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WAVE PROPAGATION IN AND VIBRATION OF A TRAVELLING BEAM WITH AND WITHOUT NON-LINEAR EFFECTS, PART II: FORCED VIBRATION

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The wave propagation in a simply supported travelling beam, studied in Part I of this paper, has been used to derive the forced responses. Based upon the wave-propagation principles, a simple method for constructing the closed-form transfer function of such a beam has been presented. The use of this transfer function offers an easy alternative to the usual modal analysis for obtaining the steady-state harmonic response. The effects of non-linearities during the steady-state oscillation, maintained by a non-resonant hard harmonic excitation, have also been studied. The present method, when compared to the conventional Galerkin's technique, requires much less computational effort.

1. INTRODUCTION

The vibration of any flexible continuous system is known to be associated with wave propagation. In Part I of this paper, the free vibration of a simply supported travelling beam has been studied using the wave-propagation theory. In such a system, the natures of the waves change depending on various system parameters, like axial speed, initial tension, etc. For any axial speed c below a certain value c_d, one evanescent and one propagating waves travel in both the upstream and downstream directions. For speed $c > c_d$, the evanescent waves disappear to give rise to two more downstream propagating waves. During the modal vibration of the beam in any of the two speed regimes, the phases of the associated waves change by an integer multiple of 2π after travelling to and fro once across the span. Even in the presence of a non-linear term the phase-closure principle holds good.

The response of a travelling string has been studied in terms of the travelling waves [1]. The calculation of the response to an initial displacement excitation becomes easy by following the waves, propagating without distortion in two opposite directions. Further, the transfer function (i.e., the Laplace transform of the Green's function), obtained by analytically solving the boundary value problem [2], has been interpreted in terms of different propagating waves. It has also been brought out that the closed-form transfer function gives a more accurate response to a harmonic force excitation than the usual 'modal analysis', where the response is *a priori* assumed to consist of a finite number of terms of an infinite series [3]. However, in a dispersive medium like a beam where the propagating waves are of significantly complicated nature, the response calculation by wave propagation principle is difficult. For such a system, the closed-form transfer function has been obtained by the analytical method [2].

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As discussed in Part I of this paper, the linear analysis becomes only approximate as the axial speed of a travelling continuous system approaches the critical speed. In this speed regime, the effects of the geometric non-linear terms, arising because of the large amplitude of vibration, cannot be neglected. The concept of non-linear complex normal mode has been derived to study the free vibration of such a non-linear system. These non-linear complex modes were also used to get the near-resonance response of a harmonically excited beam [4]. For this type of excitation, the participation of a single linear mode is of order of magnitude higher than those of the other modes. However, for a non-resonant hard excitation (i.e., an excitation with a large amplitude at a frequency far away from any of the linear natural frequencies) a number of linear modes participate significantly. Consequently, the algebra involved in calculating the response using the Galerkin's technique [5] becomes cumbersome.

In this paper, the principle of wave propagation is used to construct the transfer function of a linear travelling beam in closed form. The present method of deriving the transfer function of a travelling continuous system, though yields the same results obtained by the modal analysis, enhances the physical understanding of the system. The transfer function is used to obtain the response to any arbitrary excitation. The results of the linear analysis are then extended to obtain the response of the beam to a non-resonant hard harmonic excitation after including the non-linear terms. Since the linear response is obtained in a closed form, the computation of the non-linear response, becomes easy by considering it to be a perturbation to the linear response.

2. THEORETICAL ANALYSIS

2.1. LINEAR ANALYSIS

Consider a simply supported travelling beam of finite span. The non-dimensional equation of motion of the forced vibration of such a system, including the non-linear terms can be written as [4]

$$\frac{\partial^2 w}{\partial \tau^2} + 2c\frac{\partial^2 w}{\partial x \partial \tau} + (c^2 - T_0)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^4 w}{\partial x^4} = \varepsilon\left[\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\right]\frac{\partial^2 w}{\partial x^2} + f(x, \tau), \tag{1}$$

with the boundary conditions

$$w(0, \tau) = w(1, \tau) = 0$$

and

$$\frac{\partial^2 w(0, \tau)}{\partial x^2} = \frac{\partial^2 w(1, \tau)}{\partial x^2} = 0.$$

In equation (1), $f(x, \tau)$ represents the non-dimensional exciting force and all other symbols are explained in Part I of this paper. In what follows, the response of a linear travelling beam (i.e., with $\varepsilon = 0$ in equation (1)), first to a point impulse excitation and then to any arbitrary excitation, are derived using the method of wave propagation. To this end, the following observations of the linear free vibration of the beam are important. As shown in Part I, the nature of the waves generated during the free vibration are different in two axial speed regimes, namely, $c < c_d$ and $c \geqslant c_d$. In the first case one propagating (A_1-wave) and one evanescent (A_3-wave) waves travel in the downstream direction. Similarly the upstream waves comprise of one propagating (A_2-wave) and one evanescent (A_4-wave) waves. In the second case, in place of the two evanescent waves, two more downstream propagating waves (A_3- and A_4-waves) are generated. In either case, the following conditions are satisfied during a modal vibration.

$$\det\begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{ik_1} & e^{ik_2} & e^{ik_3} & e^{ik_4} \\ k_1^2 & k_2^2 & k_3^2 & k_4^2 \\ k_1^2 e^{ik_1} & k_2^2 e^{ik_2} & k_3^2 e^{ik_2} & k_4^2 e^{ik_4} \end{bmatrix} = 0,$$

where k_j ($j = 1, 2, 3, 4$) corresponds to the wave number associated with the A_j-wave.

The easiest way to understand the response of a continuous system by wave propagation is to analyse the equation of motion in the frequency domain. This is done by writing the equation of motion in terms of the Laplace transform of the variables, taken with respect to time, as follows:

$$s^2\hat{w} + 2sc\frac{\partial\hat{w}}{\partial x} + (c^2 - T_0)\frac{\partial^2\hat{w}}{\partial x^2} + \frac{\partial^4\hat{w}}{\partial x^4} = \hat{f}(x, s) - s\frac{\partial w}{\partial\tau}(x, 0) - w(x, 0) - 2c\frac{\partial w}{\partial x}(x, 0), \quad (2)$$

where $\hat{w}(x, s)$ and $\hat{f}(x, s)$ are the Laplace transforms of $w(x, \tau)$ and $f(x, \tau)$ respectively. The initial configuration and velocity of the beam are assumed to be trivially zero, i.e., at $\tau = 0$, $w = 0$, $\partial w/\partial\tau = 0$ and $\partial w/\partial x = 0$ for all x. Further with $f(x, \tau) = \delta(x - x_0)\delta(\tau - \tau_0)$, the equation of motion becomes

$$s^2\hat{w} + 2sc\frac{\partial\hat{w}}{\partial x} + (c^2 - T_0)\frac{\partial^2\hat{w}}{\partial x^2} + \frac{\partial^4\hat{w}}{\partial x^4} = \delta(x - x_0)\,e^{-s\tau_0}. \quad (3)$$

since $\hat{f}(x, s) = \delta(x - x_0)e^{-s\tau_0}$.

The particular integral of equation (3) can be obtained as

$$\hat{w}(x, s) = \sum_{j=1}^{4}\left(\prod_{m=1, m\neq j}^{4}\frac{i}{(k_j^* - k_m^*)}\right)e^{ik_j^* x}\int_0^x e^{-ik_j^* x_1}\delta(x_1 - x_0)e^{-s\tau_0}\,dx_1. \quad (4)$$

For $x > x_0$, the particular integral then becomes

$$\hat{w}(x, s) = \sum_{j=1}^{4} D_j e^{ik_j^*(x - x_0)}, \quad (5)$$

where

$$D_j = \prod_{m=1, m\neq j}^{4}\frac{i}{(k_j^* - k_m^*)}\,e^{-s\tau_0},$$

and k_j^*'s are the roots of the polynomial

$$s^2 + 2isck^* - (c^2 - T_0)(k^*)^2 + (k^*)^4 = 0. \quad (6)$$

For any complex value of s, the roots of the above polynomial are either real or complex. Among the complex roots, those having positive (or negative) imaginary parts correspond to the downstream (or upstream) evanescent waves. Of the real roots the negative ones correspond to the downstream propagating waves and the positive ones denote the upstream propagating waves. For $s = i\omega$, the possibilities of the two kinds of waves are already reported in Part I. Depending upon the relation between c and ω, either two propagating or four propagating waves may exist.

From the physical nature of the problem, it can be judged that all the terms in equation (5) are not present at a point $x > x_0$. This is because of the following reason. At point $x = x_0$, there exist four waves of strength $D_j(j = 1, 2, 3, 4)$, out of which only those which propagate in the downstream direction will have an effect on a point lying in the same direction of x_0. The effects of the upstream waves will not be felt at these points. For example, for $s = \mathrm{i}\omega$ ($\omega > 0$ and $c < c_d$) only the contribution of A_1- and A_3-waves are to be retained. But for $c > c_d$, all the downstream propagating waves (i.e., the D_1, D_3 and D_4 terms) will contribute to the response at a point $x > x_0$.

Thus, omission or inclusion of any term of the right-hand side of equation (5) depends upon the values of k_j^*'s. In the following, the response will be calculated only for $s = \mathrm{i}\omega$ and $c < c_d$. The other cases may be studied in an analogous manner. The particular integral (equation (5)) then becomes

$$\hat{w}(x, s) = \sum_{j=1,3} D_j \mathrm{e}^{\mathrm{i}k_j^*(x-x_0)} \quad \text{for } x \geqslant x_0, \tag{7}$$

The particular integral for $x < x_0$ can be found out by first substituting $\xi = -x$ in equation (2). Following the steps analogous to those carried out for $x > x_0$ and noting that $-k_j^*$'s are the roots of equation corresponding to equation (6), the particular integral is finally obtained as

$$\hat{w}(x, s) = -\sum_{j=1}^{4} D_j \mathrm{e}^{\mathrm{i}k_j^*(x-x_0)} \quad \text{for } x \leqslant x_0,$$

Following the same reasoning, used for $x > x_0$, the particular integral in this case has the terms corresponding to the waves going in the upstream direction, i.e.,

$$\hat{w}(x, s) = -\sum_{j=2,4} D_j \mathrm{e}^{\mathrm{i}k_j^*(x-x_0)} \quad \text{for } x \leqslant x_0. \tag{8}$$

It is to be noted that the particular integral implies the response of the beam only due to the direct influence of the force. However, the waves generated by the external excitation are reflected at the boundaries and new terms appear. These terms are taken care of by retaining the complementary function. The total response is thus given by

$$\hat{w}(x, s) = \sum_{j=1}^{4} C_j \mathrm{e}^{\mathrm{i}k_j^* x} + \sum_{j=1,3} D_j \mathrm{e}^{\mathrm{i}k_j^*(x-x_0)} \quad \text{for } x \geqslant x_0, \tag{9}$$

$$= \sum_{j=1}^{4} C_j \mathrm{e}^{\mathrm{i}k_j^* x} - \sum_{j=2,4} D_j \mathrm{e}^{\mathrm{i}k_j^*(x-x_0)} \quad \text{for } x \leqslant x_0. \tag{10}$$

One can also verify that the continuity of displacement, slope and moment exist at $x = x_0$. The unknown constants C_j's can be obtained by ensuring the response satisfies the boundary conditions. This yields

$$0 = [\Delta^*]\{C\} + [F]\{D\}, \tag{11}$$

where

$$[\Delta^*] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ (k_1^*)^2 & (k_2^*)^2 & (k_3^*)^2 & (k_4^*)^2 \\ e^{ik_1^*} & e^{ik_2^*} & e^{ik_3^*} & e^{ik_4^*} \\ (k_1^*)^2 e^{ik_1^*} & (k_2^*)^2 e^{ik_2^*} & (k_3^*)^2 e^{ik_3^*} & (k_4^*)^2 e^{ik_4^*} \end{bmatrix},$$

$$[F] = \begin{bmatrix} 0 & e^{-ik_2^* x_0} & 0 & e^{-ik_4^* x_0} \\ 0 & (k_2^*)^2 e^{-ik_2^* x_0} & 0 & (k_4^*)^2 e^{-ik_4^* x_0} \\ e^{-ik_1^*(1-x_0)} & 0 & e^{-ik_3^*(1-x_0)} & 0 \\ (k_1^*)^2 e^{-ik_1^*(1-x_0)} & 0 & (k_3^*)^2 e^{-ik_3^*(1-x_0)} & 0 \end{bmatrix},$$

$\{C\} = \{C_1, C_2, C_3, C_4\}^T$ and $\{D\} = \{D_1, -D_2, D_3, -D_4\}^T$. The values of C_j's become

$$\{C\} = -[\Delta^*]^{-1}[F]\{D\} = -[\Delta^*]_{adj}[F]\{D\}/\det[\Delta^*], \tag{12}$$

where $[\Delta^*]_{adj}$ is the adjoint matrix of $[\Delta^*]$. Substitution of the values of C_j's into equations (9) and (10), gives the transfer function for a travelling beam and will be denoted by $G(x, x_0, s)$.

The response of the beam to any arbitrary excitation $f(x, \tau)$ can be obtained as

$$\hat{w}(x, s) = \int_0^1 G(x, \zeta, s)\hat{f}(\zeta, s)\,d\zeta,$$

where $\hat{f}(x, s)$ is the Laplace transform of $f(x, \tau)$. To retrieve the time response $w(x, \tau)$, to an impulse excitation one has to take the inverse Laplace transform of $\hat{w}(x, s)$ as

$$w(x, \tau) = \lim_{Y\to\infty} \frac{1}{2\pi i} \int_{\beta - iY}^{\beta + iY} \hat{w}(x, s)e^{s\tau}\,ds \quad \text{for } \tau > 0 \tag{13}$$

where the value of β is so chosen that the integration converges. It is shown below that the integration can also be evaluated using the contour integration theory of complex variable. As seen from equations (5)–(10), the integrand contains a term $e^{-s\tau_0}$, for which the inverse transformation is carried out separately for $\tau < \tau_0$ and $\tau > \tau_0$.

For $\tau < \tau_0$, the value of β is taken to be negative and the contour is devoid of any singularity (see Figure 1). Thus,

$$w(x, \tau) = 0 \quad \text{for } \tau < \tau_0. \tag{14}$$

For $\tau > \tau_0$, one must choose β to be an arbitrary positive number. Since the contour (see Figure 2) now contains countably infinite number of singular points, the integration can be performed with the help of the following two theorems [6, 7].

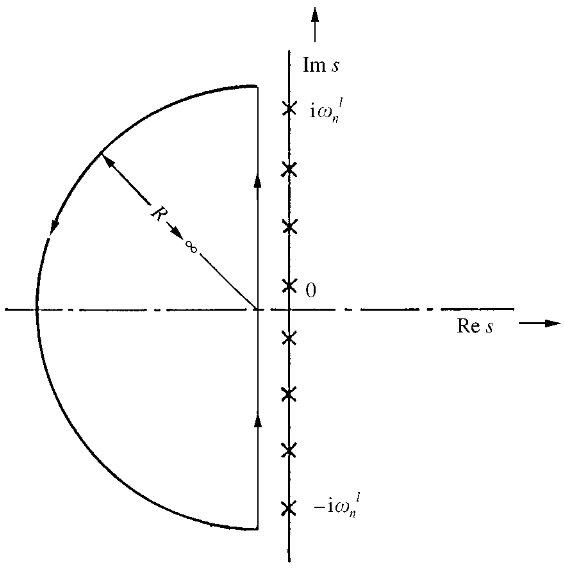

Figure 1. Contour of the integration (equation (13)) for $\tau < \tau_0$.

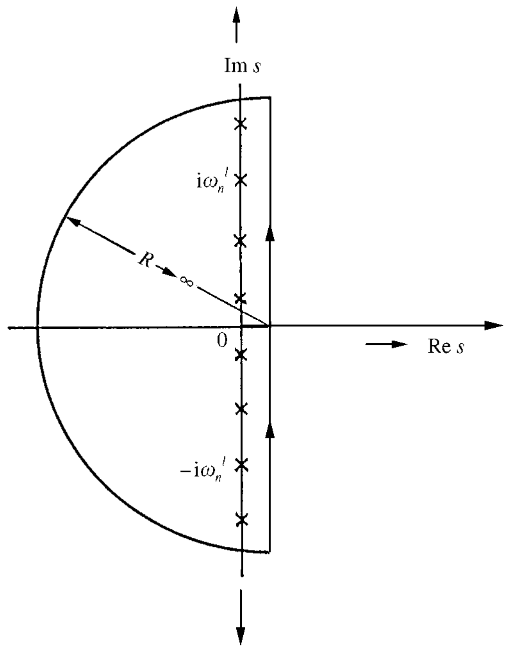

Figure 2. Contour of integration (equation (13)) for $\tau > \tau_0$.

Theorem 1. *Let $f(s)$ be a function which is analytic inside a simple closed path C and on C, except for finitely many singular points $a_1, a_2, \ldots, a_m$ inside C, then*

$$\int_C f(s)\,\mathrm{d}s = 2\pi\mathrm{i} \sum_{j=1}^{m} \operatorname{Res} f(s)|_{s=a_j}.$$

the integral being taken in the counterclockwise sense around the path C [6].

Theorem 2. *If $f_1(s)$ and $f_2(s)$ are analytic in the neighbourhood of s_0 and if $f_1(s_0) \neq 0$ but $f_2(s)$ has a simple pole at s_0, then the residue of $f_1(s)/f_2(s)$ at s_0 is equal to $f_1(s_0)/f_2'(s_0)$* [7].

The integration appearing in equation (13), when carried out, yields

$$w(x,\tau) = \sum_{n=1}^{\infty} \left\{ \frac{\sum_{j=1}^{4} C_j \mathrm{e}^{\mathrm{i}k_j^* x} + \sum_{j=1,3} D_j \mathrm{e}^{\mathrm{i}k_j^*(x-x_0)}}{\partial(\det[\Delta^*])/\partial s} \right\}\Bigg|_{s=\mathrm{i}w_n^l} \mathrm{e}^{\mathrm{i}\omega_n^l(\tau-\tau_0)} + c.c. \quad \text{for } x \geqslant x_0, \tag{15}$$

where the value of the denominator is

$$\frac{\partial}{\partial s}(\det[\Delta^*]) = \sum_{j=1}^{4} \frac{\partial}{\partial k_j} \det[\Delta^*] \frac{\mathrm{d}k_j}{\mathrm{d}s}.$$

In equation (15), *c.c.* denotes the complex conjugate of the previous term. The response can be obtained for $x \leqslant x_0$, after replacing D_1 and D_3 in equation (15) by D_2 and D_4 respectively.

It should be brought to the notice that the response can also be obtained in an infinite series form following the modal analysis, as detailed below.

In this method, the response is first assumed as

$$W(x,\tau) = \sum_{n=1}^{\infty} (\zeta_n(\tau)\Phi_n(x) + \bar{\zeta}_n(\tau)\bar{\Phi}_n(x)),$$

with zero initial conditions, where the vectors W and $\Phi_n(x)$ are defined in Part — I of this paper. The unknown functions ζ_n's are obtained by solving the following equation:

$$\frac{\mathrm{d}\zeta_n}{\mathrm{d}\tau} - \mathrm{i}\omega_n^l \zeta_n = \frac{\int_0^1 \bar{\Phi}_n \mathbf{f} \, \mathrm{d}x}{\int_0^1 \bar{\Phi}_n \mathbf{A} \Phi_n \, \mathrm{d}x},$$

where $\mathbf{f} = \{\delta(x-x_0)\delta(\tau-\tau_0), 0\}^{\mathrm{T}}$. The solution can be written in an expanded form as

$$\zeta_n(\tau) = -\mathrm{i} \frac{[\int_0^\tau \mathrm{e}^{\mathrm{i}\omega_n^l(\tau-\tau_1)} \bar{\phi}_n(x_0)\delta(\tau_1-\tau_0)\,\mathrm{d}\tau_1]}{[2\omega_n^l \int_0^1 \phi_n \bar{\phi}_n \,\mathrm{d}x - 2\mathrm{i}c\int_0^1 \bar{\phi}_n \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\,\mathrm{d}x]}$$

yielding the response as

$$w(x,\tau) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\bar{\phi}_n(x_0)}{2\omega_n^l \int_0^1 \phi_n \bar{\phi}_n \,\mathrm{d}x - 2\mathrm{i}c\int_0^1 \bar{\phi}_n \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\,\mathrm{d}x} \mathrm{e}^{\mathrm{i}\omega_n^l(\tau-\tau_0)} + c.c. \quad \text{for } \tau \geqslant \tau_0. \tag{16}$$

and zero for $\tau < \tau_0$.

Before considering the steady state response of a travelling beam, it may be worthwhile to mention that the response of a travelling string obtained by the wave propagation method can be explicitly shown to be identical to equation (16). This is detailed in Appendix A.

The computation of the steady state response of a travelling beam to a harmonic point load is very convenient by the wave-propagation method. For a linear beam, excited by a harmonic excitation with frequency Ω (i.e., $f(x,\tau) = F_0\delta(x-x_0)\cos\Omega\tau$), the response is obtained as

$$w(x,\tau) = \frac{F_0}{2} G(x, x_0, \mathrm{i}\Omega)\mathrm{e}^{\mathrm{i}\Omega\tau} + c.c., \tag{17}$$

where *c.c.* denotes the complex conjugate of the preceding term. It may be noted that the response becomes infinitely large if the beam is resonantly excited with $\Omega = \omega_n^l$ as $\det[\Delta^*] = 0$.

2.1.1. *Numerical results and discussions*

In this section, numerical results of the linear response, obtained by the wave-propagation method are presented. The initial tension and the axial speed of the beam is taken as $T_0 = 1$ and $c' = c/(c_{cr})_1 = 0{\cdot}5$ where $(c_{cr})_1 = \sqrt{T_0 + \pi^2}$.

Figure 3 shows the variation of $|G(x, x_{0,1}\Omega)|$ with the frequency of excitation (Ω) when the exciting force of unit amplitude as applied at $x_0 = 0{\cdot}3$ and the response is measured at $x = 0{\cdot}5$. As shown in the figure, the response amplitude is infinitely large if the excitation frequency Ω coincides with one of the natural frequencies of the beam and the point of either excitation or observation does not fall on the nodal points of the corresponding mode. Thus, the response amplitude at $\Omega = \omega_2^l$, as depicted in the figure, is zero since the point of measurement coincides with the nodal point of the second mode. A large response is obtained for this excitation (i.e., with $\Omega = \omega_2^l$) at any other point. Figure 4 shows a very large value of response obtained at $x = 0{\cdot}75$. It must be pointed out that interchanging the positions of the excitation and observation does not alter the response. This reciprocity relation has indeed been observed in the numerical computation.

As discussed in section 2.1, the response can also be computed in terms of the normal modes, whereby a solution is constructed in a series form. However, only a few terms of the infinite series are usually taken into account. For the excitation frequency close to a natural frequency of the beam, the corresponding normal mode contributes most significantly. But

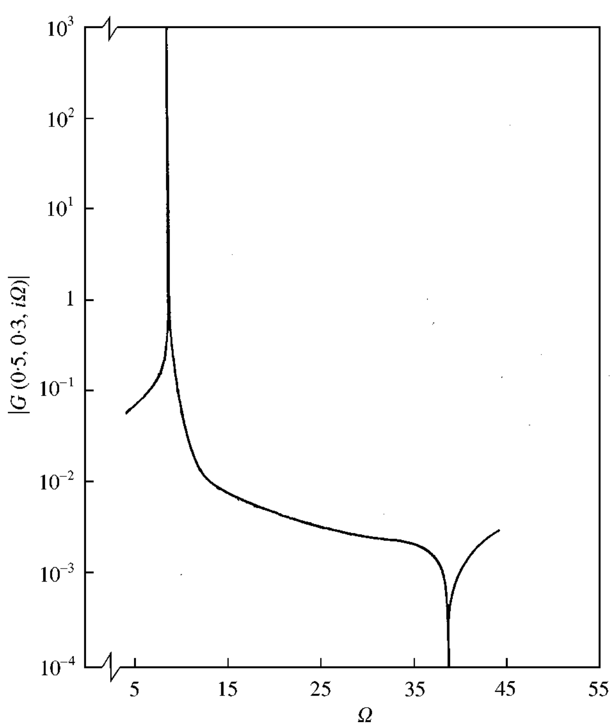

Figure 3. Linear frequency response function to a point excitation at $x_0 = 0{\cdot}3$ measured at $x = 0{\cdot}5$. $c' = 0{\cdot}5$.

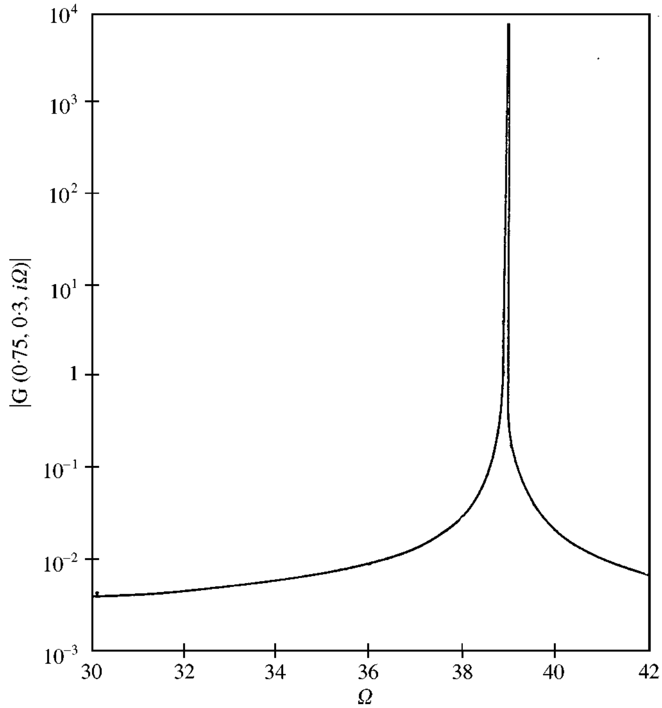

Figure 4. Linear frequency response function to a point excitation at $x_0 = 0{\cdot}3$ measured at $x = 0{\cdot}75$. $c' = 0.5$.

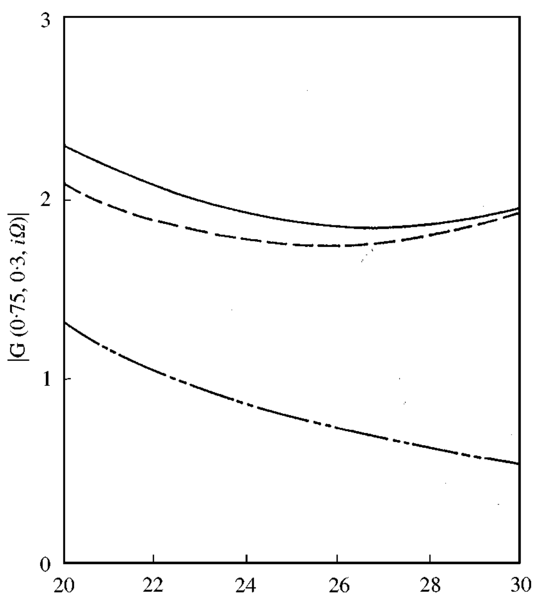

Figure 5. Linear frequency response function to a point excitation at $x_0 = 0{\cdot}3$ measured at $x = 0{\cdot}75$. $c' = 0.5$. ——: wave propagation analysis; —-·-—: one-term modal analysis; – – –: two-term modal analysis.

if the excitation frequency is away from any ω_n^l, the effects of all the neighbouring modes become significant. For a high excitation frequency a large number of modes are to be considered making the computation quite expensive. However, in the wave propagation method, the response is obtained accurately in the closed form. Figure 5 shows the comparative results obtained by the wave-propagation method and also by the one- and two-term modal approximations. For the chosen frequency range, one-term approximation is very crude. As shown in the figure, an addition of another mode makes the result closer to the value given by the present method. It has been found that for a frequency $\Omega \sim \omega_1^l$, the first-mode approximation yields quite accurate results.

2.2. NON-LINEAR ANALYSIS

In this section, the non-linear steady state periodic response of the beam subjected to a non-resonant hard excitation is analyzed. As the frequency of excitation is away from any of the linear natural frequencies, various linear modes are excited. Unlike the near-resonance excitation, in this case, the non-linear normal modes are not convenient for obtaining the response. As will be shown below, the transfer function of the beam, derived in the previous section, can be used in such a situation. Since the linear response in the present method, contrary to the modal analysis, is obtained in a closed form, the non-linear analysis becomes simplified. Also the error due to neglecting the higher order modes is not encountered in this method.

Assuming the excitation of the form $f(x,\tau) = F_0\delta(x - x_0)\cos\Omega\tau$, where the magnitude of the force, F_0, is sufficiently large for the effects of the non-linear term in equation (1) to become significant, even if the excitation frequency Ω is away from any of the natural frequencies or their combinations, the response $w(x,\tau)$ is sought in the following series form:

$$w(x,\tau) = w_0(x,\tau) + \varepsilon w_1(x,\tau) + \cdots. \tag{18}$$

Substituting equation (18) into equation (1) and equating the coefficients of the like powers of ε from both the sides, the following results are obtained:

$$\varepsilon^0: \frac{\partial^2 w_0}{\partial\tau^2} + 2c\frac{\partial^2 w_0}{\partial x\partial\tau} + (c^2 - T_0)\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^4 w_0}{\partial x^2} = f(x)\cos\Omega\tau, \tag{19}$$

$$\varepsilon^1: \frac{\partial^2 w_1}{\partial\tau^2} + 2c\frac{\partial^2 w_1}{\partial x\partial\tau} + (c^2 - T_0)\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^4 w_1}{\partial x^4} = \left[\int_0^1 \left(\frac{\partial w_0}{\partial x}\right)^2 \mathrm{d}x\right]\frac{\partial^2 w_0}{\partial x^2}. \tag{20}$$

Equation (19) is solved to yield

$$w_0(x;\tau) = \frac{\mathrm{e}^{\mathrm{i}\Omega\tau}}{2}\int_0^1 G(x,\zeta,\mathrm{i}\Omega)f(\zeta)\,\mathrm{d}\zeta + \frac{\mathrm{e}^{-\mathrm{i}\Omega\tau}}{2}\int_0^1 G(x,\zeta,-\mathrm{i}\Omega)f(\zeta)\,\mathrm{d}\zeta \tag{21}$$

or

$$w_0(x,\tau) = \frac{H(x,\mathrm{i}\Omega)}{2}\mathrm{e}^{\mathrm{i}\Omega\tau} + \frac{H(x,-\mathrm{i}\Omega)}{2}\mathrm{e}^{-\mathrm{i}\Omega\tau}, \tag{22}$$

where

$$H(x,\mathrm{i}\Omega) = \int_0^1 G(x,\zeta,\mathrm{i}\Omega)f(\zeta)\,\mathrm{d}\zeta.$$

Solution of equation (20) can be obtained by writing the right-hand side of the same equation as follows:

$$\left[\int_0^1\left(\frac{\partial w_0}{\partial x}\right)^2 \mathrm{d}x\right]\frac{\partial^2 w_0}{\partial x^2} = [M_1(x)\mathrm{e}^{\mathrm{i}\Omega\tau} + M_2\mathrm{e}^{3\mathrm{i}\Omega\tau}] + c.c., \tag{23}$$

where

$$M_1(x) = \frac{1}{8}\left[\int_0^1\left(\frac{\partial H(x,\mathrm{i}\Omega)}{\partial x}\right)^2 \mathrm{d}x\right]\frac{\partial^2 H(x,-\mathrm{i}\Omega)}{\partial x^2} + \frac{1}{4}\left[\int_0^1 \frac{\partial H(x,\mathrm{i}\Omega)}{\partial x}\frac{\partial H(x,-\mathrm{i}\Omega)}{\partial x}\mathrm{d}x\right]\frac{\partial^2 H(x,\mathrm{i}\Omega)}{\partial x^2}$$

and

$$M_2(x) = \frac{1}{8}\left[\int_0^1\left(\frac{\partial H(x,\mathrm{i}\Omega)}{\partial x}\right)^2 \mathrm{d}x\right]\frac{\partial^2 H(x,\mathrm{i}\Omega)}{\partial x^2}.$$

With the help of equation (23), $w_1(x,\tau)$ is obtained as

$$w_1(x,\tau) = \mathrm{e}^{\mathrm{i}\Omega\tau}\int_0^1 G(x,\zeta,\mathrm{i}\Omega)M_1(\zeta)\,\mathrm{d}\zeta + \mathrm{e}^{3\mathrm{i}\Omega\tau}\int_0^1 G(x,\zeta,3\mathrm{i}\Omega)M_2(\zeta)\,\mathrm{d}\zeta + c.c. \tag{24}$$

Substituting equations (22) and (24) in equation (18) the response of a non-linear beam, up to the term $o(\varepsilon)$, is obtained. Further simplification can be obtained if the force is assumed to be a point harmonic one, i.e. $f(x) = F_0\delta(x - x_0)$. In that case

$$H(x,\mathrm{i}\Omega) = F_0 G(x, x_0, \mathrm{i}\Omega),$$

and the values of $M_1(x)$ and $M_2(x)$ are evaluated by numerically computing a few definite integrals.

2.2.1. *Numerical results and discussion*

The non-linear response of a beam subjected to a point harmonic excitation is presented in this section. The axial speed, initial tension, amplitude of excitation and the small parameter ε are taken as follows:

$$c' = 0{\cdot}5, \quad T_0 = 1{\cdot}0, \quad F_0 = 1000, \quad \varepsilon = 10^{-2}.$$

The Fourier transform of the response at the point $x = 0{\cdot}75$ to a non-resonant hard excitation applied at $x_0 = 0{\cdot}3$ is shown in Figure 6. The response has both harmonic and superharmonic components. Figure 7 shows the variations of harmonic components ($w_\Omega(x,\tau)$, say) of the response amplitudes, measured at two different locations ($x = 0{\cdot}5$ and $x = 0{\cdot}75$) of the beam, to a point excitation having frequency (Ω) away from both ω_1^l ($= 8{\cdot}72855$) and ω_2^l ($= 38{\cdot}85552$). It is seen that the effect of the non-linearity is more pronounced at $x = 0{\cdot}75$, since the participation of the second mode is stronger for that location.

As discussed earlier, the non-linear response can be obtained by either the wave-propagation method or the Galerkin's technique. Figure 8 shows the amplitude of the harmonic component of the response calculated by these two methods. While using the

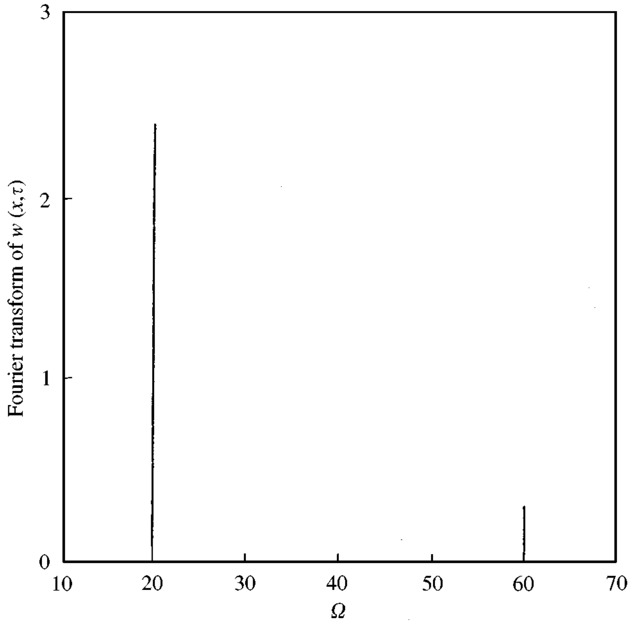

Figure 6. Fourier transform of the non-linear response at $x = 0{\cdot}75$. $F_0 = 1000$, $c' = 0{\cdot}5$, $\varepsilon = 0{\cdot}01$, $\Omega = 20$, $x_0 = 0{\cdot}3$.

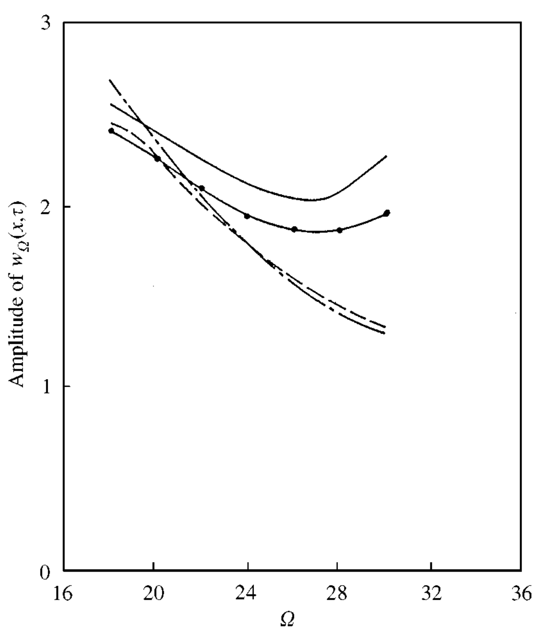

Figure 7. Variation of the harmonic components of responses to point harmonic excitation. $F_0 = 1000$, $c' = 0{\cdot}5$, $\varepsilon = 0{\cdot}01$, $x_0 = 0{\cdot}3$. ——: non-linear response measured at $x = 0{\cdot}75$; —●—: linear response measured at $x = 0{\cdot}75$; - - -: non-linear response measured at $x = 0{\cdot}5$; – · –: linear response measured at $x = 0{\cdot}5$.

Galerkin's technique, numerical results were obtained by using only two linear modes. The discrepancy between the results obtained by the two methods is seen to be more than that for a linear beam. This is perhaps due to significant coupling between various linear modes in the presence of non-linearity.

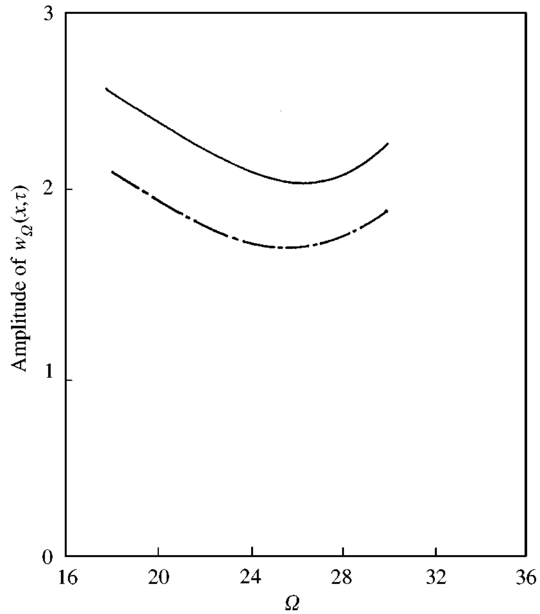

Figure 8. Variation of the harmonic component of non-linear response to point harmonic excitation measured at $x = 0{\cdot}75$. $F_0 = 1000$, $c' = 0{\cdot}5$, $\varepsilon = 0{\cdot}01$, $x_0 = 0{\cdot}3$. ——: wave propagation method; —·—: Galerkin's technique.

3. CONCLUSIONS

A method based on wave propagation has been proposed for determining the response of a travelling beam. The closed-form transfer function of the linear beam is first obtained. The equivalence of the modal and wave-propagation analyses for obtaining the linear, impulse-response is established. The transfer function is subsequently used to derive the steady state response of the same beam to a non-resonant hard excitation, after taking the effects of non-linear term. In both the linear and non-linear cases, the method proves to be more accurate and computationally more efficient than other methods such as modal analysis for a linear beam and Galerkin's technique for the non-linear one.

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APPENDIX A: FORCED RESPONSE OF A TRAVELLING STRING

For a linear travelling string, equation (3) is replaced by

$$s^2\hat{w} + 2sc\frac{\partial \hat{w}}{\partial x} + (c^2-1)\frac{\partial^2 \hat{w}}{\partial x^2} = \delta(x-x_0)\mathrm{e}^{-s\tau_0}. \tag{A1}$$

The response from the wave-propagation analysis, formulated in section 2, is obtained as

$$\hat{w}(x,s) = C_1\mathrm{e}^{\lambda_1 x} + C_2\mathrm{e}^{\lambda_2 x} + D_2\mathrm{e}^{\lambda_2(x-x_0)} \quad \text{for } x \geqslant x_0 \tag{A2}$$

$$= C_1\mathrm{e}^{\lambda_1 x} + C_2\mathrm{e}^{\lambda_2 x} - D_1\mathrm{e}^{\lambda_1(x-x_0)} \quad \text{for } x < x_0 \tag{A3}$$

where $\lambda_1 = \mathrm{i}k_1 = s/(1-c)$, $\lambda_2 = \mathrm{i}k_2 = -s/(1+c)$, $D_1 = -\mathrm{e}^{-s\tau_0}/2s$, $D_2 = \mathrm{e}^{-s\tau_0}/2s$.

Applying the boundary conditions $\hat{w}(0,s) = \hat{\omega}(1,s) = 0$, the following expressions of C_1 and C_2 are obtained

$$C_1 = \mathrm{e}^{-s\tau_0}[\mathrm{e}^{\lambda_2 - \lambda_1 x_0} - \mathrm{e}^{\lambda_2(1-x_0)}]/2s(\mathrm{e}^{\lambda_1} - \mathrm{e}^{\lambda_2})$$

and

$$C_2 = \mathrm{e}^{-s\tau_0}[\mathrm{e}^{\lambda_2(1-x_0)+\lambda_1} - \mathrm{e}^{\lambda_1(1-x_0)}]/2s(\mathrm{e}^{\lambda_1} - \mathrm{e}^{\lambda_2})$$

which, when substituted into equations (26) and (27), yield

$$\hat{w}(x,s) = \frac{[\mathrm{e}^{\lambda_2(x-x_0)} - \mathrm{e}^{\lambda_2 x - \lambda_1 x_0} - \mathrm{e}^{\lambda_2(1-x_0)-\lambda_1(1-x)} + \mathrm{e}^{\lambda_2 - \lambda_1(1+x_0-x)}]\,\mathrm{e}^{-s\tau_0}}{2s(1-\mathrm{e}^{\lambda_2-\lambda_1})} \quad \text{for } x \geqslant x_0, \tag{A4}$$

$$= \frac{[-\mathrm{e}^{\lambda_2 x - \lambda_1 x_0} + \mathrm{e}^{\lambda_2(1-x-x_0)-\lambda_1} + \mathrm{e}^{-\lambda_1(x_0-x)} - \mathrm{e}^{\lambda_2(1-x_0)-\lambda_1(1-x)}]\,\mathrm{e}^{-s\tau_0}}{2s(1-\mathrm{e}^{\lambda_2-\lambda_1})} \quad \text{for } x < x_0. \tag{A5}$$

For $\tau_0 = 0$, the above equations are identical with the transfer function obtained by Yang and Tan [2]. As mentioned therein, $s = 0$ is not a singular point since $w(x,s)$ is finite as $s \to 0$. The only singular points are $s = \pm\mathrm{i}\omega_n^l = \pm\,\mathrm{i}n\pi(1-c^2)$, when $\lambda_1 = \pm\,\mathrm{i}n\pi(1+c) = \lambda_1^{(n)}$, and $\lambda_2 = \mp\,\mathrm{i}n\pi(1-c) = \lambda_2^{(n)}$.

The temporal response is obtained by inverse Laplace transform of equations (A4) and (A5) as

$$w(x,\tau) = \sum_{n=1}^{\infty}\left[\frac{(\mathrm{e}^{\lambda_2^{(n)}x} - \mathrm{e}^{\lambda_1^{(n)}x})(\mathrm{e}^{-\lambda_2^{(n)}x_0} - \mathrm{e}^{-\lambda_1^{(n)}x_0})}{2(1-\mathrm{e}^{\lambda_2^{(n)}-\lambda_1^{(n)}}) + 2s\left(\dfrac{\partial\lambda_1^{(n)}}{\partial s} - \dfrac{\partial\lambda_2^{(n)}}{\partial s}\right)\mathrm{e}^{\lambda_2^{(n)}} - \lambda_1^{(n)}}\right]\mathrm{e}^{\mathrm{i}\omega_n^l(\tau-\tau_0)} + \, c.c. \quad \text{for } \tau \geqslant \tau_0$$

or

$$w(x,\tau) = \sum_{n=1}^{\infty}\frac{(\mathrm{e}^{\lambda_2^{(n)}x} - \mathrm{e}^{\lambda_1^{(n)}x})(\mathrm{e}^{-\lambda_2^{(n)}x_0} - \mathrm{e}^{-\lambda_1^{(n)}x_0})}{4\mathrm{i}n\pi}\mathrm{e}^{\mathrm{i}\omega_n^l(\tau-\tau_0)} + c.c. \quad \text{for } \tau \geqslant \tau_0. \tag{A6}$$

Since for a travelling string, the linear normal mode ϕ_n can be written as $\phi_n = \mathrm{e}^{\lambda_2^{(n)}x} - \mathrm{e}^{\lambda_1^{(n)}x}$, one can verify that

$$2\omega_n^l\int_0^1 \phi_n\bar{\phi}_n\,\mathrm{d}x - 2\mathrm{i}c\int_0^1 \bar{\phi}_n\frac{\mathrm{d}\phi_n}{\mathrm{d}x}\,\mathrm{d}x = 4n\pi. \tag{A7}$$

Using equations (A6) and (A7) the response is finally obtained as

$$w(x, \tau) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\bar{\phi}_n(x_0)}{2\omega_n\int_0^1 \phi_n\bar{\phi}_n \, \mathrm{d}x - 2\mathrm{i}c\int_0^1 \bar{\phi}_n \dfrac{\mathrm{d}\phi_n}{\mathrm{d}x} \, \mathrm{d}x} \mathrm{e}^{\mathrm{i}\omega_n^l(\tau-\tau_0)} + c.c. \quad \text{for } \tau \geqslant \tau_0. \tag{A8}$$

Equation (A8) is identical to equation (16).