Journal of Sound and Vibration (2000) **236**(3), 457–485
doi:10.1006/jsvi.2000.3000, available online at http://www.idealibrary.com on IDEAL®

DYNAMICS OF PRESTRESSED ROTATING ANISOTROPIC PLATES SUBJECT TO TRANSVERSE LOADS AND HEAT SOURCES, PART I: MODELLING AND SOLUTION METHOD

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(*Received* 6 *October* 1999, *and in final form* 22 *February* 2000)

This paper considers the application of the finite element method for the analysis of translating or rotating plates, based on Mindlin plate theory and the von Kármán strain expression, in the context of linear thermoelasticity. The existence of convective terms generates gyroscopic terms, unstabilizing effects in the stiffness matrix, and radial in-plane tension. Homogenization theory, applicable to not only determining the global material properties for composite materials like laminate or fiber-reinforced matrix, but also computing microscopic stress levels, was applied to obtain orthotropic material properties. The quasi-static stretching assumption was used to simplify the governing equations. A second order implicit time-integration scheme, applicable for both the linear and non-linear governing equations, was presented, which allows a time increment sufficiently large (without numerical stability problems) based on the accuracy needed. This paper (Part I) presents the problem formulation and solution methods, while a companion paper (Part II) presents and discusses results for specially orthotropic rotating disks.

1. INTRODUCTION

This paper considers a general two-dimensional coupled thermal and plate bending problem for a rotating anisotropic plate subject to transverse forces and heat sources, including transverse shear stresses and rotary inertia. Such problems arise in a variety of technologies, including subsidiary memory devices such as floppy disks, hard disks, CD-ROMs, and optical disks.

Many researchers have studied the behavior of disks or circular plates in such applications as circular saws, turbines, and more recently, computer memory devices. Recently, laminated composite media are effectively utilized in some of these applications. It is, thus, important to consider anisotropic material properties in the study of the dynamic behavior of disks. In a computer disk this anisotropy—special orthotropy—comes from two sources. One is that, the disk has different geometric shapes in the radial and circumferential directions, because the disks used for data storage store their data along tracks in the circumferential direction. These tracks are actually circumferentially aligned

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0022-460X/00/380457 + 29 $35.00/0

grooves or pits. Sometimes the disks include coating of magnetic or overcoat layers, and in the magnetic layers the crystal structures are intentionally aligned along tracks or along the polar axis, according to the recording method in the memory device, to get the largest areal capacity.

One of the most important applications of the rotating disk is in the field of subsidiary memory devices such as floppy disks, hard disks, CD-ROMs, and optical disks. Because a floppy disk rotates on demand only, there are effects due to the time-varying rotational speed. For hard or optical disks there are also transient effects due to acceleration in the rotation speed of the disk when a disk starts or stops. The floppy or flexible disk, as implied by its name, is very flexible and develops relatively large deflections even when subject to small external forces. However, it still works well because the track density for floppy disks, usually 96–192 tracks per inch, is not so large. In the case of hard disks the track density is 1000 or 2000 tracks per inch. Thus, tracking capability must be more accurate in order to recover data. The transverse deflection has the highest magnitude among the deflections in various directions of a typical plate. The transverse force is, thus, important in the analysis of the motion of a disk. In a hard-disk drive, the aerodynamic interaction between the read/write heads and disks, with the surrounding media, give rise to a transverse force on the disk.

The most advanced subsidiary memory device is the erasable optical disk drive which uses a magneto-optical disk composed of a polycarbonate substrate covered by magnetic and other layers. The material used for making the magnetic layer has the property that its coercivity decreases abruptly when temperature increases. As a result, the polarity of the material easily changes in accordance with the external magnetic field applied to the disk. During the reading of stored data, an optical sensor determines bit information by measuring the reflected path of the laser, which generates heat. During the writing of new data, it is necessary to heat the data bit section by laser, then apply the desirable external magnetic field. This gives rise to a temperature gradient on the disk. Another temperature effect comes from the heat generated by a spindle motor in a disk drive.

Several studies of stationary isotropic disks exist. For example, Chen and Doong [1] demonstrated the effect of varying loads and other parameters on the natural frequencies. Chaudhure [2] showed the effect of thickness variation on the stress resultants. Honda *et al.* [3] computed transverse deflections caused by a harmonic force.

Extensive research has also been done for rotating isotropic disks. Eversman and Dodson [4] analyzed membrane disks and showed the effect of the magnitudes of the inner and outer radius on the natural frequencies of the disk. The effects of rotational speed on the natural frequencies have been studied either by utilizing a Green's function approach [5] or by using separation of variables [6]. Similar studies were done by Barasch and Chen [7] and Iwan [8]. Benson and Bogy [9] and Benson [10] have studied the near-membrane case. It was shown through an eigenvalue study that it is necessary to retain bending terms in the governing differential equation to support an arbitrary load. Dugdale [11] showed the existence of travelling waves for a centrally clamped disk. Irie *et al.* [12, 13] obtained free or forced frequency responses. Leung and Pinnington [14] computed spectral densities for different rotational speeds. Cole and Benson [15] have obtained the deflection and eigenvalue spectra under several different loading conditions. Huang and Chiou [16] studied the dynamics of spinning disks subject to a moving magnetic head in the context of computer storage systems. A harmonic moving load was assumed and resonant conditions as well as transient response were calculated. Vera *et al.* [17] calculated the natural frequencies for transverse vibrations of annular plates with various combinations of boundary conditions. Nayfeh *et al.* [18] studied non-linear transverse vibrations of a centrally clamped circular disk rotating at constant angular velocity.

Orthotropic materials properties were examined in several studies on stationary disks. The natural frequencies were computed for both the uniform thickness case [19] and the variable thickness case [20, 21]. Oyibo and Brunelle [22] studied a simple one-dimensional problem considering initial stresses. Tutuncu [23] obtained closed-form solutions for stresses and displacements in a polar anisotropic circular plate subject to a constant temperature change. Lin and Tseng [24] studied the free-vibration problem for polar axisymmetric orthotropic laminated circular or annular plates.

Mote [25] obtained transient thermal stresses for an isotropic stationary disk for which the temperature distribution is specified as a function of time. He used a one-dimensional vibration model to compute the corresponding natural frequency variation as a function of time. Tomar and Gupta [26] considered thermal effects limited to the material properties in the study of the eigenvalues of isotropic stationary disks. Two-dimensional differential equations of heat conduction have been added to the elasticity equilibrium equation to show the effect of thermal strain on the natural frequencies of a rotating orthotropic disk with constant rotational speed [27]. Ghosh [27] simplified the problem into a one-dimensional axisymmetric model for the transverse vibration by taking average stresses over the thickness of the disk and neglecting both in-plane and transverse shear stresses. The additional assumption that the mean radial and tangential stresses are independent of the transverse deflection makes it possible to determine these stresses prior to the analysis of transverse vibrations. Thus, Ghosh [27] has dealt with the two-dimensional thermoelastic rotating disk problem, and does not consider the non-axisymmetric case to compute time responses, and neglects transverse shear stresses.

The purpose of the research reported in this paper is to formulate and solve a general two-dimensional coupled thermal and plate bending problem for an anisotropic plate in non-steady rotation subject to transverse forces and heat sources, including transverse shear stresses and rotary inertia.

In section 2, the detailed problem formulation is given. First, the equations of motion for the plate bending problem are obtained based upon the Mindlin plate theory and the von Kármán strain expressions for the case where the medium is rotating. In section 3 material properties are expressed for specially orthotropic materials. In section 4, the conduction problem is formulated considering convection terms and in section 5 the matrix forms for the combined thermoelastic plate problem are obtained. The solution method is presented in section 6 where an α-method, together with a predictor–multicorrector scheme, applicable to both the linear and non-linear cases, is utilized for the time integration. Section 7 summarizes the modelling approach and conclusions. The details of obtaining necessary matrices and vectors for solution are included in Appendix A. Appendix B summarizes a general method for obtaining homogenized material properties. A companion paper (Part II), presents results and discussions from the application of the formulation presented in this paper to rotating specially orthotropic disks.

2. THE EQUATIONS OF MOTION FOR AN ANISOTROPIC PLATE

For consistent derivation of the linearized equations, and for consideration of geometric non-linearities, the strains need to be expressed non-linearly in terms of first partial derivatives of displacements. To generalize the problem of the dynamics of rotating disks, those with anisotropic properties will be considered.

A thin disk of inner radius R_i, and outer radius R_o, is assumed to rotate about its polar axis with an angular velocity $\Omega(t)$ as shown in Figure 1. A space-fixed cylindrical

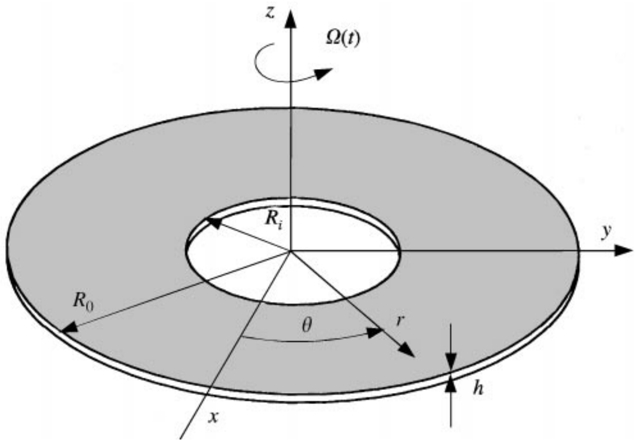

Figure 1. Disk configuration.

co-ordinate (r, θ, z), or rectangular co-ordinate (x, y, z) is chosen to represent the undeformed configuration.

The plate of interest is assumed to be in the elastic region during the analysis. A finite strain tensor can be developed in its full quadratic form in terms of first partial derivatives [28]. However, the fact that the magnitudes of the derivatives of displacements are quite different due to a small thickness of the disk enables us to use von Kármán strain expressions. Assume that the lines normal to the mid-plane remain straight under deformation, and that there exist transverse shear strains following the Mindlin plate theory. The disk is assumed to rotate at an angular velocity which is a prescribed function of time. The disk is also subject to thermal strains caused by a temperature gradient due to heat sources.

Let (u, v, w) be the displacements at an arbitrary point having the co-ordinate (x, y, z). The von Kármán strain assumes, in representing strains in quadratic form in terms of the first derivatives of displacements, that the second order terms except $w_{,x}^2$, $w_{,y}^2$, and $w_{,x}w_{,y}$, may be neglected [29]. Assuming further that the strain ε_z may be neglected, then one can write the strains ε as

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \begin{Bmatrix} u_{,x} + \frac{1}{2} w_{,x}^2 \\ v_{,y} + \frac{1}{2} w_{,y}^2 \\ \frac{1}{2}(v_{,x} + u_{,y} + w_{,x} w_{,y}) \\ \frac{1}{2}(w_{,y} + v_{,z}) \\ \frac{1}{2}(u_{,z} + w_{,x}) \end{Bmatrix}. \tag{1}$$

The strains in the cylindrical co-ordinate system are

$$\varepsilon = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_{r\theta} \\ \varepsilon_{\theta z} \\ \varepsilon_{rz} \end{Bmatrix} = \begin{Bmatrix} u_{r,r} + \frac{1}{2} w_{,r}^2 \\ \frac{1}{r} u_r + \frac{1}{r} u_{\theta,\theta} + \frac{1}{r^2} w_{,\theta}^2 \\ \frac{1}{2}(\frac{1}{r} u_{r,\theta} + u_{\theta,r} - \frac{1}{r} u_\theta + \frac{1}{r} w_{,r} w_{,\theta}) \\ \frac{1}{2}(\frac{1}{r} w_{,\theta} + u_{\theta,z}) \\ \frac{1}{2}(u_{r,z} + w_{,r}) \end{Bmatrix}, \tag{2}$$

where (u_r, u_θ, w) are the displacements at arbitrary points measured along the cylindrical co-ordinate. In index notation, one can write

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + w_{,i}w_{,j}), \qquad \varepsilon_{iz} = \tfrac{1}{2}(w_{,i} + u_{i,z}). \tag{3}$$

Throughout, the indices i, j, k, l take on the values 1, 2. The summation convention is in effect. The stress–strain relation is

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \qquad \sigma_{iz} = C_{iziz}2\varepsilon_{iz}. \tag{4}$$

In order to obtain the equations of motion using Hamilton's principle, it is necessary to compute the potential energy, the kinetic energy and the work done by non-conservative forces.

The strain energy U is computed as

$$U = \frac{1}{2}\int_v \varepsilon_{ij}C_{ijkl}\varepsilon_{kl}\,\mathrm{d}V + \frac{1}{2}\sum_{i=1}^{2}\int_v 2\varepsilon_{iz}C_{iziz}2\varepsilon_{iz}\,\mathrm{d}V. \tag{5}$$

The total potential energy, V, can be expressed as the sum of the strain energy, U, and the potential energy due to the displacement w in the presence of the gravitational field, g. Thus,

$$\delta V = \delta U + \int_v \rho g \delta w\,\mathrm{d}V. \tag{6}$$

If one considers a rotating system with a rotational speed of $\Omega(t)$, then the velocity observed by an observer on the inertial frame is expressed as

$$\mathbf{v} = \begin{Bmatrix} v_r \\ v_\theta \\ v_z \end{Bmatrix} = \begin{Bmatrix} \dot{u}_r - \Omega u_\theta \\ \dot{u}_\theta + \Omega(r + u_r) \\ \dot{w} - \Omega u_\theta \dfrac{\partial w}{\partial r} + \Omega\left(1 + \dfrac{u_r}{r}\right)\dfrac{\partial w}{\partial \theta} \end{Bmatrix}, \tag{7}$$

where (r, θ, z) is expressed in the cylindrical co-ordinate system. In a rectangular co-ordinate system (x, y, z), one can write

$$\mathbf{v} = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} = \begin{Bmatrix} \dot{u} - \Omega(y + v) \\ \dot{v} + \Omega(x + u) \\ \dot{w} - \Omega(y + v)\dfrac{\partial w}{\partial x} + \Omega(x + u)\dfrac{\partial w}{\partial y} \end{Bmatrix}. \tag{8}$$

Note that if a translating plate is considered, the above velocity expressions can easily be modified by using the translational speed of the plate. Thus, the analysis method presented here can easily be applied to translating plates [28].

In general, one may write the velocities compactly as

$$v_i = \dot{u}_i + q_i^*, \qquad v_z = \dot{w} + q_i^*\frac{\partial w}{\partial x_i}, \tag{9}$$

where

$$q_i^* = q_i + p_{ij}(x_j + u_j), \qquad q_1 = V_x, \qquad q_2 = V_y.$$

$$p_{12} = -\Omega, \qquad p_{21} = \Omega, \qquad p_{11} = p_{22} = 0.$$

Note that $p_{ij}\,(=-p_{ji})$ is antisymmetric and that q_i^* is the total velocity in the ith direction.

The kinetic energy of an infinitesimal element is

$$\mathrm{d}T = \tfrac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}m = \tfrac{1}{2}\,\rho\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}V,$$

where ρ is the mass per unit volume. The total kinetic energy is

$$T = \frac{1}{2}\int_v \rho\mathbf{v}\cdot\mathbf{v}\mathrm{d}V = \frac{1}{2}\int_v \rho(v_i v_i + v_z v_z)\,\mathrm{d}V. \tag{10}$$

The work, W_{nc}, done by the non-conservative force per unit area, $f(x_i, t)$, is

$$W_{nc} = \int_A fw\,\mathrm{d}A, \qquad \delta W_{nc} = \int_A f\delta w\,\mathrm{d}A = \int_A \frac{f}{h}\,\delta w\,\mathrm{dV}. \tag{11}$$

Hamilton's principle states that

$$\int_{t_1}^{t_2} \delta(T - V)\,\mathrm{d}t + \int_{t_1}^{t_2} \delta W_{nc}\,\mathrm{d}t = 0. \tag{12}$$

From the Euler equations for the above variational form, one obtains the following equations of motion:

$$\begin{aligned} \rho\ddot{u}_i &= a_{ij}\dot{u}_j + b_{ij}u_j + c_i + \sigma_{ij,j} + \sigma_{iz,z} \\ \rho\ddot{w} &= 2a_i\dot{w}_{,i} + b_i w_{,i} + d_{ij}w_{,ij} + c + (\sigma_{ij}w_{,i})_{,j} + \sigma_{iz,i}. \end{aligned} \tag{13}$$

where

$$a_{ij} = -2\rho p_{ij}, \qquad b_{ij} = -\rho\dot{p}_{ij}, \qquad c_i = -\rho[\dot{q}_i + \dot{p}_{ij}(x_j + u_j) + p_{ij}q_j^*]$$

and

$$a_i = -\rho q_i^*, \qquad b_i = -\rho\{\dot{q}_i + \dot{p}_{ij}(x_j + u_j)\}, \, d_{ij} = -\rho q_i^* q_j^*, \qquad c = -\rho g + f/h.$$

Note that for the purely rotating case with constant speed Ω,

$$c_i = -\rho p_{ij}p_{jk}(x_k + u_k) = \rho\Omega^2(x_i + u_i) \tag{14}$$

becomes the centrifugal force per unit volume in the ith direction.

The Mindlin plate theory can now be used to simplify the equations of motion by expressing them in terms of the mid-plane deflections in a two-dimensional domain (plate).

Let $(\bar{u}, \bar{v}, \bar{w})$ be the displacements at the mid-surface co-ordinate $(x, y, 0)$. The Mindlin plate theory is based upon the assumptions [30]

$$\varepsilon_z = 0, \qquad u = \bar{u} - z\theta_x, \qquad v = \bar{v} - z\theta_y, \tag{15}$$

where θ_x and θ_y are the rotations in the x and y directions respectively. Here the assumption that $\varepsilon_z = 0$ is also for the von Kármán strain expression, and has already been used in the previous section. Therefore,

$$w = \bar{w}$$

and the strains at an arbitrary point become

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix} = \mathbf{e} - z\kappa, \tag{16}$$

where the mid-plane strains $\mathbf{e}$ are expressed as

$$\mathbf{e} = \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \\ e_{yz} \\ e_{xz} \end{Bmatrix} = \begin{Bmatrix} \bar{u}_{,x} + \frac{1}{2}\bar{w}_{,x}^2 \\ \bar{v}_{,y} + \frac{1}{2}\bar{w}_{,y}^2 \\ \frac{1}{2}(\bar{v}_{,x} + \bar{u}_{,y} + \bar{w}_{,x}\bar{w}_{,y}) \\ \frac{1}{2}(\bar{w}_{,y} - \theta_y) \\ \frac{1}{2}(\bar{w}_{,x} - \theta_x) \end{Bmatrix}.$$

and the curvatures κ are shown to be

$$\kappa = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \kappa_{yz} \\ \kappa_{zx} \end{Bmatrix} = \begin{Bmatrix} \theta_{x,x} \\ \theta_{y,y} \\ \frac{1}{2}(\theta_{x,y} + \theta_{y,x}) \\ 0 \\ 0 \end{Bmatrix}. \tag{17}$$

In terms of cylindrical co-ordinates [29], the mid-plane strain–displacement relation is

$$\mathbf{e} = \begin{Bmatrix} e_r \\ e_\theta \\ e_{r\theta} \\ e_{\theta z} \\ e_{rz} \end{Bmatrix} = \begin{Bmatrix} \bar{u}_{,r} + \frac{1}{2}\bar{w}_{,r}^2 \\ \frac{1}{r}\bar{u} + \frac{1}{r}\bar{v}_{,\theta} + \frac{1}{r^2}\bar{w}_{,\theta} \\ \frac{1}{2}(\frac{1}{r}\bar{u}_{,\theta} + \bar{v}_{,r} - \frac{1}{r}\bar{v} + \frac{1}{2r^2}\bar{w}_{,r}\bar{w}_{,\theta}) \\ \frac{1}{2}(\frac{1}{r}\bar{w}_{,\theta} - \theta_\theta) \\ \frac{1}{2}(\bar{w}_{,r} - \theta_r) \end{Bmatrix}. \tag{18}$$

and the curvature change to displacement relation is

$$\kappa = \begin{Bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_{r\theta} \\ \kappa_{\theta z} \\ \kappa_{rz} \end{Bmatrix} = \begin{Bmatrix} \theta_{r,r} \\ \frac{1}{r}\theta_r + \frac{1}{r}\theta_{\theta,\theta} \\ \frac{1}{2}(\frac{1}{r}\theta_{r,\theta} + \theta_{\theta,r} - \frac{1}{r}\theta_\theta) \\ 0 \\ 0 \end{Bmatrix}. \tag{19}$$

In index notation,

$$u_i = \bar{u} - z\theta_i, \qquad w = \bar{w} \tag{20}$$

and

$$\varepsilon_{ij} = \tfrac{1}{2}[\bar{u}_{i,j} + \bar{u}_{j,i} + \bar{w}_{,i}\,\bar{w}_{,j} - z(\theta_{i,j} + \theta_{j,i})], \qquad \varepsilon_{iz} = \tfrac{1}{2}(\bar{w}_{,i} - \theta). \tag{21}$$

From the relations

$$\varepsilon_{ij} = e_{ij} - z\kappa_{ij}, \qquad \varepsilon_{iz} = e_{iz} - z\kappa_{iz}$$

the strain and curvature can be written as

$$\begin{aligned} e_{ij} &= \tfrac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i} + \bar{w}_{,i}\bar{w}_{,j}), \qquad e_{iz} = \tfrac{1}{2}(\bar{w}_{,i} - \theta_i). \\ \kappa_{ij} &= \tfrac{1}{2}(\theta_{i,j} + \theta_{j,i}), \qquad \kappa_{iz} = 0. \end{aligned} \tag{22}$$

Now the stress components are

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = C_{ijkl}(e_{kl} - z\kappa_{kl}), \qquad \sigma_{iz} = C_{iziz}2\varepsilon_{iz} = C_{iziz}2(e_{iz} - z\kappa_{iz}). \tag{23}$$

Define

$$\begin{aligned} s_{ij} &\equiv C_{ijkl}e_{kl}, \qquad s_{iz} \equiv C_{iziz}2e_{iz}, \\ \mu_{ij} &\equiv C_{ijkl}\kappa_{kl}, \qquad \mu_{iz} \equiv C_{iziz}2\kappa_{iz} = 0, \end{aligned} \tag{24}$$

then

$$\sigma_{ij} = s_{ij} - z\mu_{ij}, \qquad \sigma_{iz} = s_{iz}. \tag{25}$$

Note that all the components of e, κ, s, and μ are independent of z. Now define the force and moment resultants as

$$\begin{aligned} N_{ij} &\equiv \int_z \sigma_{ij}\,\mathrm{d}z = hs_{ij} = hC_{ijkl}e_{kl} = hC_{ijkl}(\bar{u}_{k.l} + \tfrac{1}{2}\bar{w}_{,k}\,\bar{w}_{,l}), \\ Q_i &\equiv \int_z \sigma_{iz}\,\mathrm{d}z = hs_{iz} = hC_{iziz}2e_{iz} = hC_{iziz}(\bar{w}_{,i} - \theta_i), \\ M_{ij} &\equiv \int_z z\sigma_{ij}\,\mathrm{d}z = -D\mu_{ij} = -DC_{ijkl}\kappa_{kl} = -DC_{ijkl}\theta_{k,l}, \end{aligned} \tag{26}$$

where $D = h^3/12$. Integrate equations (13) along the direction of the thickness $z = (-h/2, h/2)$, to get

$$\rho h\ddot{\bar{u}}_i = ha_{ij}\dot{\bar{u}}_j + hb_{ij}\bar{u}_j + hc_i + N_{ij,j} + \left[\sigma_{iz}\left(\frac{h}{2}\right) - \sigma_{iz}\left(-\frac{h}{2}\right)\right],$$
$$\rho h\ddot{\bar{w}} = 2ha_i\dot{w}_{,i} + hb_i w_{,i} + hd_{ij}w_{,ij} - \rho gh + (N_{ij}\bar{w}_{,i})_{,j} + Q_{i,i} + f. \tag{27}$$

Multiplying equation (13a) by z and integrating along the thickness, one obtains

$$\rho D\ddot{\theta}_i = Da_{ij}\dot{\theta}_j + Db_{ij}\theta_j - M_{ij,j} + Q_i - \frac{h}{2}\left[\sigma_{iz}\left(\frac{h}{2}\right) + \sigma_{iz}\left(-\frac{h}{2}\right)\right]. \tag{28}$$

Rewrite equations (27) and (28), in a compact form as

$$N_{ij,j} + f_i = 0, \qquad Q_{i,i} + (N_{ij}\bar{w}_{,i})_{,j} + f_z = 0,$$
$$M_{ij,j} - Q_i + m_i = 0, \tag{29}$$

where

$$f_i = -\rho h\ddot{\bar{u}}_i + ha_{ij}\dot{\bar{u}}_j + hb_{ij}\bar{u}_j + hc_i + \left[\sigma_{iz}\left(\frac{h}{2}\right) + \sigma_{iz}\left(-\frac{h}{2}\right)\right],$$

$$f_z = -\rho h\ddot{\bar{w}} + 2ha_i\frac{\partial\dot{\bar{w}}}{\partial x_i} + hb_i\frac{\partial\bar{w}}{\partial x_i} + hd_{ij}\frac{\partial^2\bar{w}}{\partial x_i\partial x_j} + hc, \tag{30}$$

$$m_i = \rho D\ddot{\theta}_i - Da_{ij}\dot{\theta}_j - Db_{ij}\theta_j + \frac{h}{2}\left[\sigma_{iz}\left(\frac{h}{2}\right) + \sigma_{iz}\left(-\frac{h}{2}\right)\right].$$

Now one has a set of the equilibrium equations (29) expressed in terms of the resultant forces N_{ij}, Q and the resultant moments M_{ij} instead of equations (13). Since the resultant forces and moments are functions of the mid-plane deflections (see equation (26)), one has the equilibrium equations in terms of the mid-plane deflections.

Equation (29) is a strong form of the governing equation. In order to solve for this equation numerically, a weak form is obtained and then semi-discretized using shape functions which are constant with respect to time. This leads to the so-called semi-discrete Galerkin's method, where a finite-dimensional approximate solution is sought.

To obtain the weak form, equations (29) are multiplied by weighting functions $\tilde{u}_i$, $\tilde{w}$ and $\tilde{\theta}_i$, respectively, and added together, and integrated over the domain A:

$$\int_A \{-\tilde{u}_i[N_{ij,j} + f_i] - \tilde{w}[Q_{i,i} + (N_{ij}\bar{w}_{,i})_{,j} + f_z] + \tilde{\theta}_i[M_{ij,j} - Q_i + m_i]\}\,\mathrm{d}A = 0. \tag{31}$$

Since $w = \bar{w}$, we will use these interchangeably. Here the approximate solutions

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) = (\bar{u}_1, \bar{u}_2, w, \theta_1, \theta_2)$$

satisfy the essential boundary conditions and the weighting functions

$$\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5) = (\tilde{u}_1, \tilde{u}_2, \tilde{w}, \tilde{\theta}_1, \tilde{\theta}_2)$$

are chosen to satisfy the homogeneous counterpart of the essential boundary conditions:

$$\begin{aligned} \mathbf{x} \in \mathbf{S} &= \{(x_p) | x_p = g_p \text{ on } \Gamma_{g_p}, \ p = 1, \ldots, 5\}, \\ \tilde{\mathbf{x}} \in \mathbf{V} &= \{\tilde{x}_p | \tilde{x}_p = 0 \text{ on } \Gamma_{g_p}, \ p = 1, \ldots, 5\}. \end{aligned} \tag{32}$$

Then integration by parts gives the form

$$\begin{aligned} &\int_A [\tilde{u}_{i,j} N_{ij} + \tilde{w}_{,i} Q_i + \tilde{w}_{,j} N_{ij} \tilde{w}_{,i} - \tilde{\theta}_{i,j} M_{ij} - \tilde{\theta}_i Q_i] \, \mathrm{d}A \\ &\quad = \int_A [\tilde{u}_i f_i + \tilde{w} f_z - \tilde{\theta}_i m_i] \mathrm{d}A + \int_\Gamma [-\tilde{u}_i N_{ij} n_j - \tilde{w} Q_i^* n_i - \tilde{\theta}_i M_{ij} n_j] \, \mathrm{d}\Gamma. \end{aligned} \tag{33}$$

where

$$Q_i^* n_i = Q_i n_i + N_{ij} \tilde{w}_{,i} n_j, \qquad \Gamma = \partial A = \Gamma_g \cup \Gamma_h.$$

Γ_g represents the boundary with specified displacement and Γ_h the boundary with specified force or moment.

Assume the following boundary conditions:

$$\begin{aligned} u_i = u_i^1, \qquad w = w^1, \qquad q_n = q_n^1, \qquad q_s = q_s^1 \quad \text{on } G^1, \\ u_i = u_i^2, \qquad w = w^2, \qquad M_{nn} = M_{nn}^2, \qquad M_{ns} = M_{ns}^2 \quad \text{on } G^2, \\ N_i = N_i^3, \qquad Q_n^* = Q_n^{*3} \qquad q_n = q_n^3 \qquad q_s = q_s^3 \quad \text{on } G^3, \\ N_i = N_i^4, \qquad Q_n^* = Q_n^{*4} \qquad M_{nn} = M_{nn}^4, \qquad M_{ns} = M_{ns}^4 \quad \text{on } G^4, \end{aligned} \tag{34}$$

where the superscripts denote the types of the boundary conditions, the subscripts n and s denote the normal and the tangential directions to the boundaries, and the subscript i, as usual, takes on the values 1 and 2. The first type of boundary condition, Γ^1, denotes that all the deflections are specified (essential boundary conditions). The second type, Γ^2, has partly essential (both the in-plane and the out-of-plane deflections are specified) and partly forced (the moments are specified) boundaries. The third type, Γ^3, is also of the partly essential and partly forced boundary type, but this time with specified in-plane and out-of-plane forces and specified rotations. The fourth type of boundary condition, Γ^4, is purely for a forced boundary condition.

If one decomposes the resultant moment into normal and tangential components,

$$M_{ij} n_j \tilde{\theta}_i = M_{nn} \tilde{\theta}_n + M_{ns} \tilde{\theta}_s, \tag{35}$$

the boundary terms become

$$\begin{aligned} \int_\Gamma (\cdots) \, \mathrm{d}\Gamma &= \int_{\Gamma^2} [-M_{nn}^2 \tilde{\theta}_n - M_{ns}^2 \tilde{\theta}_s] \, \mathrm{d}\Gamma + \int_{\Gamma^3} [N_1^3 \tilde{u}_1 + N_2^3 \tilde{u}_2 + Q_n^{*3} \tilde{w}] \, \mathrm{d}\Gamma \\ &\quad + \int_{\Gamma^4} [N_1^4 \tilde{u}_1 + N_2^4 \tilde{u}_2 + Q_n^{*4} \tilde{w} - M_{nn}^4 \tilde{\theta}_n + M_{ns}^4 \tilde{\theta}_s] \, \mathrm{d}\Gamma. \end{aligned} \tag{36}$$

Then, by following a standard procedure as described in Appendix A, the following system of equations can be obtained:

$$\mathbf{M}_u \ddot{u} = \mathbf{F}_u^0, \qquad \mathbf{M}_x \ddot{x} = \mathbf{F}_x^0, \tag{37}$$

with the vectors of displacements **u** and **x** being described as

$$\mathbf{u} = \{u_1, u_2, \ldots, u_N\}, \qquad \mathbf{x} = \{x_1, x_2, \ldots, x_N\},$$

where N is the total number of nodal points, and $u_\alpha = \{u_{1\alpha}, u_{2\alpha}\}^{\mathrm{T}}$, $x_\alpha = \{w_\alpha, \theta_{1\alpha}, \theta_{2\alpha}\}^{\mathrm{T}}$ ($\alpha = 1, 2, \ldots, N$).

The matrices and vectors in equation (37) are defined in Appendix A.

Next introduce the quasi-static stretching assumption. The dynamic characteristics of in-plane motion and of out-of-plane motion are quite different. The eigenvalues associated with the in-plane motions are very high, and those with the transverse motion are very low. If all the dynamics are included and computed numerically, then the so-called "stiff" system results. In order to avoid a stiff system, one can neglect the dynamics of the very-high-frequency in-plane motions. Here, only the static equilibrium state of the in-plane motions is computed, whereas the dynamics related to the transverse deflection and the rotations at the mid-plane are retained.

The general form for the governing equations then becomes

$$\mathbf{0} = \mathbf{F}_u^1(t; \mathbf{z}) = -\mathbf{N}_u(t; \mathbf{z}), \qquad \mathbf{M}_x \ddot{\mathbf{x}} = \mathbf{F}_x^1(t; \mathbf{z}) = -\mathbf{N}_x(t; \mathbf{z}), \tag{38}$$

where $\mathbf{z} = \{\mathbf{T}^{\mathrm{T}}, \mathbf{u}^{\mathrm{T}}, \mathbf{x}^{\mathrm{T}}, \dot{\mathbf{x}}^{\mathrm{T}}\}^{\mathrm{T}}$ and $\mathbf{T}$ is the temperature (see section 3):

$$\mathbf{F}_u^1(t; \mathbf{z}) = \mathbf{F}_u^0 - \int_A h a_{ij}\, \ddot{u}_j \, \mathrm{d}A, \qquad \mathbf{F}_x^1(t; \mathbf{z}) = \mathbf{F}_x^0,$$

$$\mathbf{N}_u(t; \mathbf{z}) = -\mathbf{F}_u^1(t; \mathbf{z}), \qquad \mathbf{N}_x(t; \mathbf{z}) = -\mathbf{F}_x^1(t; \mathbf{z}).$$

The consistent stiffness matrices $\mathbf{K}_u$ and $\mathbf{K}_x$ and the consistent damping matrix $\mathbf{C}_x$ at a given geometry $\mathbf{z} = \mathbf{z}_0$ are defined as

$$\mathbf{K}_u = \left.\frac{\partial \mathbf{N}_u}{\partial \mathbf{u}}\right|_{\mathbf{z}_0}, \qquad \mathbf{K}_x = \left.\frac{\partial \mathbf{N}_x}{\partial \mathbf{x}}\right|_{\mathbf{z}_0}, \quad \mathbf{C}_x = \left.\frac{\partial \mathbf{N}_x}{\partial \dot{\mathbf{x}}}\right|_{\mathbf{z}_0}. \tag{39}$$

The linearized version, about $\mathbf{z}_0$, of the equilibrium equations in matrix form is

$$\mathbf{K}_u \mathbf{u} = \mathbf{F}_u, \qquad \mathbf{M}_x \ddot{\mathbf{x}} + \mathbf{G}_x \dot{\mathbf{x}} + \mathbf{K}_x \mathbf{x} = \mathbf{F}_x, \tag{40}$$

where

$$\mathbf{F}_u = \mathbf{F}_u^1 - \mathbf{K}_u \mathbf{u}, \qquad \mathbf{F}_x = \mathbf{F}_x^1 - \mathbf{G}_x \dot{\mathbf{x}} - \mathbf{K}_x \mathbf{x}.$$

Here, $\mathbf{G}_x$ is used instead of $\mathbf{C}_x$, because damping is not included and there is only a gyroscopic operator. The consistent stiffness matrices have the form

$$\mathbf{K}_u = \mathbf{K}_u^0 + \mathbf{K}_u^G + \mathbf{K}_u^F, \qquad \mathbf{K}_x = \mathbf{K}_x^0 + \mathbf{K}_x^G + \mathbf{K}_x^F. \tag{41}$$

The expressions for all matrices and vectors are given in Appendix A.

3. MATERIAL PROPERTIES FOR AN ORTHOTROPIC PLATE

The elasticity tensor has been used in the constitutive relation in the derivation of the governing equations of motion. Since the directional properties are readily available in most cases, it is necessary to obtain the relation between the directional elastic properties and the elasticity tensor. The strain–stress relation, for orthotropic materials, is represented by [31].

$$\varepsilon_{ij} = A_{ijkl}\sigma_{kl}, \qquad 2\varepsilon_{iz} = A_{iziz}\sigma_{iz}, \tag{42}$$

where the compliance tensor A_{imjn} satisfies major and minor symmetry

$$A_{ijkl} = A_{klij} = A_{jikl} = A_{ijlk}$$

and has the components

$$A_{1111} = \frac{1}{E_1}, \qquad A_{1122} = -\frac{v_{21}}{E_2}, \qquad A_{2222} = \frac{1}{E_2}, \qquad A_{2211} = -\frac{v_{12}}{E_1},$$

$$A_{1313} = \frac{1}{4G_{13}}, \qquad A_{2323} = \frac{1}{4G_{23}}, \qquad A_{1212} = A_{2121} = A_{1221} = A_{2112} = \frac{1}{4G_{12}},$$

$$A_{imjn} = 0 \quad \text{otherwise.}$$

By inverting the influence tensor, one can obtain the constitutive relation of the form

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \qquad \sigma_{iz} = C_{iziz}2\varepsilon_{iz}, \tag{43}$$

where the elasticity tensor C_{imjn} also satisfies major and minor symmetry and has the components

$$C_{1111} = \frac{E_1}{(1 - v^2)}, \qquad C_{1122} = \frac{E_1 v_{21}}{(1 - v^2)},$$

$$C_{2222} = \frac{E_2}{(1 - v^2)}, \qquad C_{2211} = \frac{E_2 v_{12}}{(1 - v_2)},$$

$$C_{1313} = G_{13}, \qquad C_{2323} = G_{23}, \qquad C_{1212} = C_{2121} = C_{1221} = C_{2112} = G_{12},$$

$$C_{imjn} = 0 \quad \text{otherwise}$$

and $v^2 = v_{12}v_{21}$.

If one assumes that there are pre-strains then

$$\begin{gathered} \sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^0) = C_{ijkl}\varepsilon_{kl} - \sigma_{ij}^0, \\ \sigma_{iz} = C_{izjz}2(\varepsilon_{jz} - \varepsilon_{jz}^0) = C_{izjz}2\varepsilon_{jz} - \sigma_{iz}^0, \end{gathered} \tag{44}$$

where ε_{jn}^0 represents the pre-strains caused by residual stresses and the temperature distribution within the domain, and σ_{im}^0 are the pre-stresses. To consider the effects of microstructure on the global elasticity properties, the homogenized elasticity tensor, C_{imjn}^H, which can be obtained as described in Appendix B will be used for laminates.

4. HEAT CONDUCTION PROBLEM

In order to include the effect of temperature gradients on the analysis of dynamic deflections, one needs to know the dynamic distribution of temperature.

Consider a time-dependent heat conduction process. Time is denoted by t, temperature by T, the domain by $A \subset \Re^2$, the closure of the domain by $\bar{A}$, and $\Re^n$ represents the n-dimensional space of real numbers. The heat source, f, is given by

$$f\colon A \times (0, t_f) \to \Re. \tag{45}$$

Here, f includes the effects of both the external heat source and the convective heat flow between the medium and the ambient air,

$$f = f_s - h_s(T - T_0), \tag{46}$$

where f_s is the heat generated by the heat source within the domain, h_s is the heat transfer coefficient, and T_0 is the ambient air temperature above the domain. The boundary data are given as

$$g\colon \Gamma_g \times (0, t_f) \to \Re, \qquad H_T\colon \Gamma_h \times (0, t_f) \to \Re, \tag{47}$$

where t_f represents the steady state.

The heat flow, H_T, at the boundary is

$$H_T = Q_h - H_b(T - T_\infty), \tag{48}$$

where Q_h denotes the external heat flowing through the boundaries, H_b the heat transfer coefficient and T_∞ the ambient air temperature outside of the boundaries. Also, the initial condition for the problem is given by

$$T_0\colon A \to \Re. \tag{49}$$

The heat transfer coefficients, h_s and H_b, can be determined by using the rate form of the convection heat flow

$$q_n = h_v(T - T_a), \tag{50}$$

where q_n is the heat flow in the outward normal direction, h_v is the convection heat coefficient, and T_a is the temperature of the ambient air. Here the convection coefficient is a function of the fluid velocity adjacent to the surface of the solid. If one assumes that the current domain is under the effect of deflection then it is reasonable to include the effect of increase of the area due to deflection in computing the convection coefficient. This provides the feedback from the deflection to the temperature dynamics. Together with the temperature to deflection effect as initial strains (see Appendix A), this completes the coupling mechanism. The convection coefficient, considering the change in the area of the domain, then becomes

$$h_s = h_v(J^+ + J^-), \tag{51}$$

where J is the Jacobian of the deformation gradient tensor related with the area of the domain, and the superscripts + and − denote the quantities at $z = h/2$ and $z = -h/2$ respectively. Let the undeformed configuration be x_i and the current configuration be $x_i + u_i$. The deformation gradient tensor then becomes

$$F_{ij} = \frac{\partial(x_i + u_i)}{\partial x_j} = \delta_{ij} + u_{i,j}. \tag{52}$$

For the Mindlin plate, since

$$u_i = \bar{u}_i - z\theta_i, \qquad F_{ij} = \delta_{ij} + \bar{u}_{i,j} - z\theta_{i,j}, \tag{53}$$

the Jacobian, J, is

$$J = \det(F_{ij}) = \varepsilon^{ij} F_{i1} F_{j2}, \tag{54}$$

where ε^{ij} denotes the permutation symbol. Therefore,

$$J^+ + J^- = J|_{z=h/2} + J|_{z=-h/2} = 2\det(\delta_{ij} + \bar{u}_{i,j})\frac{h^2}{2}\det(\theta_{i,j})$$

and the heat transfer coefficient, h_s, on the surface becomes

$$h_s = h_v\left[2\det(\delta_{ij} + \bar{u}_{i,j}) + \frac{h^2}{2}\det(\theta_{i,j})\right]. \tag{55}$$

At the boundary Γ_h, the heat transfer coefficient, H_b, becomes

$$H_b = hh_v\frac{\mathrm{d}S}{\mathrm{d}s}, \tag{56}$$

where ds and dS are the lengths of an infinitesimal boundary element before and after deformation respectively. If a small boundary element has two node points (1) and (2) and the vector from point (1) to point (2) before and after deflection are r and R, respectively, then

$$\frac{\mathrm{d}S}{\mathrm{d}s} = \frac{\|R\|_2}{\|r\|_2},$$

where $\|x\|_p$ is a p-norm of $x \in \Re^n$ defined by

$$\|x\|_p = \left[\sum_i^n x_i^p\right]^{1/p}.$$

For $0{\cdot}6 < Pr < 60$, the following empirical relation holds [32]:

$$C_f/2 = \mathrm{St}\,\mathrm{Pr}^{2/3}, \tag{57}$$

where the friction coefficient, C_f,

$$C_f = \frac{\tau_s}{\frac{1}{2}\rho V^2} \tag{58}$$

and Pr and St denote the Prandtl and Stanton numbers defined as usual [32]. The range of Pr is 0·786–0·686 when the temperature of the air is in the range -173 to $+177°$C, hence the use of equation (57) is justified. The shear stress is known to satisfy

$$\tau_s = \mu \frac{\partial u}{\partial z}, \tag{59}$$

where μ is the viscosity and u is the velocity of the fluid. By use of the above relations, one obtains

$$h_v = \frac{k\mu}{\rho V (v^2 \alpha)^{1/3}} \frac{\partial u}{\partial z}, \tag{60}$$

where v is the kinematic viscosity.

The velocity distribution of the fluid in the neighborhood of a rotating disk with the rotational speed of $\Omega(t)$ is known (e.g., reference [33]):

$$\frac{\partial u_r}{\partial z} = 0{\cdot}510 r\Omega \left(\frac{\Omega}{v}\right)^{1/2}, \qquad \frac{\partial u_\theta}{\partial z} = -0{\cdot}6159 r\Omega \left(\frac{\Omega}{v}\right)^{1/2}.$$

Since

$$\frac{\partial u}{\partial z} = \left[\left(\frac{\partial u_r}{\partial z}\right)^2 + \left(\frac{\partial u_\theta}{\partial z}\right)^2\right]^{1/2} = 0{\cdot}8 r\Omega \left(\frac{\Omega}{v}\right)^{1/2} = 0{\cdot}8 V \left(\frac{\Omega}{v}\right)^{1/2},$$

the approximate expression for h_v is

$$h_v = \frac{0{\cdot}8 k\mu \Omega^{1/2}}{\rho v^{7/6} \alpha^{1/3}}. \tag{61}$$

The initial-boundary value problem in the strong form is stated as follows. Given f_s, g, h_s, H_b, T_0 and T_∞, find $T : \bar{A} \to \Re$ such that

$$\begin{aligned} \rho hc(\dot{T} + V_i T_{,i}) + \nabla \cdot \mathbf{q} &= f \qquad \text{on } A \times (0,\ t), \\ T &= g \qquad \text{on } \Gamma_g \times (0, t), \\ -\mathbf{q} \cdot \mathbf{n} &= H_b \quad\ \text{on } \Gamma_h \times (0, t), \\ T(\mathbf{x}, 0) = T_0(\mathbf{x}), &\qquad \mathbf{x} \in A, \end{aligned} \tag{62}$$

where f and H_T are given as in equations (46) and (48), and V_i is the velocity of the medium expressed in index notation. Density, ρ, thickness, h, and capacity, c, are all assumed positive functions of $\mathbf{x} \in \mathbf{A}$. The heat flux vector $\mathbf{q}$ is given by the generalized Fourier's law

$$\mathbf{q} = -\mathbf{K} \cdot \nabla T \quad \text{or} \quad q_i = -K_{ij} T_{,j}, \tag{63}$$

where $K_{ij} = hk_{ij}$ denotes the conductivity tensor and k_{ij} is the heat conductivity.

The corresponding weak form is obtained by multiplying equation (62) by a weighting function $\tilde{T}$ and integrating over the domain

$$\int_A \tilde{T}\,[\rho hc(\dot{T} + V_i T_{,i}) - \nabla\cdot(\mathbf{K}\cdot \nabla T)]\,\mathrm{d}A = \int_A \tilde{T} f\,\mathrm{d}A.$$

Defining

$$\begin{aligned} S^n &= \{\mathbf{u}\,|\,u_i = g_i \text{ on } \Gamma_{g_i},\ u_i \in H^1(A),\ i = 1, \ldots, n\}, \\ V^n &= \{\mathbf{v}\,|\,v_i = 0 \text{ on } \Gamma_{g_i},\ v_i \in H^1(A),\ i = 1, \ldots, n\}, \end{aligned} \tag{64}$$

then by use of the divergence theorem, one obtains the following weak form.

Given f_s, Q_b, g, h_s, H_b, T_0, and T_∞, find $T \in S^1\colon \bar{A} \to \Re$ such that, for all $\tilde{T} \in V^1$,

$$\begin{aligned} &\int_A [\tilde{T}\rho hc(\dot{T} + V_i T_{,i}) + \nabla\tilde{T}\cdot\mathbf{K}\cdot\nabla T + \tilde{T} h_s T]\,\mathrm{d}A + \int_{\Gamma_h} \tilde{T} H_b T\,\mathrm{d}\Gamma \\ &\quad= \int_A [\tilde{T}(f_s + h_s T_0)]\,\mathrm{d}A + \int_{\Gamma_h} [\tilde{T}(Q_b + H_b T_\infty)]\,\mathrm{d}\Gamma. \end{aligned} \tag{65}$$

By using the proper shape functions

$$\tilde{T} = \tilde{T}_\alpha N_\alpha, \qquad T = T_\alpha N_\alpha,$$

it follows that

$$\begin{aligned} &\left[\int_A N_\alpha \rho hc N_\beta\,\mathrm{d}A\right]\dot{T}_\beta \\ &\quad+ \left[\int_A (\nabla N_\alpha\cdot\mathbf{K}\cdot\nabla N_\beta + N_\alpha \rho hc V_i N_{\beta,i} + N_\alpha h_s N_\beta)\,\mathrm{d}A + \int_{\Gamma_h} N_\alpha H_b N_\beta\,\mathrm{d}\Gamma\right] T_\beta \\ &= \int_A N_\alpha(f_s + h_s T_0)\,\mathrm{d}A + \int_{\Gamma_h} N_\alpha(Q_b + H_b T_\infty)\,\mathrm{d}\Gamma. \end{aligned} \tag{66}$$

The corresponding matrix form is stated as follows. Given f_s, Q_b, g, h_s, H_b, h, T_0 and T_∞, find $T \in S\colon \bar{A} \to \Re$ such that

$$\mathbf{M}_T\dot{\mathbf{T}} + \mathbf{K}_T\mathbf{T} = \mathbf{F}_T, \tag{67}$$

where

$$M_{T\alpha\beta} = \int_A N_\alpha \rho hc N_\beta\,\mathrm{d}A,$$

$$K_{T\alpha\beta} = \int_A [\nabla N_\alpha\cdot K\cdot\nabla N_\beta + N_\alpha h_s N_\beta + N_\alpha \rho hc V_i N_{\beta,i}]\,\mathrm{d}A + \int_{\Gamma_h} N_\alpha H_b N_\beta\,\mathrm{d}\Gamma,$$

$$F_{T\alpha} = \int_A N_\alpha(f_s + h_s T_0)\,\mathrm{d}A + \int_{\Gamma_h} N_\alpha(Q_b + H_b T_\infty)\,\mathrm{d}\Gamma.$$

One can now solve for the temperature distribution.

Temperature distributions give rise to thermal strains. Let the orthotropic thermal expansion coefficients be α_{im},

$$\alpha = \{\alpha_{11}, \alpha_{22}, \alpha_{12}, \alpha_{23}, \alpha_{13}\}^{\mathrm{T}},$$

then the initial strains, ε^0_{im}, or the initial stresses, σ^0_{im}, caused by the temperature distribution are

$$\varepsilon^0_{im} = \alpha_{im} \nabla T, \qquad \sigma^0_{im} = \beta_{im} \nabla T, \tag{68}$$

where

$$\beta_{ij} = C_{ijkl}\alpha_{ki}, \qquad \beta_{iz} = C_{iziz} 2\alpha_{iz}.$$

5. THERMOELASTIC PLATE PROBLEM

The combined thermoelastic plate problem has now been formulated to include both the dynamics of the elastic plate and the heat conduction problem. From equations (67) and (38).

$$\mathbf{M}_T\dot{\mathbf{T}} = \mathbf{F}_T(t; \mathbf{z}) = -\,\mathbf{N}_T(t; \mathbf{z}), \qquad \mathbf{0} = \mathbf{F}_u(t; \mathbf{z}) = -\,\mathbf{N}_u(t; \mathbf{z}), \qquad \mathbf{M}_x\ddot{\mathbf{x}} = \mathbf{F}_x(t; \mathbf{z}) = -\,\mathbf{N}_x(t; \mathbf{z}), \tag{69}$$

where $\mathbf{z} = \{\mathbf{T}^{\mathrm{T}}, \mathbf{u}^{\mathrm{T}}, \mathbf{x}^{\mathrm{T}}, \dot{\mathbf{x}}^{\mathrm{T}}\}^{\mathrm{T}}$.

It is important to exactly compute the tangent stiffness and tangent damping matrices for non-linear problems, especially when used with unconditionally stable integration schemes. For the α-method, the tangent stiffness and the tangent damping matrices for the updated geometry, may be computed several times during the computation of one time step. Here, a “modified” α-method will be introduced for the sake of economy of computational effort.

Equations (69) may be expressed in the linearized form at an arbitrary point 0 as

$$\mathbf{M}_T\dot{\mathbf{T}} + \mathbf{K}_T\mathbf{T} = \mathbf{F}^0_T(t; \mathbf{z}), \qquad \mathbf{K}_u\mathbf{u} = \mathbf{F}^0_u(t; \mathbf{z}),$$

$$\mathbf{M}_x\mathbf{a} + \mathbf{C}_x\mathbf{v} + \mathbf{K}_x\mathbf{x} = \mathbf{F}^0_x(t; \mathbf{z}),$$

where

$$\mathbf{v} = \mathbf{x}, \qquad \mathbf{a} = \ddot{\mathbf{x}}, \qquad \mathbf{z}_0 = \{\mathbf{T}^{\mathrm{T}}_0, \mathbf{u}^{\mathrm{T}}_0, \mathbf{x}^{\mathrm{T}}_0, \dot{\mathbf{x}}^{\mathrm{T}}_0\}^{\mathrm{T}}, \tag{70}$$

$$\mathbf{K}_T = \left.\frac{\partial \mathbf{N}_T}{\partial \mathbf{T}}\right|_0, \qquad \mathbf{F}^0_T(t; \mathbf{z}) = \mathbf{F}_T(t; \mathbf{z}_0) - \mathbf{K}_T\mathbf{T},$$

$$\mathbf{K}_u = \left.\frac{\partial \mathbf{N}_u}{\partial \mathbf{u}}\right|_0, \qquad \mathbf{F}^0_u(t; \mathbf{z}) = \mathbf{F}_u(t; \mathbf{z}_0) - \mathbf{K}_u\mathbf{u},$$

$$\mathbf{K}_x = \left.\frac{\partial \mathbf{N}_x}{\partial x}\right|_0, \qquad \mathbf{C}_x = \left.\frac{\partial \mathbf{N}_v}{\partial \mathbf{v}}\right|_0, \qquad \mathbf{F}^0_x(t; \mathbf{z}) = \mathbf{F}_x(t; \mathbf{z}_0) - \mathbf{K}_x\mathbf{x} - \mathbf{C}_x\mathbf{v}.$$

Here the consistent (or tangent) system matrices have been moved to the left-hand side of equation (70) and the force vectors adjusted accordingly.

6. SOLUTION METHOD

Equation (70a) is of parabolic form. Using the predictor–corrector scheme, one can obtain the time response as

$$\begin{aligned}\tilde{\mathbf{T}}_{n+1} &= \mathbf{T}_n + (\mathbf{1} - \alpha)\Delta t\dot{\mathbf{T}}_n,\\ (\mathbf{M}_T + \alpha\Delta t\mathbf{K}_T)\dot{\mathbf{T}}_{n+1} &= \mathbf{F}_{n+1} - \mathbf{K}_T\tilde{\mathbf{T}}_{n+1},\\ \mathbf{T}_{n+1} &= \tilde{\mathbf{T}}_{n+1} + \alpha\Delta t\dot{\mathbf{T}}_{n+1}.\end{aligned} \tag{71}$$

For unconditional stability, it is necessary to choose $\alpha \geqslant 0{\cdot}5$.

The α-method (Hilber–Hulbert–Taylor method) can implement numerical damping without degrading the order of accuracy [34]. Equation (70c) is modified to

$$\mathbf{M}_x\mathbf{a}_{n+1} + (1+\alpha_1)\mathbf{C}_x\mathbf{v}_{n+1} - \alpha_1\mathbf{C}_x\mathbf{v}_n + (\mathbf{1}+\alpha_1)\mathbf{K}_x\mathbf{x}_{n+1} - \alpha_1\mathbf{K}_x\mathbf{x}_n = \mathbf{F}_{n+1+\alpha_1}. \tag{72}$$

If the parameters are chosen such that $\alpha_1 \in [-\frac{1}{3}, 0]$, $\gamma = (1-2\alpha_1)/2$ and $\beta = (1-\alpha_1)^2/4$, then a second order accurate, unconditionally stable scheme results.

The following "modified" α-method is used to get the dynamic response when temperature distribution, in-plane deflection, and out-of-plane deflection are all considered simultaneously. Notice that it is called a "modified" α-method since it does not compute the tangent matrices for every step. Here, the current tangent matrices are used j_{max} times to compute the deflections and the velocities, and then the matrices are updated. This reduces the computing time because one has to carry out the costly step of factorizing the system matrices whenever there are new sets of tangents. If one sets $j_{max} = 1$, then this scheme is exactly the α-method.

1. *Initial phase*: Set $n = 0$. Determine the coefficients

$$\begin{gathered}c_1 = 1+\alpha_1, \qquad c_2 = \gamma\Delta t, \qquad c_3 = \beta\Delta t^2,\\ c_4 = (1+\alpha_1)\gamma\Delta t, \qquad c_5 = (1+\alpha_1)\beta\Delta t^2, \qquad c_6 = (1+\alpha_1)\Delta t,\\ c_7 = (1+\alpha_1)(\tfrac{1}{2}-\beta)\Delta t^2, \qquad c_8 = (1+\alpha_1)(1-\gamma)\Delta t,\\ c_9 = (1+\alpha_1)\alpha\Delta t, \qquad c_{10} = \alpha\Delta t, \qquad c_{11} = (1+\alpha_1)(1-\alpha)\Delta t\end{gathered} \tag{73}$$

and the initial condition

$$\dot{\mathbf{T}}_0 = \mathbf{M}_T^{-1}[\mathbf{F}_T(0) - \mathbf{N}_T(\mathbf{T}_0)], \qquad \mathbf{a}_0 = \mathbf{M}_x^{-1}[\mathbf{F}_x(0) - \mathbf{N}_x(\mathbf{x}_0, \mathbf{v}_0)]. \tag{74}$$

If lumped mass matrices are used, then $\mathbf{M}$ is diagonal and its inversion is trivial.

2. *Predictor phase*: Set $n \leftarrow n+1$, and $t = (n-1)\Delta t + c_6$,

$$\mathbf{T}_1 = \tilde{\mathbf{T}} = \mathbf{T} + c_{11}\dot{\mathbf{T}}, \ \dot{\mathbf{T}} = 0,$$

$$\mathbf{u}_1 = \mathbf{u}, \tag{75}$$

$$\mathbf{x}_1 = \tilde{\mathbf{x}} = \mathbf{x} + c_6\mathbf{v} + c_7\mathbf{a}, \qquad \mathbf{v}_1 = \tilde{\mathbf{v}} = \mathbf{v} + c_8\mathbf{a}, \qquad \mathbf{a} = 0,$$

$$\mathbf{z}_1 = \{\mathbf{T}_1^{\mathrm{T}}, \mathbf{u}_1^{\mathrm{T}}, \mathbf{x}_1^{\mathrm{T}}, \mathbf{v}_1^{\mathrm{T}}\}^{\mathrm{T}}.$$

3. *Update and factorization of consistent matrices.* Set $j = 1$:

$$\mathbf{K}_T = \left.\frac{\partial \mathbf{N}_T}{\partial \mathbf{T}}\right|_1, \qquad \tilde{\mathbf{M}}_T = \mathbf{M}_T + c_9\mathbf{K}_T = \mathbf{L}_T\mathbf{U}_T,$$

$$\mathbf{K}_u = \left.\frac{\partial \mathbf{N}_u}{\partial \mathbf{u}}\right|_1 = \mathbf{L}_u\mathbf{U}_u, \tag{76}$$

$$\mathbf{K}_x = \left.\frac{\partial \mathbf{N}_x}{\partial \mathbf{x}}\right|_1, \qquad \mathbf{C}_x = \left.\frac{\partial \mathbf{N}_x}{\partial \mathbf{v}}\right|_1, \ \tilde{\mathbf{M}}_x = \mathbf{M}_x + c_4\mathbf{C}_x + c_5\mathbf{K}_x = \mathbf{L}_x\mathbf{U}_x.$$

4. *Corrector phase at* $t_{n+1+\alpha_1}$:

$$\Delta\mathbf{F}_T = \mathbf{F}_T(t; \mathbf{z}_1) - \mathbf{M}_T\dot{\mathbf{T}}, \qquad \Delta\mathbf{F}_x = \mathbf{F}_x(t; \mathbf{z}_1) - \mathbf{M}_x\mathbf{a}, \tag{77}$$

$$\Delta\dot{\mathbf{T}} = \tilde{\mathbf{M}}_T^{-1}\Delta\mathbf{F}_T, \qquad \mathbf{T}_1 = \mathbf{T}_1 + c_9\Delta\dot{\mathbf{T}}, \qquad \dot{\mathbf{T}} = \dot{\mathbf{T}} + \Delta\dot{\mathbf{T}},$$

$$\mathbf{u} = \mathbf{K}_u^{-1}\mathbf{F}_u(t; \mathbf{z}_1), \tag{78}$$

$$\Delta\mathbf{a} = \tilde{\mathbf{M}}_x^{-1}\Delta\mathbf{F}_x, \qquad \mathbf{x}_1 = \mathbf{x}_1 + c_5\Delta\mathbf{a}, \qquad \mathbf{v}_1 = \mathbf{v}_1 + c_4\Delta\mathbf{a}, \qquad \mathbf{a} = \mathbf{a} + \Delta\mathbf{a}.$$

5. *Convergence test*: If $e = \|\Delta\mathbf{a}\|_2 \leqslant$ error bound, go to Step 6. Else set $j \leftarrow j+1$, and if $j > j_{max}$ go to Step 3, otherwise go to Step 4.

6. *Update of states at* t_{n+1}:

$$\mathbf{T} = (\mathbf{T}_1 - \mathbf{T})/c_1 + \mathbf{T}, \qquad \mathbf{u} = (\mathbf{u}_1 - \mathbf{u})/c_1 + \mathbf{u},$$
$$\mathbf{v} = (\mathbf{v}_1 - \mathbf{v})/c_1 + \mathbf{v}, \qquad \mathbf{x} = (\mathbf{x}_1 - \mathbf{x})/c_1 + \mathbf{x}. \tag{79}$$

Go to Step 2 for the next time step.

7. SUMMARY AND CONCLUSIONS

A dynamic thermoelastic problem for rotating plates or disks subject to transverse loads, heat sources, and with anisotropic material properties is formulated. The formulation is based upon the Mindlin plate theory, the von Kármán strain expression, and the quasi-static stretching assumption. The existence of convective terms generates gyroscopic terms, unstabilizing effects in the stiffness matrix, and radial in-plane tension.

The resulting governing equations of motion are non-linear. The most important contribution of the non-linear governing equation is that it can capture the effect of in-plane deflection or stress on the transverse deflection. Other contributions of the higher order terms, for example the effect of transverse deflection on the in-plane deflection and the higher order terms in the equation of transverse deflection, were neglected.

By using the quasi-static stretching assumption, it was possible to significantly simplify the governing equations. At an equilibrium, consistent system matrices were obtained. These consistent matrices were used as the linearized version of the governing equation to solve the corresponding eigenvalue problem.

A time integration scheme, applicable for both the linear and non-linear governing equations were presented. The second order accurate implicit method is chosen in this study, because it allows a sufficiently large time increment based on the accuracy needed without causing a numerical stability problem. A companion paper (Part II) presents results of applications of this method to specially orthotropic rotating disks.

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APPENDIX A: MASS AND STIFFNESS MATRICES AND FORCE VECTORS

Substituting equations (29) into (33) using the finite-dimensional approximation of $\tilde{u}_i$, $\bar{u}_i$, $\tilde{w}$, $\bar{w}$, $\tilde{\theta}_\alpha$ and θ_α in the form

$$\tilde{u}_i = \tilde{u}_{ia} N_a, \qquad \tilde{w} = \tilde{w}_a N_a, \qquad \tilde{q}_i = \tilde{q}_{ia} M_a, \tag{A.1}$$

$$\bar{u}_j = \bar{u}_{j\beta} N_\beta, \qquad \bar{w} = \bar{w}_\beta N_\beta, \qquad \theta_j = \theta_{j\beta} M_\beta, \tag{A.2}$$

where N and M $(\alpha, \beta = 1, \dots, n)$ are the shape functions for the deflections and the rotations, respectively, one gets

$$\begin{aligned}
&\int_A [\tilde{u}_{i\alpha} N_{\alpha,j} h C_{ijkl} N_{\beta,1} \bar{u}_{k\beta} + \tilde{w}_\alpha N_{\alpha,i} h C_{iziz} (N_{\beta,i} w_\beta - M_\beta \theta_{i\beta}) \\
&+ \tilde{\theta}_{i\alpha} M_{\alpha,j} D C_{ijkl} M_{\beta,l} \theta_{k\beta} - \tilde{\theta}_{i\alpha} M_\alpha h C_{iziz} (N_{\beta,i} w_\beta - M_\beta \theta_{i\beta}) \\
&+ \tilde{w}_\alpha N_{\alpha,j} h C_{ijkl} N_{\beta,l} \bar{u}_{k\beta} + \tilde{u}_{i\alpha} N_{\alpha,j} h C_{ijkl} \tfrac{1}{2} N_{\beta,k} w_\beta N_{\gamma,l} w_\gamma \\
&+ \tilde{w}_\alpha N_{\alpha,j} h C_{ijkl} \tfrac{1}{2} N_{\beta,k} w_\beta N_{\gamma,l} w_\gamma N_{\delta,i}\, w_\delta]\,\mathrm{dA} = \int_A [\tilde{u}_{i\alpha} N_\alpha f_i \\
&+ \tilde{w}_\alpha N_\alpha f_z - \tilde{\theta}_{i\alpha} M_\alpha m_i]\,\mathrm{d}A + \tilde{u}_{i\alpha} P_{i\alpha} + \tilde{w}_\alpha P_{z\alpha} + \tilde{\theta}_{i\alpha} M_{i\alpha},
\end{aligned} \tag{A.3}$$

where $P_{i\alpha}$, $P_{z\alpha}$ and $M_{i\alpha}$ are computed from equation (36), and f_i, f_z and m_i are obtained using equation (30).

The mass matrices are given as

$$\mathbf{M}_u = [M_{ui\alpha j\beta}], \qquad \mathbf{M}_x = \left[\begin{array}{c|c} M_{\alpha\beta} & 0 \\ \hline 0 & M_{i\alpha j\beta} \end{array}\right], \tag{A.4}$$

where

$$M_{u\alpha i\beta j} = \int_A N_\alpha \rho h N_\beta \, \mathrm{d}A, \qquad M_{\alpha\beta} = \int_A N_\alpha(\rho h) N_\beta \, \mathrm{d}A,$$

$$M_{i\alpha i\beta} = \int_A N_\alpha(\rho D) N_\beta \, \mathrm{d}A, \qquad M_{x\alpha i\beta j} = \int_A N_\alpha \rho h \, N_\beta \, \mathrm{d}A,$$

and the force vectors are

$$\mathbf{F}_u^0 = \{F_{ui\alpha}^0\}, \qquad \mathbf{F}_x^0 = \begin{Bmatrix} F_{w\alpha}^0 \\ F_{\theta i\alpha}^0 \end{Bmatrix}, \tag{A.5}$$

where

$$F_{ui\alpha}^0 = P_{i\alpha} + F_{i\alpha}^{Ru} + \int_A N_\alpha f_i^0 \, \mathrm{d}A - \int_A N_{\alpha,j} h C_{ijkl} N_{\beta,l} \bar{u}_{k\beta} \, \mathrm{d}A$$

$$- \int_A N_{\alpha,j} h C_{ijkl} \frac{1}{2} N_{\beta,k} w_\beta N_{\gamma,l} w_\gamma \, \mathrm{d}A,$$

$$F_{w\alpha}^0 = P_{z\alpha} + F_\alpha^R + \int_A N_\alpha f_z^0 \, \mathrm{d}A$$

$$- \int_A [N_{\alpha,i} \, h C_{iziz}(N_{\beta,i} w_\beta - M_\beta \theta_{i\beta}) + N_{\alpha,j} h C_{ijkl} N_{\beta,l} \bar{u}_{k\beta}$$

$$+ N_{\alpha,j} h C_{ijkl} \tfrac{1}{2} N_{\beta,k} w_\beta N_{\gamma,l} w_\gamma N_{\delta,i} w_\delta] \, \mathrm{d}A,$$

$$F_{\theta i\alpha}^0 = M_{i\alpha} + F_{i\alpha}^R + \int_A [M_{\alpha,j} D c_{ijkl} M_{\beta,l} \theta_{k\beta} - M_\alpha h c_{iziz}(N_{\beta,i} w_\beta - M_\beta \theta_{i\beta})] \, \mathrm{d}A,$$

$$f_i^0 = h a_{ij} \dot{\bar{u}}_j + h b_{ij} \bar{u}_j + h c_i + \left[\sigma_{iz}\left(\frac{h}{2}\right) - \sigma_{iz}\left(-\frac{h}{2}\right)\right],$$

$$f_i^0 = 2h a_i \frac{\partial \dot{w}}{\partial x_i} + h b_i \frac{\partial w}{\partial x_i} + h d_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} + hc,$$

$$m_i^0 = D a_{ij} \theta_j + D b_{ij} \theta_j - \frac{h}{2}\left[\sigma_{iz}\left(\frac{h}{2}\right) + \sigma_{iz}\left(-\frac{h}{2}\right)\right],$$

$$F_{i\alpha}^{Fu} = \int_A h N_\alpha l_{,j} c_{ijkl} e_{kl}^0 \, \mathrm{d}A, \qquad F_\alpha^R = \int_A h N_{\alpha,i} c_{iziz} 2 e_{iz}^0 + h M_{\alpha,j} D c_{ijkl} \kappa_{kl}^0 \, \mathrm{d}A.$$

Inherent stiffness matrix components are

$$\mathbf{K}_u^0 = [K_{i\alpha k\beta}^{0u}], \qquad \mathbf{K}_x^0 = \left[\begin{array}{c|c} K_{\alpha\beta}^0 & K_{\alpha j\beta}^0 \\ \hline K_{i\alpha\beta}^0 & K_{i\alpha k\beta}^0 \end{array}\right], \tag{A.6, A.7}$$

where

$$K_{i\alpha k\beta}^{0u} = \int_A hN_{\alpha,j}C_{ijkl}N_{\beta,l}\,\mathrm{d}A, \qquad K_{\alpha\beta}^0 = \sum_{i=1}^{2}\int_A hN_{\alpha,i}C_{iziz}N_{\beta,i}\,\mathrm{d}A,$$

$$K_{\alpha j\beta}^0 = -\int_A hN_{\alpha,j}C_{iziz}M_\beta\,\mathrm{d}A, \qquad K_{i\alpha\beta}^0 = -\int_A hM_\alpha C_{iziz}M_\beta\,\mathrm{d}A,$$

$$K_{i\alpha j\beta}^0 = \int_A DM_{\alpha,k}C_{ijkl}M_{\beta,l}\,\mathrm{d}A + \delta_{ij}\int_A hM_\alpha\, C_{iziz}\, M_\beta\,\mathrm{d}A.$$

The geometric stiffness matrices are

$$\mathbf{K}_u^G = [0], \qquad \mathbf{K}_x^G = \left[\begin{array}{c|c} K_{\alpha\beta}^G & 0 \\ \hline 0 & 0 \end{array}\right], \tag{A.8, A.9}$$

where

$$K_{\alpha\beta}^G = \int_A N_{\alpha,i}(N_{ij} + hd_{ij})N_{\beta,j}\,\mathrm{d}A.$$

Stiffness matrix contributions due to body forces are

$$\mathbf{K}_u^F = [K_{i\alpha j\beta}^{Fu}], \qquad \mathbf{K}_x^F = \left[\begin{array}{c|c} K_{\alpha\beta}^F & 0 \\ \hline 0 & K_{i\alpha j\beta}^F \end{array}\right], \tag{A.10, A.11}$$

where

$$K_{i\alpha j\beta}^{Fu} = -\int_A N_\alpha(hb_{ij})N_\beta\,\mathrm{d}A, \qquad K_{\alpha\beta}^F = -\int_A N_\alpha(hb_i)N_{\beta,i}\,\mathrm{d}A, \qquad K_{i\alpha j\beta}^F = -\int_A N_\alpha\,(Db_{ij})N_\beta\,\mathrm{d}A.$$

The gyroscopic matrix $\mathbf{G}_x$ is given by

$$\mathbf{G}_x = \left[\begin{array}{c|c} G_{\alpha\beta} & 0 \\ \hline 0 & G_{i\alpha j\beta} \end{array}\right], \tag{A.12}$$

where

$$G_{\alpha\beta} = \int_A ha_i(N_{\alpha,i}N_\beta - N_\alpha N_{\beta,i})\,\mathrm{d}A, \qquad G_{i\alpha j\beta} = -\int_A N_\alpha(Da_{ij})N_\beta\,\mathrm{d}A.$$

The mass matrix is

$$\mathbf{M}_x = \left[\begin{array}{c|c} M_{\alpha\beta} & 0 \\ \hline 0 & M_{i\alpha j\beta} \end{array}\right], \tag{A.13}$$

where

$$M_{\alpha\beta} = \int_A N_\alpha(\rho h) N_\beta \, \mathrm{d}A, \qquad M_{i\alpha j\beta} = \int_A N_\alpha(\rho D) N_\beta \, \mathrm{d}A.$$

The force vectors are represented by

$$\mathbf{F}_u = \mathbf{F}_u^F + \mathbf{F}_u^B + \mathbf{F}_u^R, \qquad \mathbf{F}_x = \mathbf{F}_x^F + \mathbf{F}_x^B + \mathbf{F}_x^R. \tag{A.14}$$

The first terms comes from the effect of body forces,

$$\mathbf{F}_u^F = \{F_{i\alpha}^{Fu}\}, \qquad \mathbf{F}_x^F = \left\{\begin{array}{c} F_\alpha^F \\ \hline F_{i\alpha}^F \end{array}\right\}, \tag{A.15, A.16}$$

where

$$F_{i\alpha}^{Fu} = \int_A N_\alpha \left\{ hc_i + \left[\sigma_{iz}\left(\frac{h}{2}\right) - \sigma_{iz}\left(-\frac{h}{2}\right)\right]\right\} \mathrm{d}A,$$

$$F_\alpha^F = \int_A N_\alpha [hc] \, \mathrm{d}A,$$

$$F_{i\alpha}^F = -\int_A N_\alpha \left(\frac{h}{2}\right)\left[\sigma_{iz}\left(\frac{h}{2}\right) + \sigma_{iz}\left(-\frac{h}{2}\right)\right] \mathrm{d}A.$$

The boundary force vector is

$$\mathbf{F}_u^B = \{P_{ia}\}, \qquad \mathbf{F}_x^B = \left\{\begin{array}{c} P_{z\alpha} \\ \hline M_{i\alpha} \end{array}\right\}. \tag{A.17, A.18}$$

The force vectors owing to initial displacements are

$$\mathbf{F}_u^R = \{F_{ia}^{Ru}\}, \qquad \mathbf{F}_x^R = \left\{\begin{array}{c} F_\alpha^R \\ \hline F_{i\alpha}^R \end{array}\right\}, \tag{A.19, A.20}$$

where

$$F_{i\alpha}^{Ru} = K_{i\alpha j\beta}^0 u_{j\beta}^0, \qquad F_\alpha^R = K_{\alpha\beta}^0 w_\beta^0 - K_{\alpha j\beta}^0 \theta_{j\beta}^0, \qquad F_{i\alpha}^R = -K_{i\alpha\beta}^0 w_\beta^0 + K_{i\alpha j\beta}^0 \theta_{j\beta}^0.$$

If initial strains are given, instead of initial displacements, then the force vector components are computed from

$$F_{i\alpha}^{Ru} = \int_A hN_{\alpha,j} C_{ijkl} e_{kl}^0 \, \mathrm{d}A, \qquad F_\alpha^R = \int_A hN_{\alpha,i} C_{iziz} 2e_{iz}^0 \, \mathrm{d}A,$$

$$F_{i\alpha}^R = \int_A [-hM_\alpha C_{iziz} 2e_{iz}^0 + hM_{\alpha,j} DC_{ijkl} \kappa_{kl}^0] \, \mathrm{d}A.$$

APPENDIX B: HOMOGENIZED ELASTICITY TENSOR

If a material is not isotropic, say a laminate or a composite, then one has to determine the directional properties of the material to use in a constitutive relation. Experiment is one way to determine such properties when the material of interest is readily available. In most cases, however, it is not desirable and sometimes not possible to do an experiment. If one knows the (isotropic) material properties of each constituent for the laminate or the composite and the geometry, then it is possible to compute "averaged" properties by examining the local structure of the components. Although there are conventional ways to obtain the "averaged" properties (e.g., by the rule of mixtures) these methods are lacking in their ability to assess the differences that arise due to the shapes of each constituent.

One possibility to avoid the need for "averaged" properties is to solve for the deformation, or the stress, in sufficiently small elements. This method involves extensive computing effort, due to the minute scale of the elements. Another possibility is to choose a typical "unit cell", and by discretizing it into very small elements and imposing the conditions which must be satisfied by the unit cell, to generate global properties. The scale of the elements is still very small, but the domain is only the volume of the unit cell, hence in many cases it gives an effective and systematic way to obtain the homogenized properties. The formulation of the problem which leads to the explicit expression for the homogenized properties will be explained in this section.

Suppose that a body is made up of several different materials whose mixture is formed by the spatial repetition of a unit cell of very small order compared to the dimensions of the structural body. A composite material having a unit cell composed of, for example, two different materials A and B is illustrated in Figures B1 and B2.

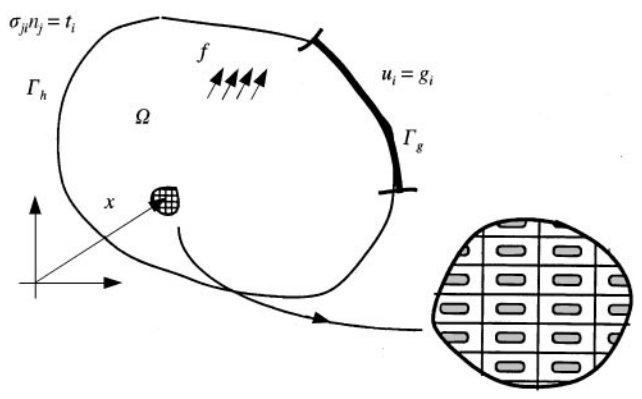

Figure B1. Composite material.

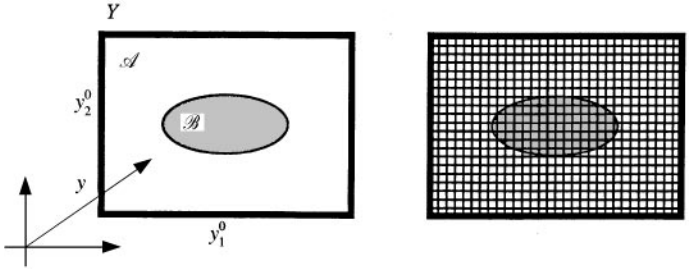

Figure B2. Unit cell.

Due to the existence of the microstructure, the whole domain V is very inhomogeneous. As a result, when this body is subject to external force f and boundary conditions, the resulting deflection, or the stress, abruptly varies from location to location. In order to obtain the deflection or stress distribution with reasonable accuracy, it is needed to have a tremendously large number of discrete elements, sufficient to cover the details of the microstructure. Thus, it is very useful to develop a method capable of reflecting the existence of the microstructure during computation of the macroscopic behavior of the body without considering the microscopic details of each material point. The homogenization method [35] is used here to characterize the equivalent elasticity tensor for macroscopic analysis. The microscopic behavior in the unit cell can also be approximately computed by postprocessing a macroscopic response. Note that the homogenization process is not a simple averaging with respect to the volume ratio of the composing materials. This method reflects not only the amounts but also the shapes of the constituents.

It is reasonable to assume that all quantities have two explicit dependencies, i.e., letting ϕ be a general function

$$\phi^{\varepsilon} = \phi^{\varepsilon}(\mathbf{x}, \mathbf{y}), \tag{B.1}$$

where the ε-superscripted quantities reflect the existence of the microstructure characterized by ε, and ε is a very small positive number representing the ratio of the macroscopic level $\mathbf{x}$ and microscopic level $\mathbf{y}$, i.e.,

$$\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}. \tag{B.2}$$

The dependency on $\mathbf{y}$ means that a quantity varies within a very small region with dimensions much smaller than those of the macroscopic level.

Let $V \subset \Re^n$ be an open set with a smooth boundary Γ. Let the domain of a unit cell be Y,

$$Y = \prod_{i=1}^{n} [0, y_i^0], \tag{B.3}$$

as shown in Figure B2. There are two open sets, namely, A for the material part A and B for the material part B. Since the microstructure is Y-periodic, the microscopic level y is also Y-periodic. Therefore, it is reasonable to assume that the dependency of all quantities on y is Y-periodic, i.e.,

$$\phi^{\varepsilon}(\mathbf{x}, \mathbf{y}) = \phi^{\varepsilon}(\mathbf{x}, \mathbf{y} + \mathbf{y}^0), \tag{B.4}$$

where

$$\mathbf{y}^0 = \begin{Bmatrix} y_1^0 \\ \vdots \\ y_n^0 \end{Bmatrix}. \tag{B.5}$$

The governing equation for the elasticity problem is stated as

$$\begin{gathered} \sigma_{ji,j}^{\varepsilon} + f_i^{\varepsilon} = \rho^{\varepsilon} \frac{\partial^2 u_i^{\varepsilon}}{\partial t^2} \quad \text{in } V \\ \sigma_{ji} n_j = t_i \quad \text{on } \Gamma_{h_i}, \qquad u_i = g_i \quad \text{on } \Gamma_{g_i}, \end{gathered} \tag{B.6}$$

where the stress $\sigma^{\varepsilon}_{ij}$ is

$$\sigma^{\varepsilon}_{ij} = C^{\varepsilon}_{ijkl}\varepsilon^{\varepsilon}_{kl}$$

and the elasticity tensor C^{ε}_{ijkl} satisfies major and minor symmetries and positive definiteness:

$$C^{e}_{ijkl} = C^{e}_{klij} = C^{e}_{jikl} = C^{e}_{ijlk},$$

$$\exists\alpha > 0 : C^{e}_{ijkl}\varepsilon_{ij}\varepsilon_{kl} = \alpha\varepsilon_{ij}\varepsilon_{ij} \quad \forall\varepsilon_{ij} = \varepsilon_{ji}$$

and is defined as

$$C^{\varepsilon}_{ijkl} = \begin{cases} C^{A}_{ijkl} & \text{if } y \in \mathscr{A}, \\ C^{B}_{ijkl} & \text{if } y \in \mathscr{B}. \end{cases} \tag{B.7}$$

Note that C^{ε}_{ijkl} is not continuous and that equation (105) must be satisfied in a weak sense. Let

$$\mathscr{S} = [\mathbf{u} \,|\, u_i = g_i \text{ on } \Gamma_{g_i}\ u_i \in \mathscr{H}^1(\mathrm{V})], \tag{B.8}$$

$$\mathscr{V} = \{\mathbf{v} \,|\, v_i = 0 \text{ on } \Gamma_{g_i}\ v_i \in \mathscr{H}^1(V)\}, \tag{B.9}$$

where $H^1(V)$ denotes the Sobolev space of the first kind.

To get the weak form the governing equations are multiplied by a weighting function v_i and integrated over the domain:

$$\int_V v^{\varepsilon}_i(\sigma^{\varepsilon}_{ji,j} + f^{\varepsilon}_i)\,\mathrm{d}V = \int_V v^{\varepsilon}_i\left(\frac{\partial^2 u^{\varepsilon}_i}{\partial t^2}\right)\mathrm{dV}.$$

By use of the divergence theorem, one obtains the following weak form.

Find $\mathbf{u} \in S(V)$ such that

$$\int_V C_{ijkl}\frac{\partial u^{\varepsilon}_k}{\partial x_l}\frac{\partial v^{\varepsilon}_i}{\partial x_j}\,\mathrm{d}V = \int_V f^{\varepsilon}_i v^{\varepsilon}_i\,\mathrm{d}V + \int_{\Gamma_h} t^{\varepsilon}_i v^{\varepsilon}_i\,\mathrm{d}V + \int_V \rho^{\varepsilon}\frac{\partial^2 u^{\varepsilon}_i}{\partial t^2}v^{\varepsilon}_i\,\mathrm{d}V, \quad \forall v \in \mathscr{V}. \tag{B.10}$$

Expanding the expressions for $\mathbf{u}^{\varepsilon}$ and $\mathbf{v}^{\varepsilon}$ asymptotically,

$$\mathbf{u}^{\varepsilon} = \mathbf{u}^0(\mathbf{x}) + \varepsilon\mathbf{u}^1(\mathbf{x}, \mathbf{y}) + \varepsilon^2(\cdots) + \cdots, \qquad \mathbf{v}^{\varepsilon} = \mathbf{v}^0(\mathbf{x}) + \varepsilon\mathbf{v}^1(\mathbf{x}, \mathbf{y}) + \varepsilon^2(\cdots) + \cdots$$

and using

$$\frac{\partial\phi(x, x/\varepsilon)}{\partial x_i} = \frac{\partial\phi}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial\phi}{\partial y_i}\bigg|_{y = x/\varepsilon}$$

one obtains

$$\begin{aligned}\int_{V^{\varepsilon}} C^{\varepsilon}_{ijkl}\frac{\partial u^{\varepsilon}_k}{\partial x_l}\frac{\partial v^{\varepsilon}_i}{\partial x_j}\,\mathrm{d}V &= \int_{V^{\varepsilon}} C^{\varepsilon}_{ijkl}\left[\frac{\partial u^0_k}{\partial x_l}\frac{\partial v^0_i}{\partial x_j} + \frac{\partial u^0_k}{\partial x_l}\frac{\partial v^1_i}{\partial y_j}\right] \\ &- \frac{\partial u^1_k}{\partial y_l}\frac{\partial v^0_i}{\partial x_j} + \frac{\partial u^1_k}{\partial y_l}\frac{\partial v^1_i}{\partial y_j} + \varepsilon(\cdots)\Bigg]\,\mathrm{d}V = \int_{V^{\varepsilon}} f^{\varepsilon}_i(v^0_i + \varepsilon v^1_i)\,\mathrm{d}V \\ &+ \int_{\Gamma_h} t_i(v^0_i + \varepsilon v^1_i)\,\mathrm{d}\Gamma + \int_{V^{\varepsilon}} \rho^{\varepsilon}\frac{\partial^2 u^0_i}{\partial t^2}v^0_i + \varepsilon(\cdots)\,\mathrm{d}V, \quad \forall v^0_i.\end{aligned} \tag{B.11}$$

Note that the choice of $\mathbf{v}$ is arbitrary. Assuming $\mathbf{v}^0 = \mathbf{0}$ yields

$$\int_{v^\varepsilon} C^\varepsilon_{ijkl}\left(\frac{\partial u^0_k}{\partial x_l}\frac{\partial v^1_i}{\partial y_j} + \frac{\partial u^1_k}{\partial y_l}\frac{\partial v^1_i}{\partial y_j}\right)\mathrm{d}V + \varepsilon(\cdots) = \varepsilon(\cdots), \quad \forall v^1. \tag{B.12}$$

Assuming $\mathbf{v}^1 = \mathbf{0}$,

$$\int_{V^\varepsilon} C^\varepsilon_{ijkl}\left(\frac{\partial u^0_k}{\partial x_l}\frac{\partial v^0_i}{\partial x_j} + \frac{\partial u^1_k}{\partial y_l}\frac{\partial v^0_i}{\partial x_j}\right)\mathrm{d}V = \int_{V^\varepsilon} f^\varepsilon_i v^0_i\,\mathrm{d}V + \int_{\Gamma_h} t_i v^0_i\,\mathrm{d}\Gamma + \int_{V^\varepsilon} \rho^\varepsilon \frac{\partial^2 u^0_i}{\partial t^2} v^0_i\,\mathrm{d}V, \quad \forall \mathbf{v}^0. \tag{B.13}$$

When $\phi(\cdot, \mathbf{y})$ is Y-periodic in $\mathbf{y}$,

$$\lim_{\varepsilon\to 0}\int_{V^\varepsilon}\phi\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right)\mathrm{d}V = \int_V\left(\frac{1}{|\mathbf{Y}|}\int_{\mathbf{Y}}\phi(\mathbf{x}, \mathbf{y})\,\mathrm{d}\mathbf{Y}\right)\mathrm{d}V.$$

Taking the $\lim_{\varepsilon\to 0}$ for equation (B.12) gives

$$\int_V\left(\frac{1}{|\mathbf{Y}|}\int_Y C^\varepsilon_{ijkl}\left(\frac{\partial u^0_k}{\partial x_l}\frac{\partial v^1_i}{\partial y_j} + \frac{\partial u^1_k}{\partial y_l}\frac{\partial v^1_i}{\partial y_j}\right)\mathrm{d}\mathbf{Y}\right)\mathrm{d}\mathbf{V} = 0, \quad \forall v^1_i \in \mathscr{V}, \tag{B.14}$$

Taking the $\lim_{\varepsilon\to 0}$ for equation (B.13) gives

$$\begin{aligned}\int_V\Bigg[&\left(\frac{1}{|\mathbf{Y}|}\int_Y C^\varepsilon_{ijkl}\left(\frac{\partial u^0_k}{\partial x_l}\frac{\partial v^0_i}{\partial x_j}\right)\mathrm{d}Y\right)\\ &+\left(\frac{1}{|\mathbf{Y}|}\int_Y C^\varepsilon_{ijkl}\frac{\partial u^1_k}{\partial y_l}\mathrm{d}\mathbf{Y}\right)\frac{\partial v^0_i}{\partial x_j}\Bigg]\mathrm{d}V = \int_{V^\varepsilon} f^\varepsilon_i\, v^0_i\,\mathrm{d}V\\ &+\int_{\Gamma_h} t_i v^0_i\,\mathrm{d}\Gamma + \int_{V^\varepsilon}\rho^\varepsilon\frac{\partial^2 u^0_i}{\partial t^2}v^0_i\,\mathrm{d}V, \quad \forall v^0_i \in \mathscr{V}.\end{aligned} \tag{B.15}$$

Using the separation of variables technique

$$u^1_i(\mathbf{x}, \mathbf{y}) = -\,\chi^{pq}_i(\mathbf{y})\frac{\partial u^0_p}{\partial x_q}(\mathbf{x}), \tag{B.16}$$

one obtains the following weak form for the characteristic function, χ:

$$\int_Y C^\varepsilon_{ijkl}\frac{\partial \chi^{pq}_k}{\partial y_l}\frac{\partial v^1_i}{\partial y_j}\,\mathrm{d}\mathbf{Y} = \int_Y C^\varepsilon_{ijpq}\frac{\partial v^1_i}{\partial y_j}\,\mathrm{d}Y, \quad \forall v^1_i \in \mathscr{V}. \tag{B.17}$$

By inserting equation (B.16) into equation (B.15), the expression for the weak form, for the global behavior, is obtained:

$$\begin{aligned}\int_V\Bigg[&\left(\frac{1}{|\mathbf{Y}|}\int_Y \mathbf{C}^\varepsilon_{ijkl}\left(\frac{\partial u^0_k}{\partial x_l} - \frac{\partial \chi^{pq}_k}{\partial y_l}\frac{\partial u^0_p}{\partial x_q}\right)\frac{\partial v^0_i}{\partial x_j}\,\mathrm{d}\mathbf{Y}\right)\mathrm{d}V = \int_{\Gamma_h} t_i v^0_i\,\mathrm{d}\Gamma\\ &+\int_V\left(\frac{1}{|\mathbf{Y}|}\int_Y f^\varepsilon_i\,\mathrm{d}\mathbf{Y}\right)v^0_i\,\mathrm{d}V + \int_V\left(\frac{1}{|\mathbf{Y}|}\int_Y \rho^\varepsilon\,\mathrm{d}Y\right)\frac{\partial^2 u^0_i}{\partial t^2}v^0_i\,\mathrm{d}V,\end{aligned} \tag{B.18}$$

$$\forall \mathbf{v}^0 \in \mathbf{V}(V)$$

or

$$\int_V C^H_{ijpq} \frac{\partial u^0_p}{\partial x_q} \frac{\partial v^0_i}{\partial x_j} \mathrm{d}V = \int_{\Gamma_h} t_i v^0_i \,\mathrm{d}\Gamma + \int_V \left(\frac{1}{|\mathbf{Y}|} \int_{\mathbf{Y}} f^\varepsilon_i \,\mathrm{d}Y \right) v^0_i \,\mathrm{d}V$$
$$+ \int_V \left(\frac{1}{|\mathbf{Y}|} \int_{\mathbf{Y}} \rho^\varepsilon \,\mathrm{d}Y \right) \frac{\partial^2 u^0_i}{\partial t^2} v^0_i \,\mathrm{d}V, \quad \forall \mathbf{v}^0 \in \mathrm{V}(V), \tag{B.19}$$

where

$$C^H_{ijpq} = \left(\frac{1}{|\mathbf{Y}|} \int_{\mathbf{Y}} \left[C^\varepsilon_{ijpq} - C^\varepsilon_{ijkl} \frac{\partial \chi^{pq}_k}{\partial y_l} \right] \mathrm{d}Y \right) \tag{B.20}$$

is the homogenized elasticity tensor.