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VIBRATION AND STABILITY ANALYSIS OF NON-UNIFORM TIMOSHENKO BEAMS UNDER AXIAL AND DISTRIBUTED TANGENTIAL LOADS

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Two sets of governing equations for transverse vibration of non-uniform Timoshenko beam subjected to both axial and tangential loads have been presented. In the first set, the axial and tangential loads were taken perpendicular to the shearing force, i.e., normal to the cross-section inclined at an angle ψ, while in the second set, the axial force is assumed to be tangential to the axis of the beam-column. For each case, there exist a pair of differential equations coupled in terms of the flexural displacement and the angle of rotation due to bending. The two coupled second order governing differential equations were combined into one fourth order ordinary differential equation with variable coefficients. The parameters of the frequency equation were determined for different boundary conditions. The exact fundamental solutions could be found by expressing the coefficients of the reduced differential equation in a polynomial form before applying the Frobenius method. Several illustrative examples of uniform and non-uniform beams with various boundary conditions such as clamped supported, elastically supported, and free end mass and pinned end mass, have been presented. The stability analysis, for the variation of the natural frequencies of the uniform and non-uniform beams with the axial force, has also been investigated. Moreover, the present work illustrates the frequency behavior of the beam under a tangential load.

1. INTRODUCTION

Present-day structures need to be lightweight but have high strength. This trend has created many vibration-induced problems such as material fatigue, noise transmission and even human discomfort. Moreover, for any complicated mechanical system which has a number of resonant frequencies, it is a difficult task to eliminate all the resonating frequencies. Therefore, the determination of these frequencies and studying the behavior of structures at these resonating frequencies is important from the point of reducing the dynamic stresses and/or amplitudes.

Beams have been used for various purposes for many structures and hence the vibration behavior of beams has a great importance in many engineering applications such as in the design of machines and structures. The fundamental vibration behavior of long slender cylindrical or prismatic beams can be investigated using the classical Euler–Bernoulli beam theory. Attempting to use this theory for studying either short beams or vibrations at higher modes can lead to a significant over-prediction of natural frequencies, since the effect of both transverse shear deformation and rotatory inertia has been ignored and, therefore, it cannot provide sufficient explanations of vibration characteristics at higher modes.

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When the ratio of the radius of gyration of the cross-section of the beam to its length is very small, say in the order of 10^{-4}, the frequency values coincide with those found from the Bernoulli–Euler classical theory. Lord Rayleigh improved the classical theory by considering the effect of rotary inertia of the cross-section. Later, Timoshenko [1, 2] developed a theory which allows one to study the vibrational behavior of either shorter beams or higher vibrational modes by approximately accounting for both the transverse shear deformation and the rotary inertia, when the cross-section remains undisturbed and the axial stresses are assumed to be zero. Moreover, Timoshenko showed that the effect of the rotatory inertia is typically less important than the shear deformation. The resulting model is characterized by two coupled differential equations in terms of two dependent variables, being the transverse deflection of the neutral axis and the rotation of the cross-section measured about the neutral axis. Furthermore, the model requires the exact value of a well-known shear-correction factor κ, being defined as the ratio of the averaged shear strain within the cross-section to the shear strain at the section centroid.

Non-uniform beams in which the cross-section is varying either in a continuous or non-continuous manner along their lengths have many structural applications. It is desirable to achieve an optimum distribution of strength and weight and in some cases to satisfy special architectural and functional requirements. Therefore, determination of natural frequencies of such beams has been the subject of interest for many mechanical, aeronautical, and structural engineers.

The influence of axial or in-plane loading on the free vibration frequencies of elastic structures such as beams, plates and cylindrical shells has been investigated before. Tension tends to increase the free-vibration frequencies, while on the other hand, compression tends to reduce them. Since buckling occurs when the lowest frequency decreases to zero, it may be possible to estimate buckling loads non-destructively by first measuring the frequencies at several load levels and then extrapolating the results.

In the recent decades, due to the important usage of beams in industry, many scientists and engineers have published papers in the field of static, free and forced vibrational analysis of beams with different geometry, boundary and loading conditions. Maurizi *et al.* [3] presented a study on the free vibration of a uniform Timoshenko beam when elastically restrained against rotation and translation and showed how frequencies are influenced by the variations in the end-restrained parameters. Modal analysis was adopted to obtain the deterministic and random vibration response of a uniform, mass-loaded, hysteretically damped beam model which simulates a robotic arm [4]. At the left end of the beam, both rotational and translational springs are attached, and the offset and rotary effects of the tip mass at the right free end are considered. Rossi and Laura [5] presented the exact, analytical solution of the free vibration of Timoshenko beams carrying elastically mounted masses, and obtained an independent solution by means of a finite element code in order to ascertain the validity and accuracy of the results predicted from the exact solution.

A number of papers have been published on the vibration of Timoshenko beams subjected to moving loads and beams with varying thickness [6–8]. Irie *et al.* [9] studied the steady state response of an internally damped Timoshenko beam with varying cross-section to a sinusoidally varying point force using the spline interpolation technique. The natural frequencies of transverse vibrations of a Timoshenko beam of non-uniform thickness clamped at one end and carrying a concentrated mass at the other end was determined for two types of structural configurations: (1) discontinuous variation of thickness, (2) continuous linear variation [10]. Lee and Kuo [11, 12] performed the statical and dynamical analysis of an elastically restrained non-uniform Bernoulli–Euler beam. The exact solution for the problem governed by a general self-adjoint fourth order ordinary differential equation with arbitrarily polynomial varying coefficients were derived in

Green's function form and concisely expressed in terms of four normalized fundamental solutions of the system. Lee and Lin [13] presented the exact solution for the free vibration of a symmetric non-uniform Timoshenko beam with a tip mass at one end and elastically restrained at the other end. They have showed that the exact fundamental solutions can be obtained by the method of Frobenius.

Many investigations have been concentrated on the vibrational behavior of column beams under axial loads. The effect of a constant axial compression load on the natural frequencies and mode shapes of a uniform single-span beam, with different combinations of end conditions has been studied by Bokaian [14]. Farghaly [15] presented closed-form, exact frequency and mode shape solutions for a uniform cantilever Euler–Bernoulli beam with an elastically mounted end mass under axial load concentrated either at the point of attachment between the beam and the end mass or at the center of gravity of that mass. The same problem but for the Timoshenko beam, based on two different sets of governing equations, has been studied by Farghaly and Shebl [16]. They showed that the difference between the natural frequencies is significant for beams subjected to larger loads and at higher modes of vibration. Esmailzadeh [6] studied the vibration of a Timoshenko beam subjected to a moving mass, and Lee [17] analyzed the dynamic response of a rotating shaft when being subjected to an axial force and a moving load by using the assumed mode method and the Timoshenko beam theory.

In the present study, two sets of governing equations for the transverse vibration of a non-uniform Timoshenko beam subjected to axial and tangential loads have been derived. In the first set, the axial and tangential loads have been taken normal to the shearing force and thus normal to the cross-section inclined at an angle ψ. In the other set, the axial force is tangential to the axis of beam-column. For each case, there exists a pair of coupled differential equations in terms of flexural displacement and angle of rotation due to bending. The two coupled governing characteristic differential equations are reduced into one fourth order ordinary differential equation with variable coefficients in terms of the angle of rotation due to bending. After substituting the homogeneous solution into the associated boundary conditions, the corresponding parameters of the frequency equation were determined for different boundary conditions. The exact fundamental solutions can be determined by expressing the coefficients of the reduced differential equation in the form of a polynomial and then applying the Frobenius method. Several illustrative examples of uniform and non-uniform beams with various boundary conditions, i.e., clamped supported, elastically supported, free end mass and pinned end mass, have been presented. Furthermore, convergence of solutions and comparison between the results of the two sets of equations have also been studied. For stability analysis, the variation of the natural frequencies of the uniform and non-uniform beams with the axial force have also been investigated. The present work illustrates the frequency behavior of a beam under tangential load.

2. EQUATIONS OF MOTION

The equations of motion of the Timoshenko beam under axial loading can be obtained on the basis of three approaches: namely; the principle of virtual work, Hamilton's stationary principle and the classical dynamic equilibrium method. The derivation of these equations is based on the assumption that the shearing force acts on the cross-sections which are normal to the deflected axis of the beam-column and therefore inclined at an angle ψ with respect to the vertical direction. Two sets of equations of motion could be derived for the following two cases: (1) the axial force is taken normal to the shearing force and thus normal to the cross-section inclined at angle ψ; (2) the axial force is taken tangential to the axis of the beam-column and thus is not normal to the direction of shear force.

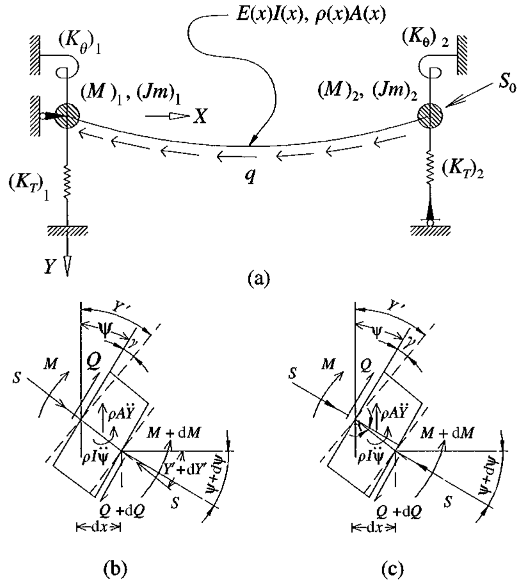

Figure 1. (a) The Timoshenko beam system under study; (b) free body diagram of an element dx—axial force tangential to the axis of the beam column; (c) free body diagram of an element dx—axial force normal to the shearing force.

Referring to Figure 1, it can be easily shown that the governing equations for vibration and stability of a non-uniform Timoshenko beam for the two cases mentioned above are as follows.

Case 1: *Axial force normal to the shearing force*,

$$\rho A \frac{\partial^2 Y}{\partial t^2} - \frac{\partial}{\partial x}\left(K'AG\left(\frac{\partial Y}{\partial x} - \psi\right)\right) + S\frac{\partial \psi}{\partial x} = 0,$$
$$\rho I \frac{\partial^2 \psi}{\partial t^2} - (K'AG + S)\left(\frac{\partial Y}{\partial x} - \psi\right) - \frac{\partial}{\partial x}\left(EI\frac{\partial \psi}{\partial x}\right) = 0. \quad (1)$$

Case 2: *Axial force tangential to the axis of beam-column*,

$$\rho A \frac{\partial^2 Y}{\partial t^2} - \frac{\partial}{\partial x}\left(K'AG\left(\frac{\partial Y}{\partial x} - \psi\right)\right) + \frac{\partial}{\partial x}\left(S\frac{\partial Y}{\partial x}\right) = 0,$$
$$\rho I \frac{\partial^2 \psi}{\partial t^2} - K'AG\left(\frac{\partial Y}{\partial x} - \psi\right) - \frac{\partial}{\partial x}\left(EI\frac{\partial \psi}{\partial x}\right) = 0. \quad (2)$$

A list of nomenclature is given in the Appendix. For harmonic vibration ($\partial^2 y/\partial t^2 = -\omega^2 y$), the dimensionless governing characteristic equations of motion are as follows:

Case 1.

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{q}{\delta}\left(\frac{\mathrm{d}y}{\mathrm{d}\zeta} - \psi\right)\right] + s\Omega^2 y - \frac{p}{\delta}\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right) + \left(\frac{p+q}{\delta}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}\zeta} - \psi\right) + v\eta\Omega^2\psi = 0. \quad (3)$$

Case 2:

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{q}{\delta}\left(\frac{\mathrm{d}y}{\mathrm{d}\zeta}-\psi\right)\right]+s\Omega^2 y-\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{p}{\delta}\frac{\mathrm{d}y}{\mathrm{d}\zeta}\right)=0,$$

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)+\frac{q}{\delta}\left(\frac{\mathrm{d}y}{\mathrm{d}\zeta}-\psi\right)+v\eta\Omega^2\psi=0. \tag{4}$$

By differentiating the second equation of each set and then combining it with the first equation, the relationship between the dimensionless flexural displacement and the angle of rotation due to bending can be obtained. Replacing the above-mentioned relations back into the second equation of relations (3) and (4), the governing characteristic differential equation of motion in terms of the angle of rotation will be obtained. The resulting fourth order differential equations with variable coefficients are as follows.

Case 1:

$$y=\frac{1}{s\Omega^2}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{q}{p+q}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)+\frac{q}{p+q}v\eta\Omega^2\psi\right)+\frac{p}{\delta}\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right],$$

$$(p+q)\frac{\mathrm{d}}{\mathrm{d}\zeta}\left\{\frac{1}{s\Omega^2}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{q}{p+q}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)+\frac{q}{p+q}v\eta\Omega^2\psi\right)+\frac{p}{\delta}\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right]\right\}+\delta\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right) \tag{5}$$

$$+(\delta v\eta\Omega^2-p-q)\psi=0.$$

Case 2.

$$y=\frac{1}{s\Omega^2}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{q-p}{q}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)+\left(\frac{q-p}{q}v\eta\Omega^2+\frac{p}{\delta}\right)\psi\right)\right],$$

$$q\frac{\mathrm{d}}{\mathrm{d}\zeta}\left\{\frac{1}{s\Omega^2}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{q-p}{q}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)+\left(\frac{q-p}{q}v\eta\Omega^2+\frac{p}{\delta}\right)\psi\right)\right]\right\}+\delta\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right) \tag{6}$$

$$+(\delta v\eta\Omega^2-q)\psi=0.$$

3. BOUNDARY CONDITIONS AND FREQUENCY EQUATION

The equations for the various end conditions, corresponding to equation (5), may be expressed in non-dimensional form as follows. For the clamped support:

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{q}{p+q}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)+\frac{q}{p+q}v\eta\Omega^2\psi\right)+\frac{p}{\delta}\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}=0 \quad \text{and} \quad \psi=0 \quad \text{for either } \zeta=0 \text{ or } 1, \tag{7}$$

while for the elastic support:

$$\frac{\beta_{\mathrm{T}}}{s\Omega^2}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{q}{p+q}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)+\frac{q}{p+q}v\eta\Omega^2\psi\right)+\frac{p}{\delta}\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right]=\frac{q}{p+q}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)+v\eta\Omega^2\psi\right]$$

$$\text{and} \quad \beta_\theta\psi=r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} \quad \text{for either } \zeta=0 \text{ or } 1. \tag{8}$$

However, for the free end mass one can write

$$\frac{\gamma}{s}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{q}{p+q}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)+\frac{q}{p+q}\nu\eta\Omega^2\psi\right)+\frac{p}{\delta}\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right]=-\frac{q}{p+q}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)+\nu\eta\Omega^2\psi\right]$$

$$\text{and } \alpha\Omega^2\psi = r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} \quad \text{for either } \zeta = 0 \text{ or } 1, \tag{9}$$

while for the pinned end mass:

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{q}{p+q}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)+\frac{q}{p+q}\nu\eta\Omega^2\psi\right)+\frac{p}{\delta}\frac{\mathrm{d}\psi}{\mathrm{d}\delta}=0 \quad \text{and} \quad r\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}=0 \quad \text{for either } \zeta=0 \text{ or } 1. \tag{10}$$

Similar relations could be written for case 2 in a similar fashion.

The general solution of the differential equation can be expected as

$$V(\zeta) = C_1V_1(\zeta) + C_2V_2(\zeta) + C_3V_3(\zeta) + C_4V_4(\zeta), \tag{11}$$

where the $V_j(\zeta)$, $j = 1, 2, 3, 4$, are the four linearly independent fundamental solutions and the C_j are the four pairs of arbitrary constants which have to be defined by imposing the boundary conditions. Fundamental solutions must satisfy the following normalization condition at the origin of the co-ordinate system:

$$\begin{bmatrix} V_1(0) & V_2(0) & V_3(0) & V_4(0) \\ V_1'(0) & V_2'(0) & V_3'(0) & V_4'(0) \\ V_1''(0) & V_2''(0) & V_3''(0) & V_4''(0) \\ V_1'''(0) & V_2'''(0) & V_3'''(0) & V_4'''(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

Note that the prime indicates differentiation with respect to the dimensionless spatial variable ζ. After substituting the general solution into the boundary conditions, a set of linear, homogeneous equations will be deduced. One obtains the frequency equation of the system if the corresponding determinant vanishes identically:

$$\begin{aligned}\Pi = &-(g_1F_3H_1 + g_4F_4H_3 + g_2F_1H_4 - g_4F_3H_4 - g_2F_4H_1 - g_1F_1H_3) \\ &+ g_5(g_1F_3H_2 + g_3F_4H_3 + g_2F_2H_4 - g_3F_3H_4 - g_2F_4H_2 - g_1F_2H_3).\end{aligned} \tag{13}$$

For a few conventional boundary conditions, parameters g_i with $i = 1, \ldots, 10$, and F_i and H_i with $i = 1, 2, 3, 4$ were evaluated and is presented in Table 1.

4. EXACT ANALYTICAL SOLUTION

In general, the closed-form fundamental solutions of a fourth order differential equation with variable coefficients are not available. If the coefficients of the reduced differential equation can be expressed in a polynomial form, then the exact fundamental solutions could be found using the method of Frobenius [13].

TABLE 1

Parameters of frequency equation for different boundary conditions

	Clamped supported–free end mass	Elastically supported–free end mass	Free end mass–Free end mass	Pinned end mass–Pinned end mass
g_1	$HT(0)$	$\beta_T HT(0)$	$\gamma HT(0)$	$HT(0)$
g_2	$HT'(0) + 2r'(0)HT(0)$	$\beta_T(HT'(0) + 2r'(0)HT(0)) + \Omega^2 HT(0)$	$(2\gamma r'(0) + 1)HT(0) + \gamma HT'(0)$	$HT'(0) + 2r'(0)HT(0)$
g_3	$\beta_T(p(0)/\delta + HT'(0)r'(0) + HT(0)(\eta\Omega^2 + r''(0)))$	$\beta_T(HT'(0) + 2r'(0)HT(0)) + \Omega^2 HT(0)$	$\gamma(p(0)/\delta + HT'(0)r'(0) + HT(0)(\eta\Omega^2 + r''(0)))$	$(p(0)/\delta + HT'(0)r'(0) + HT(0)(\eta\Omega^2 + r''(0)))$
g_4	$\Omega^2\eta(\beta_T HT'(0) + HT(0)\beta_T \mathrm{v}'(0))$	$\Omega^2(\beta_T \eta HT'(0) + HT(0)\eta(\Omega^2 + \beta_T \mathrm{v}'(0)) + p(0)/\delta)$	$\Omega^2\eta(\gamma HT'(0) + HT(0)(1 + \gamma \mathrm{v}'(0)) + p(0)/\delta$	$\Omega^2\eta(HT'(0) + HT(0)\mathrm{v}'(0))$
g_5	$-\beta_\theta$	$-\beta_\theta$	$-\alpha\Omega^2$	$-\alpha\Omega^2$
g_6	$-\alpha\Omega^2/r(1)$	$-\alpha\Omega^2/r(1)$	$-\alpha\Omega^2/r(1)$	$-\alpha\Omega^2/r(1)$
g_7	$\gamma HT(1)r(1)$	$\gamma HT(1)r(1)$	$\gamma HT(1)r(1)$	$\gamma HT(1)r(1)$
g_8	$HT(1)(2r'(1)\gamma + r(1)s(1)) + \gamma r(1) HT'(1)$	$HT(1)(2r'(1)\gamma + r(1)s(1)) + \gamma r(1)HT'(1)$	$HT(1)(2r'(1)\gamma r(1)s(1)) + \gamma r(1)HT'(1)$	$HT(1)\,2r'(1) + r(1)HT'(1)$
g_9	$HT(1)(\gamma(r''(1) + \eta\Omega^2 \mathrm{v}(1)) + r'(1)s(1)) + \gamma HT'(1) r'(1) + \gamma p(1)/\delta$	$HT(1)(\gamma(r''(1) + \eta\Omega^2\mathrm{v}(1)) + r'(1)s(1)) + \gamma HT'(1) r'(1) + \gamma p(1)/\delta$	$HT(1)(\gamma(r''(1) + \eta\Omega^2\mathrm{v}(1)) + r'(1)s(1)) + \gamma HT'(1) r'(1) + \gamma p(1)/\delta$	$HT(1)(r''(1) + \eta\Omega^2\mathrm{v}(1)) + HT'(1)r'(1) + p(1)/\delta$
g_{10}	$HT(1)\eta\Omega^2(\mathrm{s}(1)\mathrm{v}(1) + \gamma\mathrm{v}'(1)) + \gamma\eta\Omega^2\mathrm{v}(1)HT'(1) + p(1)s(1)/\delta$	$HT(1)\eta\Omega^2(s(1)\mathrm{v}(1) + \gamma\mathrm{v}'(1)) + \gamma\eta\Omega^2\mathrm{v}(1)HT'(1) + p(1)s(1)/\delta$	$HT(1)\eta\Omega^2(s(1)\mathrm{v}(1) + \gamma\mathrm{v}'(1)) + \gamma\eta\Omega^2\mathrm{v}(1)HT'(1) + p(1)s(1)/\delta$	$HT(1)\eta\Omega^2\mathrm{v}'(1) + \eta\Omega^2\mathrm{v}(1) HT'(1)$

Note: $HT = q/(p+q)$, $F_j = V_j'(1) + g_6 V_j(1)$, $H_j = g_7 V_j'''(1) + g_8 V_j''(1) + g_9 V_j'(1) + g_{10} V_j(1)$, $j = 1, 2, 3, 4$.

The general form of the fourth order ordinary differential equation with variable coefficients is

$$p_4(\zeta)V''''(\zeta) + p_3(\zeta)V'''(\zeta) + p_2(\zeta)V''(\zeta) + p_1(\zeta)V'(\zeta) + p_0(\zeta)V(\zeta) = 0. \tag{14}$$

Most often, it is possible to have a power series representation of the variable coefficients:

$$p_4(\zeta) = \sum_{j=0}^{n_4} a_j\zeta^j, \qquad p_3(\zeta) = \sum_{j=0}^{n_3} b_j\zeta^j, \qquad p_2(\zeta) = \sum_{j=0}^{n_2} c_j\zeta^j, \qquad p_1(\zeta) = \sum_{j=0}^{n_1} d_j\zeta^j, \quad \text{and}$$

$$p_0(\zeta) = \sum_{j=0}^{n_0} e_j\zeta^j. \tag{15}$$

Also, one can assume that the four fundamental solutions of equation (14) can be written in the form

$$V_i = \sum_{j=0}^{\infty} k_{i,j}\zeta^j, \quad i = 1, 2, 3, 4. \tag{16}$$

The recurrence formula

$$k_{i,z+4} = \frac{1}{(z+4)(z+3)(z+2)(z+1)a_0}\left[\sum_{j=0}^{z} e_j k_{i,z-j} + \sum_{j=0}^{z} (z-j+1)d_j k_{i,z-j+1} + \sum_{j=0}^{z} (z-j+2)(z-j+1)c_j k_{i,z-j+2} + \sum_{j=0}^{z} (z-j+3)(z-j+2)(z-j+1)b_j k_{i,z-j+3} + \sum_{j=1}^{z} (z-j+4)(z-j+3)(z-j+2)(z-j+1)a_j k_{i,z-j+4}\right],$$

$$i = 1, 2, 3, 4, \quad z = 0, 1, 2, \ldots, \tag{17}$$

can be obtained by replacing equation (16) into equation (14) and collecting the coefficients of powers of ζ. By using equation (17) and with due attention to the normalization condition (equation (12)), we evaluate the coefficients $k_{i,j}$. The natural frequencies will be obtained by substituting the fundamental solutions into the associated frequency equation.

5. RESULTS AND DISCUSSION

The Maple V software has been used for all the computational processes in this work. The results and a brief discussion of some examples with the verification problems are as follows.

5.1. ILLUSTRATIVE EXAMPLE 1

The dependent of the accuracy of results to the number of terms of power series taken is shown in Table 2. The material properties of the Timoshenko beam are taken as constant. The beam is assumed to have a linearly varying thickness with constant width. Therefore,

TABLE 2

First four frequencies of a linearly varying thickness, clamped Timoshenko beam with tip mass end. $\eta = 0{\cdot}0016$, $\mu = 0{\cdot}6$, $\gamma = 0{\cdot}54$, $\lambda = -0{\cdot}2$, $\alpha = 0$

		$N = 15$	$N = 20$	$N = 25$	$N = 30$	$N = 35$	$N = 40$
$p = 0$	Ω_1	1·85	1·85				
	Ω_2	14·44	14·44				
	Ω_3	34·08	39·87	40·07	40·07		
	Ω_4	63·57	62·07	71·83	74·22	74·24	74·24
$p = 0{\cdot}006$	Ω_1	1·18	1·18				
Case 1 — Eq. (5)	Ω_2	13·74	13·74				
	Ω_3	33·26	39·14	39·36	39·36		
	Ω_4	64·26	62·04	41·24	73·49	73·51	73·51
$p = 0{\cdot}006$	Ω_1	1·18	1·18				
Case 2 — Eq. (6)	Ω_2	13·73	13·73				
	Ω_3	33·24	39·11	39·33	39·33		
	Ω_4	64·18	61·99	71·16	73·41	73·43	73·43

$q(\zeta) = s(\zeta) = (1 + \lambda\zeta)$, $v(\zeta) = r(\zeta) = (1 + \lambda\zeta)^3$. The boundary conditions are taken as clamped at one end and a lumped mass at the other. The axial force acts at the point of attachment of the tip mass, and the dimensionless parameters of the beam are $\eta = 0{\cdot}0016$, $\mu = 0{\cdot}6$, $\gamma = 0{\cdot}54$, $\lambda = -0{\cdot}2$, $\alpha = 0$.

Results obtained for the case $p = 0$ are exactly the same as those presented by Lee and Lin [13], which verifies the developed computer program for the non-uniform beams. It can be seen that the axial compressive force would decrease all the natural frequencies of the beam. Table 2 shows that the results obtained from the two systems are the same only for lower modes of vibration. However, at higher modes the differences grow rapidly. Moreover, it can be seen that for a convergent solution, especially at higher modes, the minimum number of terms for the power series, N, must be 35.

Variation of the first natural frequency of the beam with the axial load for two different values of the tip mass is plotted in Figure 2. As expected, the frequencies of the beam with the mass ratio $\mu = 0{\cdot}6$, is lower than for the case of zero mass ($\mu = 0$). Moreover, the critical buckling load is independent of the tip mass value since the axial force acts at the point of the attachment.

5.2. ILLUSTRATIVE EXAMPLE 2

Table 3 shows the first four natural frequencies for the uniform and non-uniform clamped–free Timoshenko beam under the action of an axial load. For recovery of the previous results, dimensionless parameters of the uniform beam were selected in the form of $\eta = 0{\cdot}01$, $\mu = \gamma = \alpha = 0$, $\delta = 0{\cdot}03$, $\beta_T = 100$, $\beta_\theta = 2$ which are the same as those presented in Table 1 of reference [16]. The comparison of the results presented in Table 3 with the corresponding values taken from reference [16] shows good agreement and, therefore, verifies both the formulation and the developed computer program. Results for the case of a constant width, and a second order varying thickness Timoshenko beam ($q(\zeta) = s(\zeta) = (1 + 0{\cdot}2\zeta^2)$, $v(\zeta) = r(\zeta) = (1 + 0{\cdot}2\zeta^2)^2$) are presented in Table 3. The other parameters of the beam are taken the same as that of the uniform beam.

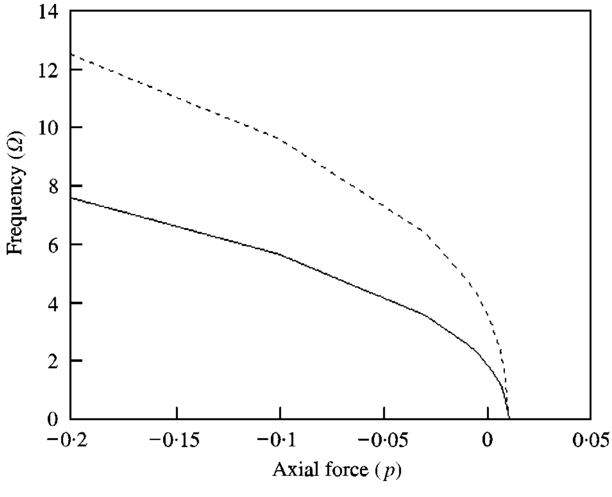

Figure 2. Variation of the natural frequency versus the axial load. Cantilever Timoshenko beam with constant width and linearly varying thickness; $\eta = 0{\cdot}0016$, $\gamma = 0{\cdot}54$, $\lambda = -0{\cdot}2$, $\alpha = 0$, $\mu = 0{\cdot}6$ and $\mu = 0$; ——, mass ratio $\mu = 0{\cdot}6$; - - -, mass ratio $\mu = 0$.

TABLE 3

Comparison between the first four frequencies of the two sets of equations, clamped Timoshenko beam: (a) uniform, (b) non-uniform; with second order varying thickness ($q(\zeta) = s(\zeta) = (1 + 0{\cdot}2\zeta^2)$, $v(\zeta) = r(\zeta) = (1 + 0{\cdot}2\zeta^2)^2$); $\eta = 0{\cdot}01$, $\mu = \gamma = \lambda = \alpha = 0$, $\delta = 0{\cdot}03$

	Ω_1	Ω_2	Ω_3	Ω_4
(a) Uniform beam				
$p = 0$	3·235	14·591	31·835	48·541
$p = 0{\cdot}03$ Case 1 — Eq. (5)	2·499	13·893	31·275	48·155
$p = 0{\cdot}03$ Case 2 — Eq. (6)	2·494	13·798	31·016	47·670
$p = 0{\cdot}3$ Case 1 — Eq. (5)		6·027	25·555	43·656
$p = 0{\cdot}3$ Case 2 — Eq. (6)		4·837	22·276	38·078
(b) Non-uniform beam				
$p = 0$	3·085	14·439	31·999	47·759
$p = 0{\cdot}03$ Case 1 — Eq. (5)	2·433	13·850	31·548	47·560
$p = 0{\cdot}03$ Case 2 — Eq. (6)	2·434	13·764	31·296	47·082
$p = 0{\cdot}3$ Case 1 — Eq. (5)		7·327	26·971	44·868
$p = 0{\cdot}3$ Case 2 — Eq. (6)		6·081	23·847	52·271

Table 3 indicates that an increase in the axial load would increase the difference between the result of the two system of equations. Furthermore, this may be significant for the higher modes of vibration.

5.3. ILLUSTRATIVE EXAMPLE 3

The vibration and the stability of an elastically supported Timoshenko beam carrying an attached mass when being subjected to an axial tensile or compressive loads are investigated here. For a uniform beam, the material and the geometric parameters are taken as $\eta = 0{\cdot}0001$, $\mu = 1{\cdot}0$, $\gamma = 1{\cdot}0$, $\alpha = 0{\cdot}01$, $\delta = 0{\cdot}0003051$.

A non-uniform beam has been assumed with the same cross-section at $\zeta = 0$ but having both the width and the depth of the beam decrease linearly with the same taper ratio as $\lambda = -0{\cdot}5$. The parameters of the non-uniform beam are chosen the same as those of the uniform beam except for the mass ratio $\mu = 1{\cdot}7143$ and $q(\zeta) = s(\zeta) = (1 - 0{\cdot}5\zeta)^2$, $v(\zeta) = r(\zeta) = (1 - 0{\cdot}5\zeta)^4$.

Variation of the first and the second frequencies of the beam with the axial force are presented in Figure 3. The results obtained for the uniform beam are in good agreement with those of the previous studies (Figure 2 of reference [16]). It can be seen that the critical buckling loads of a uniform beam are greater than those of the non-uniform one, as expected, since the stiffness of the uniform beam is now higher. On the other hand, the mass of the uniform beam is also greater than that of a non-uniform beam. Therefore, depending on the value of the axial load, the natural frequencies of the uniform beam may be either smaller or greater than those of the non-uniform beam.

5.4. ILLUSTRATIVE EXAMPLE 4

The dimensionless frequencies of the first two modes versus the axial force (p) for both the free–free uniform and non-uniform Timoshenko beams are illustrated in Figure 4. In these cases, two identical masses with negligible rotatory inertia ($\alpha = 0$) are attached to both ends

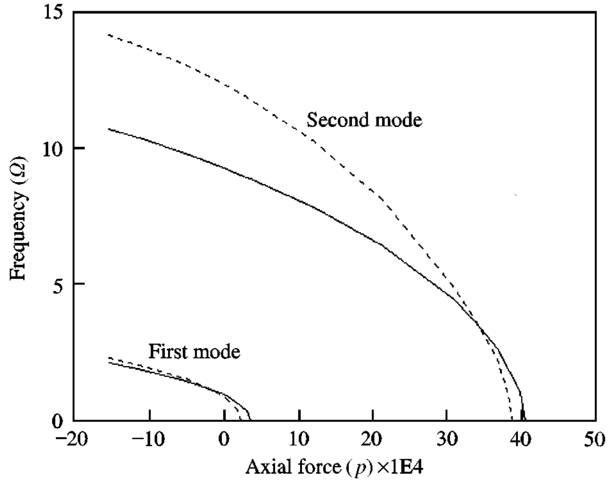

Figure 3. Variation of the natural frequency versus the axial load. Elastically restrained uniform and non-uniform Timoshenko beams with attached beam mass at one end; $\eta = 0{\cdot}0001$, $\gamma = 1{\cdot}0$, $\alpha = 0{\cdot}01$, $\delta = 0{\cdot}0003051$, $\beta_T = 100$, $\beta_\theta = 2$, $q(\zeta) = s(\zeta) = (1 - 0{\cdot}5\zeta)^2$, $v(\zeta) = r(\zeta) = (1 - 0{\cdot}5\zeta)^4$: ——, uniform beam; - - - , non-uniform beam.

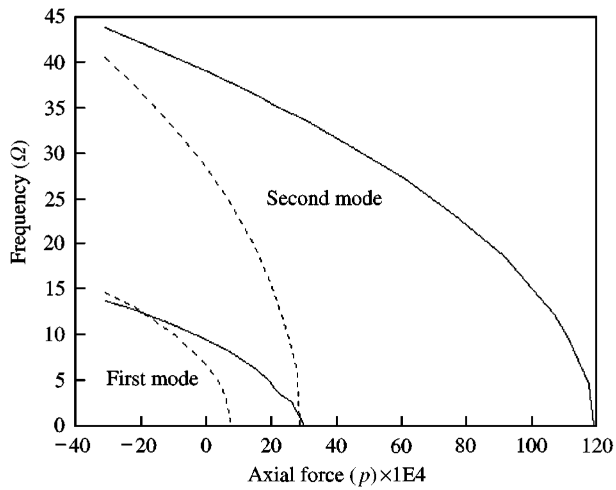

Figure 4. Variation of the natural frequency versus the axial load. Free–free uniform and non-uniform Timoshenko beams with attached mass at both ends; $\eta = 0{\cdot}0001$, $\gamma = 1{\cdot}0$, $\alpha = 0$, $\delta = 0{\cdot}0003051$, $q(\zeta) = s(\zeta) = (1 - 0{\cdot}5\zeta)^2$, $v(\zeta) = r(\zeta) = (1 - 0{\cdot}5\zeta)^4$: ——, uniform beam; - - -, non-uniform beam.

of the beam. The geometry of the non-uniform beam is the same as that of the previous example.

5.5. ILLUSTRATIVE EXAMPLE 5

Consider the non-uniform beam with the geometry and boundary conditions defined as in Example 1. The beam is subjected to an axial force at one end ($p_0 = 0{\cdot}006$) and a uniformly distributed tangential load (q) along the beam. Therefore, the variation of the axial load along the length of the beam can be written as $p(\zeta) = 0{\cdot}006 + (1 - \zeta)q$.

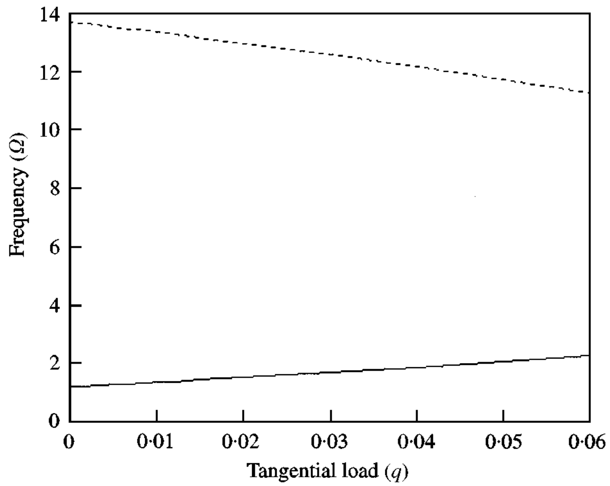

Figure 5. Variation of the natural frequency versus the tangential load. Cantilever Timoshenko beam with constant width and linearly varying thickness; $\eta = 0{\cdot}0016$, $\gamma = 0{\cdot}54$, $\lambda = -0{\cdot}2$, $\alpha = 0$, $\mu = 0{\cdot}6$ and $\mu = 0$: ——, first mode; - - -, second mode.

Figure 5 indicates that an increase in the tangential load (q) increases the first natural frequency but decreases the second. Moreover, it can be seen that the natural frequencies change linearly with the variation of the tangential load.

6. CONCLUSIONS

The exact solution of the vibration and the stability analysis for a non-uniform Timoshenko beam subjected to axial and distributed tangential loads has been presented. Two sets of governing equations are used in the formulation. In the first set, the axial and tangential loads are taken as perpendicular to the shearing force but in the second set, it is assumed that the axial force is tangential to the axis of the beam-column. For both cases, the two governing differential equations were reduced into one fourth order ordinary differential equation with variable coefficients. For different boundary conditions, the parameters of the frequency equation were determined by substituting the homogeneous solution into the associated boundary conditions. By applying the Frobenius method, the exact fundamental solution was found. Comparisons between the results from the two sets of frequency equations have been made for typical examples of uniform and non-uniform beams. The differences between the natural frequencies become significant for beams subjected to high loads and for high modes of vibration. The influences of the geometrical non-uniformity, end masses, axial force and tangential load on the natural frequencies and critical load of the beams have been investigated.

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APPENDIX: NOMENCLATURE

$A(x)$	cross-sectional area of the beam
$E(x)$	modulus of elasticity of beam material
$G(x)$	shear modulus of beam material
$I(x)$	area moment inertia of the beam
$J(x)$	mass moment of inertia of the beam per unit length
J_m	rotatory inertia attached at one or both ends of the beam
K_T, K_θ	translational and rotational spring constants, respectively
L	length of the beam
M	concentrated mass attached at one or both ends of the beam
M_b	total mass of the beam
$m(x)$	mass of the beam per unit length
$p(\zeta)$	dimensionless axial force, $S(\zeta)/Q(0)$
$Q(x)$	beam shear rigidity, $\kappa G(x)A(x)$
$q(\zeta)$	dimensionless shear rigidity, $Q(\zeta)/Q(0)$
q	tangential load
$R(x)$	beam bending rigidity, $E(x)I(x)$
$r(\zeta)$	dimensionless bending rigidity, $R(\zeta)/R(0)$
$S(x)$	axial force
$s(\zeta)$	dimensionless mass, $m(\zeta)/m(0)$
$v(\zeta)$	dimensionless mass moment inertia, $J(\zeta)/J(0)$
x	length variable of the beam

Y	beam lateral displacement
y	dimensionless displacement, Y/L
α	dimensionless rotatory inertia of the attached mass(es), $J_m/[m(0)L^3]$
β_T, β_θ	dimensionless translational and rotational spring constants, respectively, $K_T L^3/R(0)$, $K_\theta L/R(0)$
γ	dimensionless concentrated mass, $M/[m(0)L]$
δ	dimensionless ratio of bending rigidity to shear rigidity at $x = 0$, $R(0)/[Q(0)L^2]$
η	dimensionless ratio of mass moment inertia to mass at $x = 0$, $J(0)/[m(0)L^2]$
κ	shear correction factor of the beam
λ	taper ratio of the beam
μ	dimensionless ratio of the attached mass to total mass of the beam, M/M_b
ζ	dimensionless distance to the left end of the beam, x/L
ω	angular frequency of beam vibration
ψ	angle of rotation due to bending
Ω^2	dimensionless frequency, $m(0)\omega^2 L^4/R(0)$