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WAVE PROPAGATION IN AND VIBRATION OF A TRAVELLING BEAM WITH AND WITHOUT NON-LINEAR EFFECTS, PART I: FREE VIBRATION

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The relation between the wave propagation and the free vibration in a travelling beam with simple supports has been thoroughly investigated. The frequency equation of such a beam has been derived using the phase-closure principle. Since the characters of the waves change drastically as the axial speed is increased beyond a certain value, the phase-closure principle has been applied differently in these two speed regimes. The justifications for some approximate methods of obtaining the natural frequencies are also discussed. Lastly, the non-linear normal modes are derived again using the phase-closure principle. The computation of the forced response using the wave propagation approach is discussed in Part II.

1. INTRODUCTION

The connection between the vibration and the wave propagation in a continuous system has been recognized long ago. It is well known that in a string, two waves, travelling in opposite directions at the same speed, are reflected at the boundaries to generate the normal modes of vibration. The free vibration of an Euler–Bernoulli beam has also been studied through the wave superposition method. A phase-closure principle has been derived [1], that shows that the normal modes correspond to a phase change of an integer multiple of 2π for the propagating as well as the evanescent waves, as they return to their starting point after traversing to and fro along the beam. Although there is no advantage of calculating the natural frequencies using this principle, the physical understanding of the motion of the beam in its normal modes is enhanced.

If the continuous system is an axially moving slender member, like a band-saw, a travelling threadline, a magnetic tape, etc., then the speed of the wave propagation does no longer remain equal in the upstream and downstream directions. However, the phase-closure principle still remains valid during the modal vibration of an axially moving string [2]. Due to the dispersive nature of the medium, the wave propagation in an axially moving beam is more complicated than that in a moving string. Consequently, only an approximate, rather than an exact, result has been obtained for the natural frequencies of such a beam [3].

It has been observed that even in the presence of non-linear terms, all the points of a vibrating beam may oscillate having the same time period, which, however, is not a system property and depends on the extent of motion. A "natural" frequency is associated with such a time period and is known as the non-linear natural frequency. The configurations of the beam at these frequencies are called the "non-linear normal modes". The non-linear normal modes of a continuous system have been derived in various ways [4–9]. If the beam travels with a speed close to the "critical speed" [10], the effects of non-linearities have to be accounted for. A concept of "non-linear complex normal mode", similar to the non-linear normal mode of an axially stationary beam, has been derived [11].

In this paper, both the linear and non-linear complex normal modes of a simply supported travelling beam are re-examined in the light of harmonic wave propagation. This methodology provides justification for various approximate methods of calculating the natural frequencies. The analysis adds to the physical understanding of the oscillatory motion of the system, which is often missing in the traditional approach of solving a differential equation with given boundary conditions.

2. LINEAR ANALYSIS

In this section, the nature of the waves associated with the modal vibration of a linear travelling beam is first presented. The study is then extended to set up both the approximate and exact frequency equations.

2.1. WAVE-PROPAGATION AND REFLECTION AT A BOUNDARY

The equation of motion for the transverse vibration of a travelling beam can be written as [11]

$$\rho A\left[\frac{\partial^2 w^*}{\partial t^2} + 2c^*\frac{\partial^2 w^*}{\partial \xi\,\partial t} + c^{*^2}\frac{\partial^2 w^*}{\partial \xi^2}\right] - T_0^*\frac{\partial^2 w^*}{\partial \xi^2} + EI_z\frac{\partial^4 w^*}{\partial \xi^4} = 0, \tag{1}$$

where $w^*(\xi, t)$ is the transverse displacement of the beam moving with an axial speed c^* and having an initial tension T_0^*. The geometric properties are given by A, the area of cross-section and I_z, the second moment of area of the cross-section about the neutral axis, and the material characteristics by ρ, the mass density and E, the Young's modulus. The spatial and temporal co-ordinates are denoted by ξ and t respectively.

The following non-dimensionalization scheme is then used: $w = w^*/(l\gamma^2)$, $x = \xi/l$, $\tau = (E/\rho)^{1/2}\gamma t/l$, $c = c^*(E/\rho)^{-1/2}/\gamma$, $r^2 = I_z/A$, $\gamma = r/l$ and $T_0 = T_0^*/(EA\gamma^2)$, with l as the distance between the two simple supports. In terms of the non-dimensional quantities, the equation of motion is recast as

$$\frac{\partial^2 w}{\partial \tau^2} + 2c\frac{\partial^2 w}{\partial x\,\partial \tau} + (c^2 - T_0)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^4 w}{\partial x^4} = 0. \tag{2}$$

To study the harmonic wave propagation, one substitutes

$$w(x, \tau) = e^{i\omega\tau}e^{ikx} \quad \text{where } i = \sqrt{-1}, \tag{3}$$

into equation (2) and obtains the following relation:

$$-\omega^2 - 2\omega kc - (c^2 - T_0)k^2 + k^4 = 0. \tag{4}$$

Using Descartes' rule [12], it can be seen that the roots of this quartic equation belong to one of the following categories:

(i) one pair of real roots (with one negative (k_1, say) and the other positive (k_2, say)) with the other roots as complex conjugates (k_3 and k_4, say),

(ii) four real roots with one positive (k_2 say) and the other three negative (k_1, k_3 and k_4 say).

In what follows, for the sake of simplicity, the analysis is carried on with $T_0 = 0$. However, the nature of the wave propagation remains the same for any other value of T_0.

For $T_0 = 0$, the roots k_j's ($j = 1, 2, 3, 4$) are obtained as

$$k_1 = c/2 - \sqrt{\omega + c^2/4}, \quad k_2 = c/2 + \sqrt{\omega + c^2/4},$$

$$k_3 = -c/2 + \mathrm{i}\sqrt{\omega - c^2/4} \quad \text{and} \quad k_4 = -c/2 - \mathrm{i}\sqrt{\omega - c^2/4}. \tag{5}$$

Thus, the response is given by

$$w(x, \tau) = [A_1 \mathrm{e}^{\mathrm{i}k_1 x} + A_2 \mathrm{e}^{\mathrm{i}k_2 x} + A_3 \mathrm{e}^{\mathrm{i}k_3 x} + A_4 \mathrm{e}^{\mathrm{i}k_4 x}]\mathrm{e}^{\mathrm{i}\omega\tau}. \tag{6}$$

The wave numbers k_1 and k_2 correspond to the waves propagating along the downstream and upstream directions respectively, and will be denoted by A_1- and A_2-waves. Their phase velocities CP_1 and CP_2 differ by

$$CP_1 - CP_2 = \left|\frac{\omega}{k_1}\right| - \left|\frac{\omega}{k_2}\right| = c.$$

For $c < 2\sqrt{\omega}$, the other two waves, i.e., the A_3- and A_4-waves can be called "evanescent waves", since their group velocities become imaginary (implying no propagation of energy) as shown below:

$$CG_3 = \frac{\partial \omega}{\partial k_3} = -2\mathrm{i}\sqrt{\omega - c^2/4}, \qquad CG_4 = \frac{\partial \omega}{\partial k_4} = 2\mathrm{i}\sqrt{\omega - c^2/4}.$$

It should be noted that unlike in an axially stationary beam ($c = 0$) [1], in the present case the phases of the evanescent waves change as they move along the beam. However, for $c \geqslant 2\sqrt{\omega}$, all the group velocities are real and both the A_3- and A_4-waves become downstream propagating waves. The existence of such four propagating waves have been reported in reference [13]. Since the wave propagation characteristics change drastically in the two velocity regimes mentioned above, the phase-closure principle will be applied separately for these cases, namely for $0 < c < c_d$ and $c_d < c < (c_{cr})_1 = \pi$ with $c_d = 2\sqrt{\omega}$. But before applying the phase-closure principle, the reflection of waves at the boundaries with simple supports will be briefly discussed.

2.1.1. *Reflection of waves for* $0 < c < c_d$

If the downstream propagating wave, i.e., the A_1-wave is the only wave impinging upon a simply supported boundary (see Figure 1), then the reflected waves will be both the upstream propagating (A_2-) and evanescent (A_4-) waves. The total displacement at a point is

$$w(x, \tau) = [A_1 \mathrm{e}^{\mathrm{i}k_1 x} + A_2 \mathrm{e}^{\mathrm{i}k_2 x} + A_4 \mathrm{e}^{\mathrm{i}k_4 x}]\mathrm{e}^{\mathrm{i}\omega\tau},$$

which together with the boundary conditions

$$w(0, \tau) = \frac{\partial^2 w}{\partial x^2}(0, \tau) = 0,$$

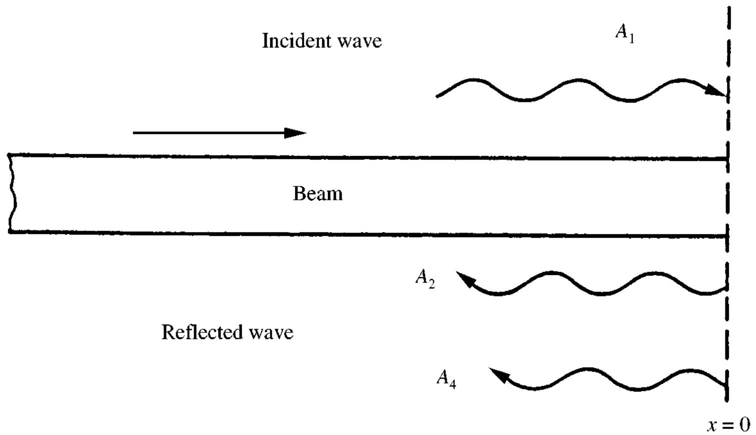

Figure 1. Reflection of wave at a simple support.

give the relations

$$A_1 + A_2 + A_4 = 0 \tag{7}$$

and

$$k_1^2 A_1 + k_2^2 A_2 + k_4^2 A_4 = 0. \tag{8}$$

Using equations (5), (7) and (8) one obtains

$$A_2 = -\left[\frac{(\omega - c/2\sqrt{\omega + c^2/4}) + \mathrm{i}c/2\sqrt{\omega - c^2/4}}{(\omega + c/2\sqrt{\omega + c^2/4}) + \mathrm{i}c/2\sqrt{\omega - c^2/4}}\right]A_1 \tag{9}$$

and

$$A_4 = -\left[\frac{c\sqrt{\omega + c^2/4}}{(\omega + c/2\sqrt{\omega + c^2/4}) + \mathrm{i}c/2\sqrt{\omega - c^2/4}}\right]A_1. \tag{10}$$

It can be seen that unlike in an axially stationary beam, the phase of the propagating wave does not change by π after reflection at the boundary. The change of phase is given by

$$\varepsilon_R = \pi + \left(\tan^{-1}\frac{c/2\sqrt{\omega - c^2/4}}{\omega - c/2\sqrt{\omega + c^2/4}} - \tan^{-1}\frac{c/2\sqrt{\omega - c^2/4}}{\omega + c/2\sqrt{\omega c^2/4}}\right)$$
$$= \pi + \varepsilon_1$$

where

$$\varepsilon_1 = \tan^{-1}\left[\frac{2(c/2)^2\sqrt{\omega^2 - (c^2/4)^2}}{\omega^2 + 2(c^2/4)^2}\right]. \tag{11}$$

2.1.2. *Reflection of waves for* $c_d \leqslant c < \pi$

In this speed regime, the reflection can be studied by considering only the upstream propagating A_2-wave impinging on the boundary. Three waves will be generated, namely the A_1-, A_3- and A_4-waves. From the boundary conditions one gets

$$(A_1 + A_3 + A_4) = - A_2.$$

Since the A_2-wave is the only wave reflected, the total phase change will be π. The individual phase relationships between the A_1-, A_3- and A_4-waves need not be considered here.

2.2. NATURAL FREQUENCIES OF A TRAVELLING BEAM USING PHASE-CLOSURE PRINCIPLE

In this section, the wave propagation theory is used to obtain the natural frequencies. In what follows, the derivation will be carried out first by neglecting the evanescent waves and again by retaining these if $0 < c < c_d$. For $c_d \leqslant c < \pi$, since there exists no evanescent wave, only the exact natural frequencies are obtained.

2.2.1. *Natural frequencies neglecting evanescent waves for* $0 < c < c_d$

As explained in section 2.1, the evanescent waves are characterized by their spatial exponential decay in the medium. Consequently, if the wave number of these waves is high, then the decay will be substantial and it is justified to neglect them altogether. In other words, the approximation becomes more accurate while determining the higher order frequencies. However, as can be seen from equation (5), the evanescent waves decay less with increasing axial speed. Therefore, the approximate derivation of the natural frequencies neglecting the evanescent waves is valid for low axial speeds.

With the above limitations, the change of phase, as the propagating waves travel once around the span, is

$$\Delta\theta_{phase} = 2(\pi + \varepsilon_1) - |k_1| - |k_2|,$$

where the first term on the right-hand side (within the parenthesis) signifies the change in the phase due to reflection at the boundaries and the other terms denote the phase changes as the propagating waves move along the beam. According to the phase closure principle, for the *n*th normal mode

$$2(\pi + \varepsilon_1) + 2\sqrt{\omega + c^2/4} = 2(n + 1)\pi$$

or

$$\varepsilon_1 + \sqrt{\omega + c^2/4} = n\pi. \tag{12}$$

Equation (12) together with equation (11) can be used to determine the natural frequencies.

If, as a further approximation, ε_1 is altogether neglected, the frequency equation becomes

$$\omega = n^2\pi^2 - \frac{c^2}{4}, \tag{13}$$

as obtained by Nelson [3], although from an altogether different approach.

2.2.2. *Calculation of the natural frequencies including the evanescent waves for* $0 < c < c_d$

In this section, the exact natural frequencies are derived considering the effects of the evanescent waves. For an axially stationary beam, the natural frequencies are obtained by considering the phase closure of either the propagating or the evanescent waves [1]. However, as already mentioned, in a travelling beam, the evanescent waves also contribute to the phase change during propagation. So, the phase closure principle has to be used by considering both the propagating and the evanescent waves.

Considering both forms of wave, the reflection at a simply supported boundary can be analysed by considering the support at any point, say at $x = 0$. The waves must satisfy the relations

$$A_1 + A_2 + A_3 + A_4 = 0 \tag{14}$$

and

$$k_1^2 A_1 + k_2^2 A_2 + k_3^2 A_3 + k_4^2 A_4 = 0. \tag{15}$$

Using these relations, one can see that if the A_1- and A_3-waves impinge on a boundary, the resulting A_2- and A_4-waves are given by

$$\begin{Bmatrix} A_2 \\ A_4 \end{Bmatrix} = -\begin{bmatrix} 1 & 1 \\ k_2^2 & k_4^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ k_1^2 & k_3^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_3 \end{Bmatrix}. \tag{16}$$

Similarly,

$$\begin{Bmatrix} A_1 \\ A_3 \end{Bmatrix} = -\begin{bmatrix} 1 & 1 \\ k_1^2 & k_3^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ k_2^2 & k_4^2 \end{bmatrix} \begin{Bmatrix} A_2 \\ A_4 \end{Bmatrix}. \tag{17}$$

Now, to use the phase-closure principle one needs to follow the waves until they reach their starting point after travelling to and fro once along the beam. This is carried out as explained below:

(i) Let A_1- and A_3-waves of respective strengths a_1 and a_3 start from $x = 0$ and travel towards $x = 1$. If the strengths of the waves at $x = 1$ become a_1' and a_3' respectively, one can verify that

$$a_1' = \mathrm{e}^{\mathrm{i}k_1} a_1$$

$$a_3' = \mathrm{e}^{\mathrm{i}k_3} a_3$$

or in the matrix form

$$\begin{Bmatrix} a_1' \\ a_3' \end{Bmatrix} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}k_1} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}k_3} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_3 \end{Bmatrix} = [R_1] \begin{Bmatrix} a_1 \\ a_3 \end{Bmatrix} \quad \text{(say)}.$$

(ii) The A_1- and A_3-waves reflect at the simply supported boundary placed at $x = 1$ to yield the A_2- and A_2-waves of strength a_2 and a_4 respectively. Since the phase change due to the reflection is independent of the location of the boundary, the transformation matrix is again given by equation (16). Thus,

$$\begin{Bmatrix} a_2 \\ a_4 \end{Bmatrix} = [R_2] \begin{Bmatrix} a_1' \\ a_3' \end{Bmatrix},$$

with

$$[R_2] = -\begin{bmatrix} 1 & 1 \\ k_2^2 & k_4^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ k_1^2 & k_3^2 \end{bmatrix}.$$

(iii) The upstream propagating (i.e., the A_2-wave) and evanescent (i.e., the A_4-wave) waves reach from $x = 1$ to $x = 0$ with strength a'_2 and a'_4 respectively. The transformation matrix is

$$[R_3] = \begin{bmatrix} e^{-ik_2} & 0 \\ 0 & e^{-ik_4} \end{bmatrix} = \begin{bmatrix} e^{ik_2} & 0 \\ 0 & e^{ik_4} \end{bmatrix}^{-1}.$$

(iv) The A_2- and A_4-waves get reflected at $x = 0$ and become A_1- and A_3-waves with strengths a''_1 and a''_3, respectively. The transformation matrix, is given by equation (17). So

$$[R_4] = -\begin{bmatrix} 1 & 1 \\ k_1^2 & k_3^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ k_2^2 & k_4^2 \end{bmatrix}.$$

The condition for the beam to vibrate in one of its normal modes can be stated as

$$a''_1 = a_1 \tag{18}$$

and

$$a''_3 = a_3. \tag{19}$$

In terms of the transformation matrices, the phase-closure principle thus yields

$$[R_4][R_3][R_2][R_1]\begin{Bmatrix} a_1 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_3 \end{Bmatrix}. \tag{20}$$

For non-trivial solutions of a_1 and a_3, the following condition must be satisfied:

$$\det[[R_4][R_3][R_2][R_1] - [I]] = 0. \tag{21}$$

As shown in Appendix A, the above equation yields

$$\det \begin{vmatrix} 1 & 1 & 1 & 1 \\ k_1^2 & k_2^2 & k_3^2 & k_4^2 \\ e^{ik_1} & e^{ik_2} & e^{ik_3} & e^{ik_4} \\ k_1^2 e^{ik_1} & k_2^2 e^{ik_2} & k_3^2 e^{ik_3} & k_4^2 e^{ik_4} \end{vmatrix} = 0, \tag{22}$$

which is the usual frequency equation reported in the literature [14].

2.2.3. *Calculation of the natural frequencies for $c_d \leqslant c < \pi$*

In this speed regime, three waves, namely the A_1-, A_3- and A_4-waves propagate in the downstream direction and the remaining A_2-wave moves in the opposite direction. When a simply supported boundary is placed at a point, say $x = 0$, the four waves satisfy the following relations:

$$A_1 + A_2 + A_3 + A_4 = 0$$

and

$$k_1^2 A_1 + k_2^2 A_2 + k_3^2 A_3 + k_4^2 A_4 = 0.$$

If an A_2-wave hits upon the boundary, the resulting A_1-, A_3- and A_4-waves are obtained from the above two equations. But since the number of unknowns exceeds that of the equations, the strengths of the waves cannot be calculated individually. However, their relative strengths A_1/A_4 and A_3/A_4 can be obtained as

$$\begin{Bmatrix} A_1/A_4 \\ A_3/A_4 \end{Bmatrix} = -\begin{bmatrix} 1 & 1 \\ k_1^2 & k_3^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ k_2^2 & k_4^2 \end{bmatrix} \begin{Bmatrix} A_2/A_4 \\ 1 \end{Bmatrix}. \tag{23}$$

On the other hand, if the propagating A_1-, A_3- and A_4-waves get reflected from the boundary, the resulting A_2-wave can be obtained from the following relation:

$$A_2 = -(A_1 + A_3 + A_4). \tag{24}$$

It is to be pointed out that this motion is over-constrained. The waves must also satisfy the following equation due to the disappearance of the bending moment at the simply supported boundaries:

$$A_1 k_1^2 + A_2 k_2^2 + A_3 k_3^2 + A_4 k_4^2 = 0. \tag{25}$$

To get the normal modes, the propagation of the waves are to be followed as described below:

(i) Let an A_2-wave of strength a_2 be reflected from the boundary placed at $x = 0$. Three waves, namely, the A_1-, A_3- and A_4-waves, are generated. The strengths of these waves cannot be determined uniquely, but the relative strengths a_1/a_4 and a_3/a_4 can be obtained in the following matrix form:

$$\begin{Bmatrix} a_1/a_4 \\ a_3/a_4 \end{Bmatrix} = -\begin{bmatrix} 1 & 1 \\ k_1^2 & k_3^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ k_2^2 & k_4^2 \end{bmatrix} \begin{Bmatrix} a_2/a_4 \\ 1 \end{Bmatrix} = [R_1] \begin{Bmatrix} a_2/a_4 \\ 1 \end{Bmatrix}.$$

(ii) The above A_1-, A_3- and A_4-waves of respective strengths a_1, a_3 and a_4 start travelling from $x = 0$ and reach $x = 1$ with strengths a_1', a_3' and a_4' respectively. It is seen that

$$a_1' = e^{ik_1} a_1, \quad a_3' = e^{ik_3} a_3, \quad a_4' = e^{ik_4} a_4.$$

Since only the values of the relative strengths a_1/a_4 and a_3/a_4 are known at $x = 0$, the corresponding values at $x = 1$ are obtained as

$$\begin{Bmatrix} a_1'/a_4' \\ a_3'/a_4' \end{Bmatrix} = \begin{bmatrix} e^{i(k_1 - k_4)} & 0 \\ 0 & e^{i(k_3 - k_4)} \end{bmatrix} \begin{Bmatrix} a_1/a_4 \\ a_3/a_4 \end{Bmatrix} = [R_2] \begin{Bmatrix} a_1/a_4 \\ a_3/a_4 \end{Bmatrix} \quad \text{(say)}.$$

(iii) The propagating A_1-, A_3- and A_4-waves are reflected from $x = 1$ and produce an A_2-wave of strength a_2'. As equations (24) and (25) are to be satisfied at the boundary, one gets the following relation in terms of the relative strengths:

$$\begin{Bmatrix} a_2'/a_4' \\ 1 \end{Bmatrix} = -\begin{bmatrix} 1 & 1 \\ k_2^2 & k_4^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ k_1^2 & k_3^2 \end{bmatrix} \begin{Bmatrix} a_1'/a_4' \\ a_3'/a_4' \end{Bmatrix} = [R_3] \begin{Bmatrix} a_1'/a_4' \\ a_3'/a_4' \end{Bmatrix}.$$

(iv) Finally, the A_2-wave of strength a_2' reaches the point $x = 0$ with strength a_2''. Thus

$$a_2'' = e^{-ik_2} a_2'$$

from which the following transformation matrix is constructed:

$$\begin{Bmatrix} a_2''/a_4 \\ 1 \end{Bmatrix} = \begin{bmatrix} e^{-i(k_2-k_4)} & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} a_2'/a_4' \\ 1 \end{Bmatrix} = [R_4] \begin{Bmatrix} a_2'/a_4' \\ 1 \end{Bmatrix}.$$

According to the phase-closure principle

$$a_2'' = a_2$$

or

$$a_2''/a_4 = a_2/a_4.$$

In terms of the transformation matrices, the above requirement is equivalent to

$$[[R_4][R_3][R_2][R_1] - I] \begin{Bmatrix} a_2/a_4 \\ 1 \end{Bmatrix} = 0.$$

For non-trivial solutions

$$\det\,[[R_4][R_3][R_2][R_1] - [I]] = 0. \tag{26}$$

One can easily verify, following the results given in Appendix A, that equation (26) gives the same result as equation (22).

3. NON-LINEAR ANALYSIS

In this section, the non-linear complex normal modes are derived using the phase-closure principle. The non-linear normal modes are defined as the configuration of the non-linear system which oscillates periodically in such a way that all the points reach the extremum positions simultaneously. All the points also cross their equilibrium positions at the same instant. Since the points of a travelling beam do not reach their equilibrium positions simultaneously, the non-linear normal modes of such a system are defined in a little different manner and are called the non-linear complex normal modes [11]. The non-linear equation of motion for the transverse vibration of a travelling beam without initial tension can be written in the following non-dimensional form [11]:

$$\frac{\partial^2 w}{\partial \tau^2} + 2c\,\frac{\partial^2 w}{\partial x \partial \tau} + c^2\,\frac{\partial^2 w}{\partial x^2} + \frac{\partial^4 w}{\partial x^4} = \varepsilon \left[\int_0^1 \left(\frac{\partial w}{\partial x} \right)^2 \mathrm{d}x \right] \frac{\partial^2 w}{\partial x^2}, \tag{27}$$

where $\varepsilon(=\gamma^2/2)$ is a small parameter, i.e., $\varepsilon \ll 1$. In what follows, the non-linear normal mode is first derived by making $c = 0$, i.e., for a stationary beam and then for a travelling beam (i.e., $c \neq 0$).

3.1. NON-LINEAR NORMAL MODE FOR A STATIONARY BEAM

For a stationary beam, the equation of motion is

$$\frac{\partial^2 w}{\partial \tau^2} + \frac{\partial^4 w}{\partial x^4} = \varepsilon \left[\int_0^1 \left(\frac{\partial w}{\partial x} \right)^2 \mathrm{d}x \right] \frac{\partial^2 w}{\partial x^2}. \tag{28}$$

To derive the non-linear normal modes, i.e., the shape of the beam which vibrates at a single frequency ω, one assumes

$$w(x, \tau) = a\psi(x)\cos\omega\tau, \tag{29}$$

where $\psi(x)$ and ω both depend on a. Putting equation (29) into equation (28) and neglecting the higher frequencies the following equation is obtained:

$$\left(-\omega^2\psi + \frac{\partial^4\psi}{\partial x^4}\right)a\cos\omega\tau = \frac{3}{4}\varepsilon a^3\left[\int_0^1\left(\frac{\partial\psi}{\partial x}\right)^2 \mathrm{d}x\right]\frac{\partial^2\psi}{\partial x^2}\cos\omega\tau. \tag{30}$$

For a normal mode to exist, $\psi(x)$ is such that the phase of any wave should change by an integral multiple of 2π by travelling once to and fro along the beam. In the absence of non-linearity (i.e., with $\varepsilon = 0$) the phase closure occurs when $\omega = \omega_n^l$ and $\psi = \phi_n$, where ω_n^l and ϕ_n are the linear natural frequency and the mode shape respectively. In general, in the presence of the non-linear term, different linear modes are simultaneously excited. Hence, the normal mode corresponds to the frequency at which the phases of the waves corresponding to all the linear modes are closed simultaneously. Thus, to get the normal oscillation, one has to search for such a frequency. To this end, the following observation from the linear analysis is important.

For the free vibration at any linear mode, say the nth mode, one can write

$$\phi_n(x) = A_1\mathrm{e}^{-\mathrm{i}kx} + A_2\mathrm{e}^{\mathrm{i}kx} + A_3\mathrm{e}^{-kx} + A_4\mathrm{e}^{kx},$$

where the wave number $k = \sqrt{\omega_n^l}$ and the coefficients A_1, A_2, A_3 and A_4 bear a constant ratio amongst each other. Thus, four (two propagating and two evanescent) waves of definite wave numbers travel once across the beam in such a way that the phase of each wave gets closed after a time interval $2\pi/\omega_n^l$. However, for a frequency different from ω_n^l, say Ω, the same waves can be closed as they traverse once across the beam only by applying a suitable external force given by

$$f_n = [(\omega_n^l)^2 - \Omega^2]\phi_n(x)\cos\Omega\tau. \tag{31}$$

With the above forcing, the wave numbers and the relative strengths of the constituent waves do not change, but the phase velocity of each wave is changed to

$$CP' = \frac{\Omega}{k} = \frac{\omega_n^l}{k}\cdot\frac{\Omega}{\omega_n^l} = CP\times\frac{\omega_n^l}{\Omega},$$

where CP is the phase-velocity of the wave propagating in the nth normal mode and having the wave number k.

Now returning to the non-linear case, by assuming weak non-linearity one can write

$$(\omega)^2 = (\omega_n^l)^2 + O(\varepsilon) \tag{32}$$

and

$$a\psi = \sum_{m=1}^{\infty} a_m\phi_m, \tag{33}$$

where $a_n = a$ and $a_m = O(\varepsilon)$ for $m \neq n$.

It can be said that the force arising out of the non-linear term (i.e., the right-hand side of equation (30)) is just suitable for simultaneous application of the phase closure for all the waves participating in various linear modes. Keeping in view equation (31) for a linear normal mode and replacing $\phi_n(x)$ by $a\psi(x)$ and Ω by ω (see equation (29)) one can write

$$\sum_{m=1}^{\infty} [(\omega_m^l)^2 - \omega^2] a_m \phi_m = \frac{3}{4} \varepsilon a^3 \left[\int_0^1 \left(\frac{\partial \psi}{\partial x} \right)^2 \mathrm{d}x \right] \frac{\partial^2 \psi}{\partial x^2}. \tag{34}$$

Applying equations (32)–(34) together with the orthogonality relationships among the linear modes i.e. $\int_0^1 \phi_n \phi_m \,\mathrm{d}x = 0$ for $m \neq n$, one gets the following result upto the term $o(\varepsilon)$:

$$\omega^2 = (\omega_n^l)^2 - \frac{3}{4} a^2 \varepsilon \frac{(\int_0^1 \phi_n (\mathrm{d}^2\phi_n/\mathrm{d}x^2)\,\mathrm{d}x)(\int_0^1 (\mathrm{d}\phi_n/\mathrm{d}x)^2\,\mathrm{d}x)}{\int_0^1 \phi_n^2 \,\mathrm{d}x}, \tag{35}$$

$$a_m = \frac{3}{4} a^3 \varepsilon \frac{(\int_0^1 \phi_m (\mathrm{d}^2\phi_n/\mathrm{d}x^2)\,\mathrm{d}x)(\int_0^1 (\mathrm{d}\phi_n/\mathrm{d}x)^2\,\mathrm{d}x)}{[(\omega_m^l)^2 - (\omega_n^l)^2]\int_0^1 \phi_m^2 \,\mathrm{d}x}. \tag{36}$$

The above results are in conformity with those obtained by a perturbation analysis [8]. For a simply supported beam one eventually gets $a_m = 0$.

3.2. NON-LINEAR NORMAL MODES FOR A TRAVELLING BEAM

The linear normal modes of a travelling beam differ from those of a stationary one by the fact that in case of the former, the phase difference between the displacement and velocity at any point depends on its axial co-ordinate x. This implies the importance of considering both the displacement and the velocity of any point during the modal response of a travelling beam. This fact is taken into account by treating the displacement and velocity as two independent quantities. This can be done by casting the equation of motion in the following usual state-space form applicable for a gyroscopic system [15]:

$$\mathbf{A} \frac{\partial W}{\partial \tau} + \mathbf{B} W = \mathbf{N}, \tag{37}$$

where

$$\mathbf{A} = \begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix}, \quad W = \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}, \quad \mathbf{N} = \left\{ \left(\int_0^1 \left(\frac{\partial w_2}{\partial x} \right)^2 \mathrm{d}x \right) \frac{\partial^2 w_2}{\partial x^2}, 0 \right\}^{\mathrm{T}},$$

with $K \equiv c^2 \partial^2/\partial x^2 + \partial^4/\partial x^4$, $G \equiv 2c(\partial/\partial x)$, $w_1 = \partial w/\partial \tau$, $w_2 = w$ and I as the identity operator.

As in the axially stationary beam, the phases of the waves corresponding to all the linear normal modes get closed during vibration in any of the non-linear normal modes. Assuming,

$$W(x, \tau) = \frac{a}{2} \Psi_n(x) \mathrm{e}^{\mathrm{i}\omega\tau} + \frac{\bar{a}}{2} \bar{\Psi}_n(x) \mathrm{e}^{-\mathrm{i}\omega\tau}, \tag{38}$$

where the bar at the top denotes the complex conjugate, the following equation is satisfied during the modal vibration:

$$\left(\mathrm{i}\omega\frac{a}{2}\mathbf{A}\Psi_n+\frac{a}{2}\mathbf{B}\Psi_n\right)\mathrm{e}^{\mathrm{i}\omega\tau}-\varepsilon\mathbf{N}_1\mathrm{e}^{\mathrm{i}\omega\tau}=0. \tag{39}$$

where

$$\mathbf{N}_1=\left\{\frac{a^2\bar{a}}{8}\left(2\,\frac{\mathrm{d}^2\psi_n}{\mathrm{d}x^2}\int_0^1\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\,\frac{\mathrm{d}\bar{\psi}_n}{\mathrm{d}x}\,\mathrm{d}x+\frac{\mathrm{d}^2\bar{\psi}_n}{\mathrm{d}x^2}\int_0^1\left(\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)^2\mathrm{d}x\right),\,0\right\}^{\mathrm{T}}.$$

From the linear analysis, it is observed that to close the phase of waves associated with Φ_n and $\bar{\Phi}_n$ in time $2\pi/\Omega$, the required forces are

$$\{f\}=\mathrm{i}\,(\Omega-\omega_n^l)\mathbf{A}\Phi_n\mathrm{e}^{\mathrm{i}\Omega\tau}$$

and

$$\{f\}=-\,\mathrm{i}\,(\Omega+\omega_n^l)\mathbf{A}\bar{\Phi}_n\mathrm{e}^{\mathrm{i}\Omega\tau},$$

respectively, where

$$\Phi_n=\begin{Bmatrix}\mathrm{i}\omega_n^l\phi_n\\ \phi_n\end{Bmatrix}$$

with ϕ_n and ω_n^l as the nth linear mode shape and the corresponding natural frequency respectively. As the force due to the non-linearity closes the phases of the waves of all the linear modes simultaneously, one can write

$$\varepsilon\mathbf{N}_1=\sum_{m=1}^{\infty}\mathrm{i}\,(\omega_m^l-\omega)\,\frac{a_m}{2}\,\mathbf{A}\Phi_m-\sum_{m=1}^{\infty}\mathrm{i}\,(\omega_m^l+\omega)\,\frac{b_m}{2}\mathbf{A}\bar{\Phi}_m. \tag{40}$$

As in the previous section, for weak non-linearity, one assumes

$$\omega=\omega_n^l+O(\varepsilon) \tag{41}$$

and

$$a\Psi=\sum_{m=1}^{\infty}a_m\Phi_m+\sum_{m=1}^{\infty}b_m\bar{\Phi}_m, \tag{42}$$

where $a_n=a$, $a_m=O(\varepsilon)$ for $m\neq n$ and $b_m=O(\varepsilon)$ for all m. Combining equations (40)–(42) and the orthogonality relations given by

$$\int_0^1\Phi_m^{\mathrm{T}}\mathbf{A}\Phi_n\,\mathrm{d}x=0\quad\text{for all } m \text{ and } n$$

and

$$\int_0^1\bar{\Phi}_m^{\mathrm{T}}\mathbf{A}\Phi_n\,\mathrm{d}x=0\quad\text{for all } m\neq n,$$

one gets the following results, valid upto $o(\varepsilon)$:

$$\omega = \omega_n^l + \frac{2\varepsilon}{ai} \frac{\int_0^1 \bar{\Phi}_n^{\mathrm{T}} \mathbf{N}_1^l \, \mathrm{d}x}{\int_0^1 \bar{\Phi}_n^{\mathrm{T}} \mathbf{A} \Phi_n \, \mathrm{d}x},$$

$$a_m = \frac{2\varepsilon}{i} \frac{\int_0^1 \bar{\Phi}_m^{\mathrm{T}} \mathbf{N}_1^l \, \mathrm{d}x}{(\omega_n^l - \omega_m^l) \int_0^1 \bar{\Phi}_m^{\mathrm{T}} \mathbf{A} \Phi_m \, \mathrm{d}x}, \quad m \neq n$$

and

$$b_m = \frac{2\varepsilon}{i} \frac{\int_0^1 \Phi_m^{\mathrm{T}} \mathbf{N}_1^l \, \mathrm{d}x}{(\omega_n^l + \omega_m^l) \int_0^1 \bar{\Phi}_m^{\mathrm{T}} \mathbf{A} \Phi_m \, \mathrm{d}x}, \quad m = 1, 2, 3, \ldots$$

where $\mathbf{N}_1^l$ is the vector $\mathbf{N}_1$ with ψ_n changed to ϕ_n. It can be verified, by expanding Φ_j's and $\mathbf{N}_1^l$ that the above results are identical to those obtained by a perturbation analysis [11].

4. CONCLUSIONS

The phase-closure principle has been used to derive the ‘natural’ frequencies of a travelling beam, first ignoring and then considering the effects of non-linearities. Justifications of some approximate methods of calculating the linear natural frequencies have been discussed. The wave-propagation principle, though does not yield any new result, enhances the physical understanding of the oscillatory behaviour of a continuous system. The knowledge of the wave propagation has been further used, in the next part of this paper, to obtain the forced response of the beam.

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APPENDIX A: PROOF OF THE IDENTITY OF EQUATIONS (21) AND (22)

The identity can be easily proved by first noting the following fact [16]. For a block matrix

$$[M] = \begin{bmatrix} [M_1] & [M_2] \\ [M_3] & [M_4] \end{bmatrix},$$

$$\det[M] = \det[M_1]\det[M_4]\det(I - [M_4]^{-1}[M_3][M_1]^{-1}[M_2]).$$

Assuming the members of matrix $[M]$ as

$$[M_1] = \begin{bmatrix} e^{ik_2} & e^{ik_4} \\ k_2^2 e^{ik_2} & k_4^2 e^{ik_4} \end{bmatrix}, \quad [M_2] = \begin{bmatrix} e^{ik_1} & e^{ik_3} \\ k_1^2 e^{ik_1} & k_3^2 e^{ik_3} \end{bmatrix},$$

$$[M_3] = \begin{bmatrix} 1 & 1 \\ k_2^2 & k_4^2 \end{bmatrix}, \quad [M_4] = \begin{bmatrix} 1 & 1 \\ k_1^2 & k_3^2 \end{bmatrix},$$

and performing a few row operations, one can easily verify that equation (22) turns out to be identical with the following:

$$\det[M] = 0.$$

Further, it can also be noticed that $[R_4][R_3][R_2][R_2] = I - [M_4]^{-1}[M_3][M_1]^{-1}[M_2]$, where $[R_j]$'s are those appearing in equation (21). Thus, $\det[M] = 0$ also implies equation (22).