 = 0{\cdot}4$, $\lambda_I = 3{\cdot}776$, $\lambda_d = 3{\cdot}671$, and $2c = 0{\cdot}2$.

vibrational mode shape functions as accurate as possible. Thus, by considering λ as in equation (30), the accuracy of the vibrational mode shape functions (27a) and (27b) is definitely more than that of comparison functions. Therefore, one may call such new functions "modified comparison functions" [9].

3.3. USING "MODIFIED COMPARISON FUNCTIONS" TO CALCULATE NATURAL FREQUENCIES OF A RECTANGULAR PLATE HAVING A CRACK WITH FINITE LENGTH

The maximum kinetic and potential energy of a vibrating rectangular plate in dimensionless co-ordinate can be obtained as [9]

$$KE = \frac{1}{2} M\phi\,\omega^2 a^4 \iint_A W^2\,\mathrm{d}\zeta\,\mathrm{d}\eta = (KE)^* \omega^2, \tag{32}$$

$$PE = \frac{D_E}{2\phi} \iint_A \left[\phi^2 \left(\frac{\partial^2 W}{\partial \zeta^2} \right)^2 + \frac{1}{\phi^2} \left(\frac{\partial^2 W}{\partial \eta^2} \right)^2 + 2\nu \frac{\partial^2 W}{\partial \zeta^2} \frac{\partial^2 W}{\partial \eta^2} + 2(1-\nu) \left(\frac{\partial^2 W}{\partial \zeta \partial \eta} \right)^2 \right] \mathrm{d}\zeta\,\mathrm{d}\eta. \tag{33}$$

By considering equations (32) and (33), one may get the angular natural frequency, ω, as

$$\omega = \sqrt{PE/(KE)^*}. \tag{34}$$

When the stored elastic potential energy at the location of crack is added to the total potential energy of the plate, equation (33) will be applicable to a cracked plate. By assuming that the crack is open in all conditions, the stored potential energy at the crack location may be calculated as [9]

$$\mathrm{d}P_c = M_\eta \theta_\zeta\,\mathrm{d}\zeta|_{\eta = \eta_0}, \tag{35}$$

where M_η is the bending moment causing the crack to be opened, and θ_ζ is the slope discontinuity at two sides of the crack at ζ position. By substituting into equation (35), one may get

$$P_c\Big|_{\eta=\eta_0} = \frac{6HD_E \alpha_{bb}^0}{b^2 \phi^2} \int_0^1 \left(\frac{\partial^2 W}{\partial \eta^2} + \nu\phi^2 \frac{\partial^2 W}{\partial \zeta^2} \right)^2 E(\zeta) f(\zeta)\,\mathrm{d}\zeta \Big|_{\eta=\eta_0}. \tag{36}$$

The stored potential energy of the cracked plate can be obtained through sum of the stored potential energy of the plate at regions (1) and (2) and the stored potential energy at the crack location

$$P = (PE)_{1m} + (PE)_{2m} + P_c. \tag{37}$$

The reference kinetic energy may be calculated through the sum of the reference kinetic energy of the plate at regions (1) and (2):

$$K^* = (KE)^*_{1m} + (KE)^*_{2m}. \tag{38}$$

By considering equations (37) and (38), the natural frequency of the cracked plate is obtained as

$$f = \frac{1}{2\pi}\sqrt{\frac{P}{K^*}}. \tag{39}$$

3.4. RESULTS

Referring to Figure 1, a cracked plate with the following characteristics is considered and different cases of crack locations and dimensions are studied:

$$E = 200\ \text{GPa},\ \rho = 7860,\ v = 0{\cdot}3,\ H = 4\ \text{mm},\ a = 18\ \text{cm},\ b = 27\ \text{cm}.$$

The influence of a crack with a relative length of 0·2 and a relative depth of 0·6 on the first and second natural frequencies is shown in Figures 4a and 4b. From the Figures, it is evident that the presence of the crack with a finite length at $\eta_0 = 0$ and $\eta_0 = 1$ has no influence on the natural frequencies; it is because of simple supports on those edges.

In Figure 4b, in addition to $\eta_0 = 0$ and 1, the presence of the crack at $\eta_0 = 0{\cdot}5$ does not affect the second natural frequency due to the existence of the nodal line at $\eta_0 = 0{\cdot}5$.

Also, to study the influence of ζ_0 as another parameter one realizes that as shown in Figures 4a and b for different values of $\eta_0 =$ constant, by increasing ζ_0 from 0·1 to 0·5, the influence of the crack on natural frequencies also increased.

On the other hand, for any values of ζ_0, maximum influence on natural frequencies occurs when η_0 is located at the mid-point of two successive nodal lines.

Figure 5 shows the influence of the location and dimensions of a crack on the value of the stored relative potential energy at crack location, P_c/P, in the first vibrational mode.

It is obvious from this figure that for any values for η_0, by increasing ζ_0 from 0·1 to 0·5, the ratio of P_c/P is increased, and its maximum value occurs at the center of the plate ($\zeta_0 = \eta_0 = 0{\cdot}5$).

4. CONCLUSION

The presence of a crack on the plate increases local flexibility of the plate at the crack location, and as the crack dimensions get larger and it is far from nodal lines, the more

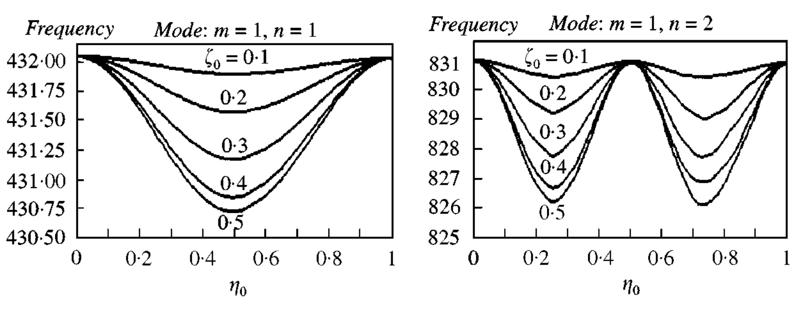

Figure 4. The influence of a crack on (a) the first and (b) the second natural frequencies for the case of $2c = 0{\cdot}2$, $h/H = 0{\cdot}6$ and different values of ζ_0 from 0·1 to 0·5.

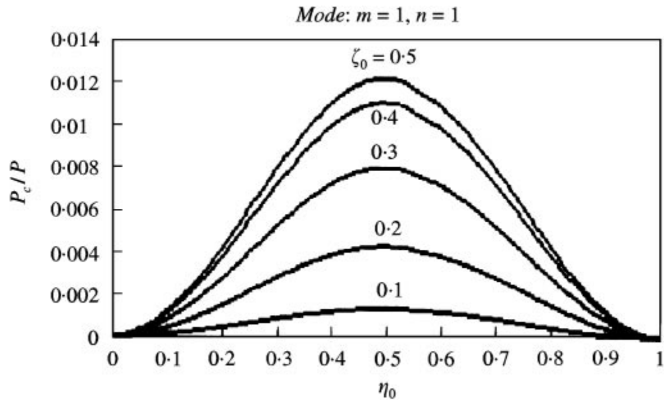

Figure 5. Variation of P_c/P against η_0 for the case of $2c = 0{\cdot}2$, $h/H = 0{\cdot}6$ and different values of ζ_0 from 0·1 to 0·5 in the first vibrational mode.

influence it will have on the natural frequencies of the plate [14]. By knowing that quite small cracks produce a low influence on natural frequencies, it is required that the selected functions be as close to the eigenfunctions as possible. Therefore, using admissible functions is not recommended due to introducing very large approximations into the solution. In addition, the influence of the crack on natural frequencies is very little against such approximations, and in some cases, the solutions obtained for the natural frequencies of cracked plate is bigger than that of the natural frequencies of the intact plate, which is contrary to the physical fact and is nonsense.

Using comparison functions introduces unacceptable errors into the calculations; for instance, the presence of a crack with a quite small length and quite big depth on a plate makes the solutions divergent. However, for cracks with large length and small depth, quite appropriate results may be obtained.

In conclusion, it is not possible to use the admissible and comparison functions for all cases of cracked plates; on the other hand, it is very difficult to obtain the eigenfunctions. Therefore, using new functions, the so-called "modified comparison functions", were suggested and used for the first time in this paper.

The main advantages of the "modified comparions functions" are

— being free of the aforementioned disadvantages of the admissible and comparison functions;
— having an appropriate accuracy which is more accurate than the comparison functions;
— can be obtained more easily than the eigenfunctions.

By considering the "modified comparison functions", the results obtained for variations of natural frequencies and stored potential energy at the crack location show that, the more the crack is able to store the potential energy, P_c, the more the decrease in the natural frequency occurs. Therefore, the location of the crack is one of the most important parameters which influences the natural frequencies of the cracked plate in a special manner.

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APPENDIX: NOMENCLATURE

a, b	plate dimensions
$2C$	crack length
$2c$	relative crack length
D_E	plate flexural rigidity
E	Young's modulus
$\left\{\begin{array}{l}E(\zeta), e_1, \ldots, e_9\\F_1(\zeta), F_2(\zeta)\end{array}\right.$	functions which appear in deriving the vibrational mode shapes
$F(\zeta)$	crack shape function
f	natural frequency of the plate
g_b	a dimensionless function
H	plate thickness
$h(x)$	crack depth in terms of x
h_0	crack depth at center of crack
k	stress-intensity factor for the crack having finite length
k_∞	stress-intensity factor for the crack having infinite length
KE	kinetic energy
K^*	reference kinetic energy
M	mass per unit area of the plate
$M_{1\eta}, \ldots$	bending moment
PE	potential energy
P_c	potential energy at crack location
S	simple support
$T(t)$	time-dependent function
$V_{1\eta}, \ldots$	shear force
$w, W, W_{1m}, \ldots$	transverse deflection
x, y, z	spatial co-ordinates of the plate
x_0, y_0	co-ordinates of the crack center
$Y_m(\eta), \ldots,$	vibrational mode shape function in the η direction
α_{bb}	dimensionless bending compliance coefficient

α_{bb}^0	dimensionless bending compliance coefficient at the crack center
β_m, γ_m	non-dimensional frequency parameters
ζ, η	dimensionless co-ordinates
ζ_0, η_0	dimensionless co-ordinates of the crack center
θ	slope discontinuity at crack location
$\lambda^2, \lambda_-^2, \lambda_+^2$	frequency parameters
v	Poisson's ratio
ξ	relative depth of all-over crack
σ_b^0	nominal bending stress at a point on the surface of the plate and crack location
σ_∞^0	nominal bending stress at a point on the surface of the plate and location of all-over crack
ϕ	plate aspect ratio
ω	angular frequency