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GENERALIZED HARMONIC OSCILLATORS

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The main purpose of this Letter is to introduce a new class of non-linear oscillator equations which model generalized one-dimensional harmonic oscillators and provide a preliminary analysis of some of their important properties. These equations take the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\,g(y)x, \tag{1}$$

where $f(x)$ and $g(y)$ are assumed to be continuous with continuous first derivatives, and also satisfy the conditions

$$f(0) > 0, \qquad g(0) > 0. \tag{2}$$

If $f(x)$ and $g(x)$ are positive constants, then equation (1) reduces to the system equations for the linear harmonic oscillator [1, 2]. The special case, where $f(x) = 1$, has previously been studied by Mickens and Jackson [3]. For this case

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\,g(y)x, \tag{3}$$

and the corresponding second order, non-linear differential equation is [4]

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + g\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)x = 0. \tag{4}$$

A number of important dynamical systems can be modelled by equation (1); see the references in reference [3]. Also, a linear transformation of the Lotka–Volterra equations from population dynamics puts them in this form [4].

In the (x, y) phase space [5], the trajectories are determined by the first order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\left[\frac{g(y)x}{f(x)y}\right]. \tag{5}$$

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The fixed-points or equilibria $(\bar{x}, \bar{y})$ correspond to the simultaneous solutions of the equations

$$g(\bar{y})\bar{x} = 0, \qquad f(\bar{x})\bar{y} = 0. \tag{6}$$

For all $g(y)$ and $f(x)$ of interest, $(\bar{x}, \bar{y}) = (0, 0)$ is always a fixed-point. Let the zeros of $f(x)$ and $g(y)$ be simple and denoted, respectively, by

$$\{\bar{x}^{(i)}: i = 1, 2, \ldots, I\}, \qquad \{\bar{y}^{(j)}: j = 1, 2, \ldots, J\}. \tag{7}$$

It follows that other fixed-points occur at

$$\{\bar{x}^{(i)}, \bar{y}^{(j)}\}, \quad i = 1, 2, \ldots, I, \quad j = 1, 2, \ldots, J. \tag{8}$$

Thus, equation (1) has a total of $(IJ + 1)$ fixed-points.

The conditions given in equations (2) lead to the conclusion that the fixed-point, $(\bar{x}, \bar{y}) = (0, 0)$, is a center. However, a detailed, but preliminary, examination of the other fixed-points, using qualitative (geometrical) methods from the theory of differential equations [5, 6] indicate that the other IJ fixed-points are either saddle points or nodes. If, in fact, this result holds in general, it can be concluded that the generalized harmonic oscillator, defined by equations (1) and (2), only has periodic motions about the fixed-point $(\bar{x}, \bar{y}) = (0, 0)$.

It should be observed that equations (1) can be rewritten in the form of a single second order differential equation. To see this, note that

$$\frac{d^2x}{dt^2} = f(x)\frac{dy}{dt} + f'(y)y\frac{dx}{dt}, \tag{9}$$

where $f'(y) \equiv df/dy$. Using the fact that

$$y = \frac{1}{f(x)}\frac{dx}{dt}, \tag{10}$$

and the second of equations (1), equation (9) takes the form

$$\frac{d^2x}{dt^2} - \left[\frac{f'(x)}{f(x)}\left(\frac{dx}{dt}\right)^2 + f(x)g\left\{\frac{1}{f(x)}\frac{dx}{dt}\right\}x = 0. \tag{11}$$

This is a rather complicated second order, non-linear differential equation. However, it does have a first-integral [2]. This function can be determined by integrating equation (5) which is separable; doing this gives

$$K(y) + V(x) = \mathit{constant}, \tag{12}$$

where

$$K(y) = \int^y \frac{z\,dz}{g(z)}, \qquad V(x) = \int^x \frac{w\,dw}{f(w)}. \tag{13}$$

The first-integral, given in equation (12), has a structure similar to that for a conservative mechanical system in which the first-integral is the energy [2, 5]. Consequently, $K(y)$ and $V(x)$ can be taken, respectively, as the generalized kinetic and potential energies for the generalized harmonic oscillator described by either equations (1) or (11). Such a correspondence has already been made, as indicated earlier, for the special case where $f(x) = 1$ [13], i.e.,

$$\frac{d^2x}{dt^2} + g\left(\frac{dx}{dt}\right)x = 0. \tag{14}$$

The existence of a first-integral allows the following conclusions to be drawn: for given functional forms for $f(x)$ and $g(y)$, the trajectories in phase space are given by

$$H(x, y) = K(y) + V(x) = constant, \tag{15}$$

where the "constant" is determined by the initial conditions (x_0, y_0). Closed curves thus correspond to periodic solutions [5, 6]. Note, however, that our preliminary studies indicate that closed trajectories will only occur in the neighborhood of the fixed-point at $(\bar{x}, \bar{y}) = (0, 0)$.

The question now is how to obtain analytical information on the periodic solutions about the fixed-point at the origin? To start, perturbation methods can be used to construct approximations to the periodic solutions for infinitesimal initial conditions [1, 5]. However, in most applications where generalized harmonic oscillations occur, finite motions in a finite domain about the origin are needed. One possibility is to use the method of harmonic balance [5], although it is expected that many terms will be needed to yield an accurate representation of both the period and the time dependence of the oscillatory behavior [7].

In summary, a new class of non-linear oscillators has been introduced and several of their important properties have been indicated. Future work will entail the detail analysis of a particular example. The simplest non-trivial case corresponds to $f(x)$ and $g(y)$ both having two simple zeros in their respective variables.

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