Journal of Sound and Vibration (2000) **236**(3), 401–411
doi:10.1006/jsvi.2000.2982, available online at http://www.idealibrary.com on IDEAL®

NON-LINEAR VIBRATIONS OF A BEAM RESTING ON A TENSIONLESS WINKLER FOUNDATION

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(*Received* 26 *May* 1999, *and in final form* 19 *January* 2000)

The forced vibrations of an elastic beam resting on a non-linear tensionless Winkler foundation subjected to a concentrated dynamic load at its centre are described in this paper. The problem is a non-linear one because of the tensionless character of the foundation and the non-linear term in the foundation model. A non-linear governing differential equation for the forced vibrations of the beam is derived in matrix form by employing the Galerkin method. The free vibration mode functions of the completely free beam are adopted as the co-ordinate functions of the displacement function of the beam. Firstly, the static solution is obtained and the contact length is determined. This is then used as an initial configuration of the forced vibrations. The results which represent the static and dynamic responses of the beam for linear and non-linear cases are presented in the Figures.

1. INTRODUCTION

The response of an elastic beam supported by an elastic foundation has been studied by many investigators. Different foundation models such as Winkler, Pasternak, Vlasov, Filonenko–Borodich have been used in these studies [1]. But the Winkler model, in which the medium is taken as a system composed of infinitely close linear springs, is the simplest one and is often adopted. It is well known that, in this model, the foundation applies only a reaction normal to the beam which is proportional to the beam deflection. Non-linear Winkler foundation models, such as hyperbolic and cubic types have also been used for determining the behaviour of beams and plates [2–5]. In addition to this, in some studies, the non-linear load–deflection curve which represents the non-linear characteristics of the foundation has been approximated by a linearized curve [6, 7]. It has generally been assumed that the foundation reacts in compression as well as in tension in many of the studies on the static and dynamic response of an elastic beam on an elastic foundation. The response of beams supported by tensionless foundations is complicated by the need to determine the contact region. Because of this mathematical difficulty, the static and dynamic response of a beam on a tensionless foundation has received only limited attention.

Weitsman [8] studied the static behaviour of a beam resting on a tensionless Winkler foundation for a concentrated load and a uniformly distributed load. Weitsman [9], Rao [10] and Choros and Adams [11] studied a beam on an elastic foundation subjected to a moving load. Celep *et al.* [12] studied the dynamic response of a beam on a tensionless foundation subjected to a concentrated load, a uniformly distributed load and a concentrated moment. Yim and Chopra [13–15] studied the effects of foundation lift-off by idealizing the structure as a single degree of freedom system and presented a simplified

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earthquake analysis of structures by considering the foundation lift-off. In some studies, the behaviour of the circular and rectangular plates has been investigated for various loadings taking into account the tensionless character of the Winkler foundation [16–19]. Studies on the behaviour of the dynamic responses of beams or plates resting on non-linear Winkler foundations are fewer. Dumir *et al.* [20] and Bhaskar and Dumir [21] studied the non-linear vibrations of orthotropic circular and rectangular plates, respectively. In both of the preceding studies, the foundation has been assumed to be linear Winkler, Pasternak and non-linear Winkler foundations. Coşkun and Engin [22] studied the non-linear vibrations of an elastic beam resting on a non-linear tensionless Winkler foundation.

In this study, the static and dynamic responses of a free beam on a non-linear tensionless Winkler foundation are investigated. The position of the lift-off points (the length of the contact regions) and the vertical displacements are obtained for linear and non-linear cases respectively. The dependence of the contact regions on the external load in the non-linear case, and the effect of the non-linearity on the contact regions and the vertical displacements are presented in the Figures.

2. ANALYSIS OF THE PROBLEM

A homogeneous beam of length $2L$ on an elastic foundation subjected to a dynamic load $P_0(t)$ is given in Figure 1. The foundation is taken as a non-linear tensionless Winkler foundation with two parameters as follows:

$$k = k_1 w + k_3 w^3, \tag{1}$$

where k_1 and k_3 are the linear and non-linear foundation parameters respectively. The equation of motion for the beam shown in Figure 1 is

$$EIw_{xxxx} + \rho A w_{tt} = -(k_1 w + k_3 w^3)H(x,t) + P_0(t)\delta(x), \tag{2}$$

where EI is the beam rigidity, $w(x, t)$ the displacement function, ρ the mass density, A the crosssectional area, δ the Dirac delta function and $H(x, t)$ is an auxiliary function. $H(x, t)$ represents the tensionless character of the foundation and is defined as $H(x, t) = 1$ if $w(x, t) \geqslant 0$ and $H(x, t) = 0$ if $w(x, t) < 0$, where $w(x, t) = 0$ corresponds to the lift-off point of

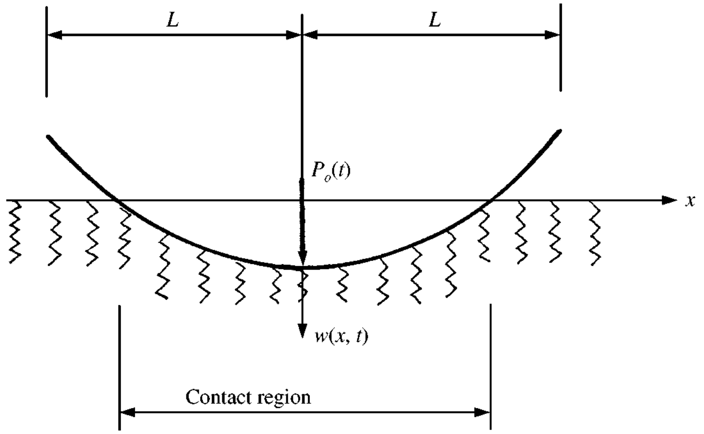

Figure 1. Beam on a non-linear tensionless Winkler foundation.

the beam. Since the geometrical and loading configurations of the problem are symmetric, the deflection of the beam is assumed to be symmetric as well. To obtain the dimensionless equation of motion, various non-dimensional parameters are introduced as

$$\eta = \frac{w}{L}, \quad \xi = \frac{x}{L}, \quad \tau = \frac{t}{L^2}\sqrt{EI/\rho A}, \quad k_l = \frac{k_1 L^4}{EI}, \quad k_n = \frac{k_3 L^6}{EI}, \quad P = \frac{P_0 L^2}{EI}. \tag{3}$$

By applying the non-dimensional parameters into equation (2), the non-dimensional equation of motion for the beam can be written as

$$\eta_{\xi\xi\xi\xi} + \eta_{\tau\tau} = -(k_l \eta + k_n \eta^3) H(\xi, \tau) + P(\tau)\delta(\xi). \tag{4}$$

Because of the random variation of $P(\tau)$, equation (4) may be solved by using various transform techniques like the Fourier and Laplace transforms. However, the inverse Fourier or the inverse Laplace transform of a function cannot be obtained in a closed form in some cases. Due to the difficulty in obtaining the exact solution of equation (4), an approximate solution is sought [12, 23] as

$$\eta(\xi, \tau) = T_1(\tau) + \sum_{n=1}^{\infty} W_n(\xi) T_{n+1}(\tau), \tag{5}$$

where T_1 and T_{n+1} are the time-dependent parts of the solution function, and $W_n(\xi)$ are the free vibration mode functions of a completely free beam which satisfy the boundary conditions of the present problem. The term T_1 corresponds to the rigid translation of the beam. Rigid rotation is not taken into account due to the symmetry. The mode functions $W_n(\xi)$ are the solutions of the differential equation

$$W_n^{iv} - \lambda_n^4 W_n = 0 \tag{6}$$

for the boundary conditions $W_{xx}(-1) = W_{xx}(1) = W_{xxx}(-1) = W_{xxx}(1) = 0$, which represent the flexural moment and the shear force for both ends of the completely free beam. λ_n are the solutions of the frequency equation

$$\cosh 2\lambda \cos 2\lambda = 1 \tag{7}$$

which has been obtained by using the boundary conditions in the solution of equation (6). Some of the roots of equation (7) which represent the dimensionless frequencies are $\lambda_1 = 2{\cdot}365020$, $\lambda_2 = 3{\cdot}926602$, $\lambda_3 = 5{\cdot}497803$, $\lambda_4 = 7{\cdot}068582$, $\lambda_5 = 8{\cdot}639379$. Thus, the mode functions used in equation (5) can be expressed as:

$$W_n(\xi) = \cos\lambda_n(1+\xi) + \cosh\lambda_n(1+\xi) - \frac{(\cosh 2\lambda_n - \cos 2\lambda_n)}{(\sinh 2\lambda_n - \sin 2\lambda_n)}$$
$$\times \{\sin\lambda_n(1+\xi) + \sinh\lambda_n(1+\xi)\}, \quad -1 \leqslant \xi \leqslant 1. \tag{8}$$

In studying the free vibrations of beams, the interest is in the natural frequencies and normal modes for various boundary conditions such as free–free, pinned–pinned, and so on, which constitute a set of homogeneous boundary conditions. These homogeneous boundary conditions which are written at the ends of the beam establish the orthogonality condition. Thus, all mode functions in equation (8) are orthogonal to each other in the interval

$(-1, +1)$. Substituting equation (5) into equation (4) and considering equation (6), the following differential equation can be obtained:

$$\ddot{T}_1(\tau) + \sum_{n=1}^{\infty} W_n(\xi)\ddot{T}_{n+1}(\tau) + \sum_{n=1}^{\infty} \lambda_n^4 W_n(\xi) T_{n+1}(\tau)$$

$$+ k_l\left[T_1(\tau)H(\xi,\tau) + \sum_{n=1}^{\infty} W_n(\xi)T_{n+1}(\tau)H(\xi,\tau)\right]$$

$$+ k_n\left[T_1(\tau) + \sum_{n=1}^{\infty} W_n(\xi)T_{n+1}(\tau)\right]^3 H(\xi,\tau) = P(\tau)\delta(\xi). \tag{9}$$

Multiplying equation (6) by $W_m(\xi)$, $m = 0, 1, 2,$ in which $W_0(\xi) = 1$, and using the orthogonality condition, the following system of differential equations is obtained in the matrix form:

$$\mathbf{A}\ddot{\mathbf{T}} + \mathbf{BT} = \mathbf{P} - \mathbf{Q}, \tag{10}$$

where **A** is a diagonal matrix, **B** is a symmetric one, **P** is the load vector and **Q** is a vector which represents the effect of non-linearity. The dots in equation (10) represent differentiations with respect to the dimensionless time τ. The matrix elements are defined as follows:

$$A_1 = 2, \quad A_{n+1} = \int_{-1}^{+1} W_n^2(\xi)\,\mathrm{d}\xi, \quad B_{1,1} = k_l \int_{-1}^{+1} H(\xi,\tau)\,\mathrm{d}\xi,$$

$$B_{1,n+1} = k_l \int_{-1}^{+1} W_n(\xi)H(\xi,\tau)\,\mathrm{d}\xi,$$

$$B_{n+1,m+1} = \lambda_n^4 A_{n+1}\delta_{nm} + k_l \int_{-1}^{+1} W_n(\xi)W_m(\xi)H(\xi,\tau)\,\mathrm{d}\xi, \quad P_1 = P(\tau), \quad P_{n+1} = P_n(\tau)W_n(0),$$

$$Q_1 = k_n \int_{-1}^{+1} (T_1^3 + 3T_1^2 W_m T_{m+1} + 3T_1 W_m W_n T_{m+1} T_{n+1}$$

$$+ W_m W_n W_p T_{m+1} T_{n+1} T_{p+1}) H(\xi,\tau)\,\mathrm{d}\xi,$$

$$Q_{s+1} = k_n \int_{-1}^{+1} (T_1^3 W_s + 3T_1^2 W_m W_s T_{m+1} + 3T_1 W_m W_n W_s T_{m+1} T_{n+1}$$

$$+ W_m W_n W_p W_s T_{m+1} T_{n+1} T_{p+1}) H(\xi,\tau)\,\mathrm{d}\xi, \; n, m, p, s \geqslant 1$$

$$\mathbf{T} = [T_1, T_2, T_3, \ldots]^{\mathrm{T}}. \tag{11}$$

3. NUMERICAL SOLUTION AND DISCUSSION

By assuming $\mathbf{T}$ = constant in equation (10), the unknown factors of $\mathbf{T}$ can be obtained as $\mathbf{T} = \mathbf{B}^{-1}(\mathbf{P} - \mathbf{Q})$ to represent the static behaviour of the beam. The linear static solution of the system is obtained by taking $k_n = 0$, initially. Since the contact length is not

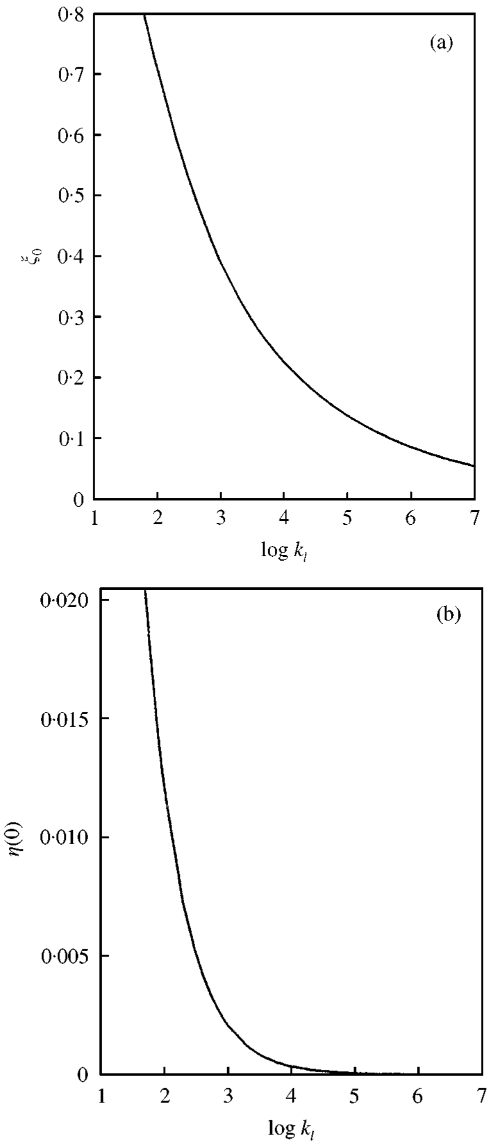

Figure 2. (a) Lift-off points ξ_0 of the beam versus linear foundation parameter k_l for non-linear foundation parameter $k_n = 0$ and dimensionless load $P = 1$. (b) Middle displacements $\eta(0)$ of the beam versus linear foundation parameter k_l for non-linear foundation parameter $k_n = 0$ and dimensionless load $P = 1$.

known in advance, the solution is sought by an iterative scheme. Therefore, the integrals in equation (11) are evaluated at each step and then equation (10) is solved by using the Gauss–Jordan method and the Newton–Raphson technique. After the linear solution is completed, the non-linear solution is carried out by taking k_n into account. These solutions provide the position of the lift-off points ξ_0 (contact lengths) and the vertical displacements of the beam for the linear and non-linear cases, respectively. The variation of the position of the lift-off points and the middle displacements for $k_n = 0$ are given in Figures 2(a) and 2(b) respectively. As it is known from Weitsman [8], the positions of the contact points do not depend on the magnitude of the load. However, contact points vary with the foundation

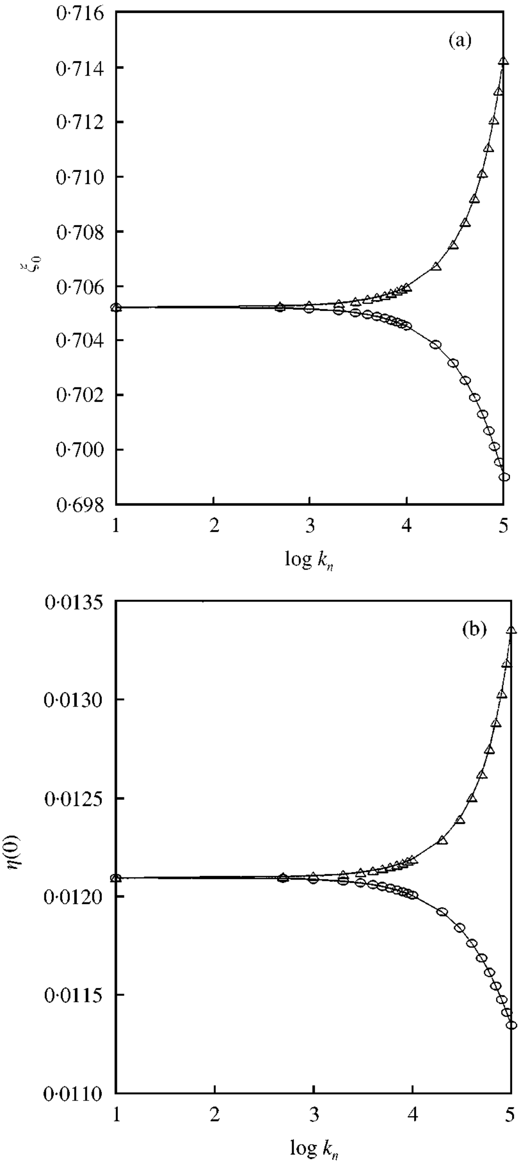

Figure 3. (a) Lift-off points ξ_0 of the beam versus non-linear foundation parameter k_n for linear foundation parameter $k_l = 100$ and dimensionless load $P = 1$. —○—, $k_n > 0$; —△—, $k_n < 0$. (b) Middle displacements $\eta(0)$ of the beam versus non-linear foundation parameter k_n for linear foundation parameter $k_l = 100$ and dimensionless load $P = 1$. —○—, $k_n > 0$; —△—, $k_n < 0$.

rigidity. When the foundation becomes stiffer, the contact length and the vertical displacements decrease as shown in Figures 2(a) and 2(b). This is because of the fact that the contact length and the vertical displacements must be small in order to resist the same external load as the foundation becomes stiffer. Full contact develops for very low foundation stiffnesses and the contact occurs only at a point under the load for a rigid foundation.

The variation of the contact points and the middle displacement of the beam are given in Figures 3(a) and 3(b). When positive non-linearity is assumed ($k_n > 0$), the contact lengths

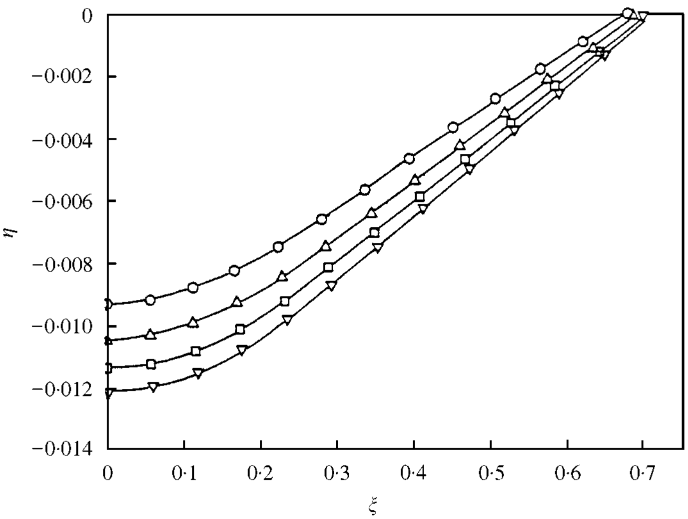

Figure 4. Elastic curves of the beam for $k_l = 100$, $P = 1$ and various values of non-linear foundation parameter k_n. —▽—, $k_n = 0$; —□—, $k_n = 100\,000$; —△—, $k_n = 300\,000$; —○—, $k_n = 700\,000$.

and the middle displacements decrease with respect to the linear case. However, these increase if negative non-linearity is taken into account. In this case it is found that the system behaves like a lumped-mass system with a non-linear spring coefficient which can be of softening or hardening type. This was as expected due to the fact that the foundation is more rigid when $k_n > 0$ and is less rigid when $k_n < 0$. In the solution of the problem, the vertical equilibrium of the beam is controlled at every step by considering

$$P = \int_{-1}^{+1} [k_l W(\xi) + k_n W^3(\xi)] H(\xi)\,\mathrm{d}\xi. \tag{12}$$

The increase or decrease in the contact lengths and the vertical displacements due to the non-linear foundation parameter k_n can be expressed by considering the vertical equilibrium of the beam. The elastic curves of the beam are presented in Figure 4. The effect of the non-linearity on the displacements and the contact length can clearly be seen in this Figure. The variation of the middle displacements and the contact points with respect to the load for positive and negative non-linearities are shown in Figures 5(a) and 5(b) respectively. The vertical displacements are exactly proportional to the load and the lift-off points do not depend on the load in the linear case [8]. However, as is seen from the Figures, the displacements do not vary linearly with the load and the contact lengths depend on the magnitude of the load due to the effect of the non-linearity. It is noted that the effect of the non-linear foundation on the contact length is very small as can be seen in Figures 3(a) and 5(b).

Equation (10) is a system of non-linear differential equations of second order, since the lift-off points are not known in advance and the presence of the non-linear foundation parameter k_n. By assuming $\mathbf{T} \neq$ constant, equation (10) is solved for $k_n = 0$, initially. The external load is considered as an impulsive type $P(\tau) = \mathrm{e}^{-|\tau - \tau_1|}$ and the solution is carried out by employing the Runge–Kutta method. An acceptable accuracy for the numerical results is obtained by considering three mode functions in addition to the rigid translation. While using the numerical procedure, the lift-off points are calculated from the condition of

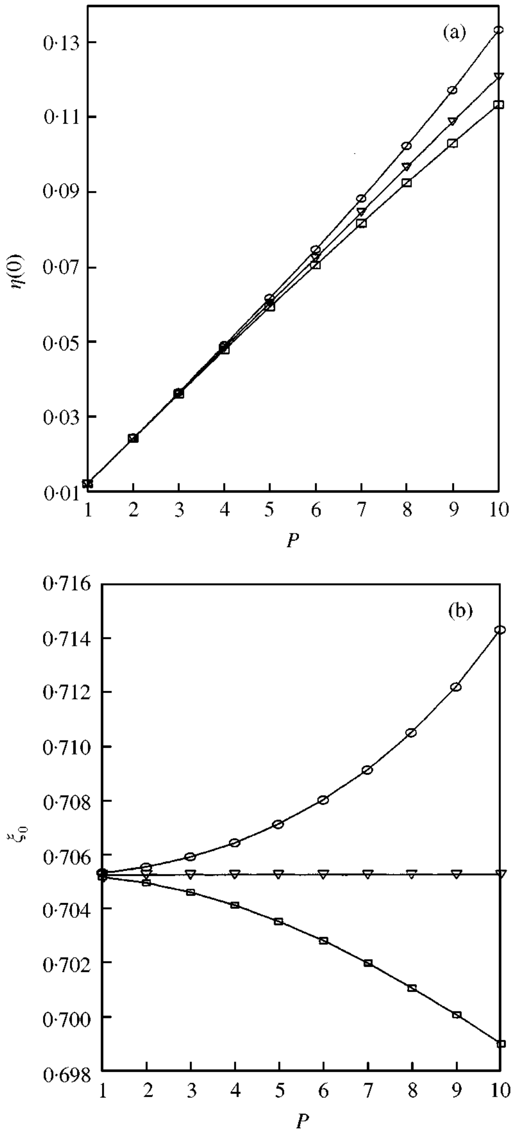

Figure 5. (a) Middle displacements $\eta(0)$ of the beam versus dimensionless load P for $k_l = 100$ and $k_n = 1000$. —▽—, $k_n = 0$; —□—, $k_n > 0$; —○—, $k_n < 0$. (b) Lift-off points ξ_0 of the beam versus dimensionless load P for $k_l = 100$ and $k_n = 1000$. —▽—, $k_n = 0$; —□—, $k_n > 0$; —○—, $k_n < 0$.

$\eta(\xi, \tau) = 0$ at each step for the time intervals of $\Delta\tau$. Control terms are set wherever it is necessary for this purpose. After the solution is completed for $k_n = 0$, the solution including the parameter k_n is carried out by employing an iterative procedure. The variation of the lift-off points with respect to the dimensionless time is given in Figure 6. The contact length increases with the dimensionless time but decreases as the foundation becomes stiffer. This is an expected result if one considers the vertical equilibrium of the beam. Figures 7(a) and 7(b) show the time variations of the middle displacements of the beam for the linear and non-linear cases respectively. The middle displacements increase with the decrease of the foundation rigidity for the linear case as it is seen in Figure 7(a). However, the middle displacements decrease with the increase of the non-linearity effect (Figure 7(b)). For

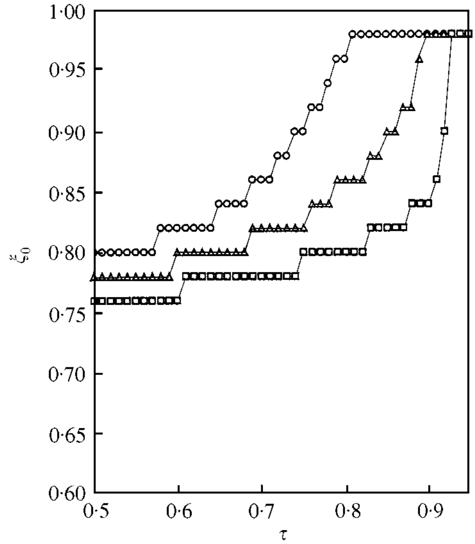

Figure 6. Lift-off points ξ_0 of the beam versus dimensionless time τ for dimensionless load $P = 10 \times e^{-|\tau - 0\cdot5| \times 10}$ at various values of k_l. —○—, $k_l = 1$; —△—, $k_l = 50$; —□—, $k_l = 100$.

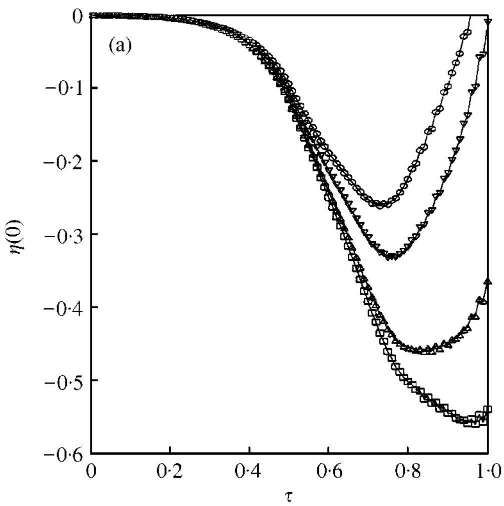

Figure 7. (a) Middle displacements $\eta(0)$ of the beam versus dimensionless time τ for dimensionless load $P = 10 \times e^{-|\tau - 0\cdot5| \times 10}$ at various values of k_l. —□—, $k_l = 1$; —△—, $k_l = 10$; —▽—, $k_l = 50$; —○—, $k_l = 100$. (b) Middle displacements $\eta(0)$ of the beam versus dimensionless time τ for $k_l = 10$ and $P = 10 \times e^{-|\tau - 0\cdot5| \times 10}$ at various values of k_n. —+—, $k_n = 0$; —□—, $k_n = 10$; —△—, $k_n = 100$; —○—, $k_n = 100$.

this case, it can be concluded that the non-linearity affects the middle displacements as in the static case.

4. CONCLUSIONS

The non-linear vibrations of an elastic beam resting on a non-linear tensionless Winkler foundation were studied by employing the Galerkin method. The static as well as the

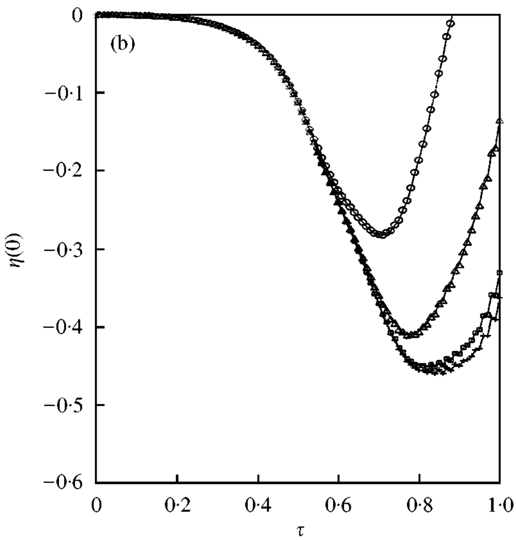

Fig. 7. Continued.

dynamic analyses of the problem become non-linear due to the presence of the foundation lift-off and the nonlinear term in the foundation model. These two factors affect the response of the beam by changing the vertical displacements and the contact length of the beam for both static and dynamic cases. It is found that the contact length of the beam depends on the magnitude of the load due to the effect of the non-linear term in the foundation model, whether it is positive or negative. It is also observed that the effect of the non-linear foundation on the contact length is very small.

ACKNOWLEDGMENTS

The author wishes to thank Prof. Dr. H. Engin from the İstanbul Technical University, for his helpful discussions, and the reviewers for their kind and useful comments.

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