Journal of Sound and Vibration (2000) **236**(5), 895–900
doi:10.1006/jsvi.1999.2727, available online at http://www.idealibrary.com on IDEAL®

LETTERS TO THE EDITOR

THE LO THEORY FOR SHELLS

H. Hassis

Ecole Nationale d'ingénieurs de Tunis-Laboratoire Modélisation et Calcul de Structures, B.P. 37 *Le Belvédère* 1002 *Tunis, Tunisia*

(*Received* 2 *June* 1999)

1. INTRODUCTION

Lo *et al.* [1] presented a high order theory for plates. It is appropriated to the following displacement forms:

$$U_1(x^1, x^2, x^3) = u_1(x^1, x^2) + x^3\beta_1(x^1, x^2) + (x^3)^2\zeta_1(x^1, x^2) + (x^3)^3\phi_1(x^1, x^2),$$

$$U_2(x^1, x^2, x^3) = u_2(x^1, x^2) + x^3\beta_2(x^1, x^2) + (x^3)^2\zeta_2(x^1, x^2) + (x^3)^3\phi_2(x^1, x^2),$$

$$U_3(x^1, x^2, x^3) = u_3(x^1, x^2) + x^3\beta_3(x^1, x^2) + (x^3)^2\zeta_3(x^1, x^2), \tag{1}$$

where (x^1, x^2) are the surface co-ordinates and x^3 is the normal co-ordinate to the surface.

The Lo theory, proposed for plates, is here extended to shell structures.

2. DEFORMATION TENSOR–STRESS TENSOR–STRESS RESULTANT

2.1. DEFORMATION TENSOR

Using the development presented in reference [2], and for small displacement, the deformation tensor associated with equation (1) is written as

$$\varepsilon_{\alpha\beta} = \gamma_{\alpha\beta}(\mathbf{u}_\omega) + x^3\gamma_{\alpha\beta}(\boldsymbol{\beta}) - x^3\rho_{\alpha\beta}(\mathbf{u}_\omega) + (x^3)^2\gamma_{\alpha\beta}(\boldsymbol{\zeta}) + (x^3)^3\gamma_{\alpha\beta}(\boldsymbol{\phi}),$$

$$2\varepsilon_{\alpha 3} = (\beta_\alpha + u_\lambda C_\alpha^\lambda + u_{3,\alpha}) + (x^3\beta_{3,\alpha} + (x^3)^2\zeta_{3,\alpha})$$

$$+ (2x^3\zeta_\alpha + 3(x^3)^2\phi_\alpha + (x^3)^2\zeta_\lambda C_\alpha^\lambda + (x^3)^3\phi_\lambda C_\alpha^\lambda),$$

$$\varepsilon_{33} = \beta_3 + 2x^3\zeta_3, \tag{2}$$

where

$$\mathbf{u}_\omega = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \mathbf{u}_\omega = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ 0 \end{pmatrix}, \quad \boldsymbol{\zeta} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ 0 \end{pmatrix}, \quad \boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ 0 \end{pmatrix}$$

with

$$\begin{cases} \gamma_{\alpha\beta}(\mathbf{V}) = \frac{1}{2}(V_{\beta\perp\alpha} + V_{\alpha\perp\beta}) - V^3 C_{\alpha\beta} \\ \rho_{\alpha\beta}(\mathbf{V}) = \frac{1}{2}(V_{\lambda\perp\alpha} C^\lambda_\beta + V_{\lambda\perp\beta} C^\lambda_\alpha) - V^3 C^\lambda_\alpha C_{\lambda\beta}, \end{cases} \tag{3}$$

where C^β_α is the coefficients of the curvature tensor and $U_{\alpha\perp\beta}$ is the covariant derivate.

2.2. STRESS RESULTANT

The stress resultants are defined by

$$[N] = \begin{bmatrix} N^{11} & N^{12} \\ N^{21} & N^{22} \end{bmatrix}, \ [M] = \begin{bmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{bmatrix}, \ [P] = \begin{bmatrix} P^{11} & P^{12} \\ P^{21} & P^{22} \end{bmatrix}, \ [\bar{M}] = \begin{bmatrix} \bar{M}^{11} & \bar{M}^{12} \\ \bar{M}^{21} & \bar{M}^{22} \end{bmatrix},$$

$$\mathbf{Q} = \begin{pmatrix} Q^1 \\ Q^2 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} S^1 \\ S^2 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} R^1 \\ R^2 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} L^1 \\ L^2 \end{pmatrix} \tag{4}$$

with

$$\begin{bmatrix} N^{11} & N^{12} & N^{22} & N_z & Q^1 & Q^2 \\ M^{11} & M^{12} & M^{22} & M_z & R^1 & R^2 \end{bmatrix} = \int_{h/2}^{h/2} \begin{bmatrix} 1 \\ x_3 \end{bmatrix} [\sigma^{11}\, \sigma^{12}\, \sigma^{22}\, \sigma^{33}\, \sigma^{13} \sigma^{23}]\, \mathrm{d}x_3,$$

$$\begin{bmatrix} P^{11} & P^{12} & P^{22} & S^1 & S^2 \\ \bar{M}^{11} & \bar{M}^{12} & \bar{M}^{22} & L^1 & L^2 \end{bmatrix} = \int_{h/2}^{h/2} \begin{bmatrix} (x_3)^2 \\ (x_3)^3 \end{bmatrix} [\sigma^{11}\, \sigma^{12}\, \sigma^{22}\, \sigma^{13} \sigma^{23}]\, \mathrm{d}x_3. \tag{5}$$

The stress resultants in terms of displacements are given by

$$N = h[2\mu\, \gamma(\mathbf{u}_\omega) + \lambda \mathrm{Tr}\{\gamma(\mathbf{u}_\omega)\} \cdot \mathbf{1}] + \frac{h^3}{12}[2\mu\, \gamma(\boldsymbol{\zeta}) + \lambda \mathrm{Tr}\{\gamma(\boldsymbol{\zeta})\} \cdot \mathbf{1}] + \lambda h \beta_3\, \mathbf{1}, \tag{6}$$

$$M = \frac{h^3}{12}[2\mu(\gamma(\boldsymbol{\beta} - \rho(\mathbf{u}_\omega)) + \lambda \mathrm{Tr}(\gamma(\boldsymbol{\beta}) - \rho(\mathbf{u}_\omega)) \cdot \mathbf{1}]$$
$$+ \frac{h^5}{80}[2\mu(\gamma(\boldsymbol{\phi})) + \lambda \mathrm{Tr}(\gamma(\boldsymbol{\phi}) \cdot \mathbf{1}] + 2\lambda \frac{h^3}{12} \zeta_3\, \mathbf{1}, \tag{7}$$

$$P = \frac{h^3}{12}[2\mu\, \gamma(\mathbf{u}_\omega) + \lambda \mathrm{Tr}\{\gamma(\mathbf{u}_\omega)\} \cdot \mathbf{1}] + \frac{h^5}{80}[2\mu\, \gamma(\boldsymbol{\zeta}) + \lambda \mathrm{Tr}\{\gamma(\boldsymbol{\zeta})\} \cdot \mathbf{1}] + \lambda \frac{h^3}{12} \beta_3\, \mathbf{1}, \tag{8}$$

$$\bar{M} = \frac{h^5}{80}[2\mu(\gamma(\boldsymbol{\beta} - \rho(\mathbf{u}_\omega)) + \lambda \mathrm{Tr}(\gamma(\boldsymbol{\beta}) - \rho(\mathbf{u}_\omega)) \cdot \mathbf{1}]$$
$$+ \frac{h^7}{448}[2\mu(\gamma(\boldsymbol{\phi})) + \lambda \mathrm{Tr}(\gamma(\boldsymbol{\phi})) \cdot \mathbf{1}] + 2\lambda \frac{h^5}{80} \zeta_3\, \mathbf{1}, \tag{9}$$

$$\mathbf{Q} = \mu h[\boldsymbol{\beta} + \nabla \mathbf{u}_3 + C \cdot \mathbf{u}_\omega] + \mu \frac{h^3}{12}[\nabla\, \zeta_3 + 3\boldsymbol{\phi} + C \cdot \boldsymbol{\zeta}], \tag{10}$$

$$\mathbf{R} = \mu \frac{h^3}{12}[\nabla\, \boldsymbol{\beta}_3 + 2\boldsymbol{\zeta}] + \mu \frac{h^5}{80}[C \cdot \boldsymbol{\phi}], \tag{11}$$

$$\mathbf{S} = \mu \frac{h^3}{12}[\boldsymbol{\beta} + \nabla \mathbf{u}_3 + C \cdot \mathbf{u}_\omega] + \mu \frac{h^5}{80}[\nabla\, \zeta_3 + 3\boldsymbol{\phi} + C \cdot \boldsymbol{\zeta}], \tag{12}$$

$$\mathbf{L} = \mu \frac{h^5}{80} [\nabla \boldsymbol{\beta}_3 + 2\boldsymbol{\zeta}] + \mu \frac{h^7}{448} [C \cdot \boldsymbol{\phi}], \tag{13}$$

$$N_z = h\, [(2\mu + \lambda)\, \beta_3 + \lambda \mathrm{Tr}(\gamma(\mathbf{u}_\omega))] + \lambda \frac{h^3}{12} \mathrm{Tr}(\gamma(\boldsymbol{\zeta})), \tag{14}$$

$$M_z = \frac{h^3}{12} [2(2\mu + \lambda)\zeta_3 + \lambda \mathrm{Tr}(\gamma(\boldsymbol{\beta}) - \rho(\mathbf{u}_\omega))] + \lambda \frac{h^5}{80} \mathrm{Tr}(\gamma(\boldsymbol{\phi})), \tag{15}$$

where **1** is the unit tensor and (μ, λ) are the Lamé coefficients, ∇ and Tr are, respectively, the gradient and the trace operators.

3. EQUATIONS OF MOTION

The principle of the virtual work is used to derive the governing equations of motion.

For Shellls subjected to the volume and surface densities $\mathbf{f}_v$ and $\mathbf{f}_s$, it is found that (for more details see Appendix A):

$$[N^{\alpha\beta}_{\perp\beta} - (M^{\delta\beta}\, C^\alpha_\beta)_{\perp\delta} - Q^\delta C^\alpha_\delta + F^\alpha] = \rho h\, \ddot{u}^\alpha + \rho \frac{h^3}{12} \ddot{\zeta}^\alpha,$$

$$M^{\alpha\beta}_{\perp\beta} - Q^\alpha + m^\alpha = \rho \frac{h^3}{12} \ddot{\beta}^\alpha + \rho \frac{h^5}{80} \ddot{\phi}^\alpha,$$

$$N^{\alpha\beta}\, C_{\alpha\beta} - (M^{\alpha\beta}\, C^\delta_\alpha\, C_{\delta\beta}) + Q^\alpha_{\perp\alpha} + F^3 = \rho h\, \ddot{u}^3 + \rho \frac{h^3}{12} \ddot{\zeta}_3,$$

$$R^\alpha_{\perp\alpha} - N_z + F^3_1 = \rho \frac{h^3}{12} \ddot{u}_3 + \rho \frac{h^5}{80} \ddot{\zeta}_3,$$

$$P^{\alpha\beta}_{\perp\beta} - 2R^\alpha - S^\delta\, C^\alpha_\delta + F^\alpha_1 = \rho \frac{h^3}{12} \ddot{u}^\alpha + \rho \frac{h^5}{80} \ddot{\zeta}^\alpha,$$

$$S^\alpha_{\perp\alpha} - 2M_z + F^3_2 = \rho \frac{h^3}{12} \beta_3,$$

$$\bar{M}^{\alpha\beta}_{\perp\beta} - 3S^\alpha - L^\delta C^\alpha_\delta + F^\alpha_2 = \rho \frac{h^5}{80} \ddot{\beta}^\alpha + \rho \frac{h^7}{448} \ddot{\phi}^\alpha \tag{16.a}$$

or

$$\mathrm{div}\, N - \mathrm{div}\, (C \cdot M) - C \cdot \mathbf{Q} + \mathbf{F}_\omega = \rho h\, \pi(\ddot{\mathbf{u}}_\omega) + \rho \frac{h^3}{12} \ddot{\boldsymbol{\zeta}},$$

$$\operatorname{div} M - \mathbf{Q} + \mathbf{m}_\omega = \frac{\rho h^3}{12}\,\ddot{\boldsymbol{\beta}} + \rho\,\frac{h^5}{80}\,\ddot{\boldsymbol{\phi}},$$

$$N : C - (C \cdot M) : C + \operatorname{div}(\mathbf{Q}) + F^3 = \rho h\,\ddot{\mathbf{u}}_3 + \rho\,\frac{h^3}{12}\,\ddot{\zeta}_3,$$

$$\operatorname{div} \mathbf{R} - N_z + F_1^3 = \rho\,\frac{h^3}{12}\,\ddot{u}_3 + \rho\,\frac{h^5}{80}\,\ddot{\zeta}_3,$$

$$\operatorname{div} P - 2\mathbf{R} - C \cdot \mathbf{S} + \mathbf{F}_1 = \rho\,\frac{h^3}{12}\,\pi(\ddot{\mathbf{u}}_\omega) + \rho\,\frac{h^5}{80}\,\ddot{\boldsymbol{\zeta}},$$

$$\operatorname{div} \mathbf{S} - 2M_z + F_2^3 = \rho\,\frac{h^3}{12}\,\beta_3,$$

$$\operatorname{div} \bar{M} - 3\mathbf{S} - C \cdot \mathbf{L} + \mathbf{F}_2 = \rho\,\frac{h^5}{80}\,\ddot{\boldsymbol{\beta}} + \rho\,\frac{h^7}{448}\,\ddot{\boldsymbol{\phi}}, \tag{16.b}$$

where ρ is the mass density, h is the thickness of the shell, $\pi(\ddot{\mathbf{u}}_\omega)$ is the projection on surface of the surface's acceleration $\ddot{\mathbf{u}}_\omega$, $\mathbf{F}_\omega$ is the middle surface forces vector, $\mathbf{m}_\omega$ is the middle surface moments vector. $\mathbf{F}_1$, $\mathbf{F}_2$, F_1^3 and F_2^3 are defined by

$$F_1^\alpha = \int_{-h/2}^{h/2} (x^3)^2\, f_v^\alpha\, \mathrm{d}x^3, \quad F_2^\alpha = \int_{-h/2}^{h/2} (x^3)^3\, f_v^\alpha\, \mathrm{d}x^3, \quad F_1^3 = \int_{-h/2}^{h/2} x^3 f_v^3\, \mathrm{d}x^3,$$

$$F_2^3 = \int_{-h/2}^{h/2} (x^3)^2\, f_v^3\, \mathrm{d}x^3. \tag{17}$$

For a shell loaded by a surface density forces applied on $\partial\omega\ x\ ]-h/2, h/2[$ ($\partial\omega$ is the boundary of the surface ω), the boundary conditions are

$$\begin{aligned}
&- N^{\alpha\beta} v_\beta + (M^{\delta\beta} C_\beta^\alpha)\, v_\delta + F_s^\alpha = 0,\\
&- M^{\alpha\beta} v_\beta + m_s^\alpha = 0,\\
&- Q^\alpha v_\alpha + F_s^3 = 0,\\
&- R^\alpha v_\alpha + (F_1^3)_s = 0,\\
&- P^{\alpha\beta} v_\beta + (F_1^\alpha)_s = 0,\\
&- S^\alpha v_\alpha + (F_2^3)_s = 0,\\
&- \bar{M}^{\alpha\beta} v_\beta + (F_2^\alpha)_s = 0
\end{aligned} \tag{18.a}$$

or

$$\begin{aligned}
&- N \cdot \mathbf{v} + C \cdot M \cdot \mathbf{v} + (\mathbf{F}_\omega)_s = \mathbf{0},\\
&- M \cdot \mathbf{v} + (\mathbf{m}_\omega)_s = \mathbf{0},\\
&- \mathbf{Q} \cdot \mathbf{v} + F_S^3 = 0,\\
&- \mathbf{R} \cdot \mathbf{v} + (F_1^3)_s = 0,
\end{aligned}$$

$$-\mathbf{P}\cdot\mathbf{v}+(\mathbf{F}_1)_s=\mathbf{0},$$

$$-\mathbf{S}\cdot\mathbf{v}+(F_2^3)_s=0,$$

$$-\bar{M}\cdot\mathbf{v}+(\mathbf{F}_2)_s=\mathbf{0}, \tag{18.b}$$

where $(\mathbf{F}_\omega)_s$ is the in-surface boundary forces vector and $(\mathbf{m}_\omega)_s$ is the in-surface boundary moments vector. $(\mathbf{F}_s)_1$, $(\mathbf{F}_s)_2$, $(F_1^3)_s$ and $(F_2^3)_s$ are defined by

$$(F_1^\alpha)_s=\int_{-h/2}^{h/2}(x^3)^2\, f_s^\alpha\,\mathrm{d}x^3,\quad (F_2^\alpha)_s=\int_{-h/2}^{h/2}(x^3)^3\, f_s^\alpha\,\mathrm{d}x^3,$$

$$(F_1^3)_s=\int_{-h/2}^{h/2}x^3 f_s^3\,\mathrm{d}x^3,\quad (F_2^3)_s=\int_{-h/2}^{h/2}(x^3)^2\, f_s^3\,\mathrm{d}x^3. \tag{19}$$

REFERENCES

1. K. H. Lo, R. M. Christensen and E. M. Wu 1997 *Journal of Applied Mechanics* **44**, 663–676. A high order theory of plate deformation. Part 1: Homogeneous plates.
2. H. Hassis 1999 *Journal of Sound and Vibration* **225**, 633–653. A "warping-Kirchhoff" and a "warping-Mindlin" theory for shell deformation (accepted).

APPENDIX A

Using an integration over the normal co-ordinate, the external virtual work is (for the densities $\mathbf{f}_v$ and $\mathbf{f}_s$):

$$W_e=\int_\omega [F^\alpha u_\alpha^* + m^\alpha\beta_\alpha^* + F_1^\alpha\zeta_\alpha^* + F_2^\alpha\phi_\alpha^* + F^3u_3^* + F_1^3\beta_3^* + F_2^3\zeta_3^*]\,\mathrm{d}\omega$$

$$+\int_{\partial\omega}[F_s^\alpha u_\alpha^* + m_s^\alpha\beta_\alpha^* + (F_1^\alpha)_s\,\zeta_\alpha^* + (F_2^\alpha)_s\,\phi_\alpha^* + F_s^3u_3^* + (F_1^3)_s\beta_3^* + (F_2^3)_s\zeta_3^*]\,\mathrm{d}\Gamma,$$

where

$$F^i=\int_{-h/2}^{h/2} f_v^i\,\mathrm{d}x^3,\quad F_s^i=\int_{-h/2}^{h/2} f_s^i\,\mathrm{d}x^3,\quad i=1,2\text{ or }3.$$

Using an integration over the normal co-ordinate, the inertial virtual work

$$W_j=-\int_\omega\left[\rho h(\ddot{u}^\alpha u_\alpha^* + \ddot{u}^3u_3^*) + \rho\frac{h^3}{12}(\ddot{\beta}^\alpha\beta_\alpha^* + \ddot{u}^\alpha\zeta_\alpha^* + \ddot{\zeta}^\alpha u_\alpha^* + \ddot{u}_3\zeta_3^* + \ddot{\beta}_3\beta_3^*\right.$$

$$\left.+\ddot{\zeta}_3u_3^*)\right]\mathrm{d}\omega-\int_\omega\left[\rho\frac{h^5}{80}(\ddot{\beta}^\alpha\phi_\alpha^* + \ddot{\phi}^\alpha\beta_\alpha^* + \ddot{\zeta}^\alpha\zeta_\alpha^* + \ddot{\zeta}_3\zeta_3^*) + \rho\frac{h^7}{448}(\ddot{\phi}^\alpha\phi_\alpha^*)\right]\mathrm{d}\omega.$$

Using an integration over the normal co-ordinate and using the stress resultants defined by equation (5), the internal virtual work is

$$W_i = -\int_\omega [N^{\alpha\beta}\gamma_{\alpha\beta}(\mathbf{u}^*_\omega) + M^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\beta}^*) - M^{\alpha\beta}\rho_{\alpha\beta}(\mathbf{u}^*_\omega) + P^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\zeta}^*)$$
$$+\bar{M}^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\phi}^*)]\,\mathrm{d}\omega - \int_\omega [Q^\alpha(\beta^*_\alpha + C^\lambda_\alpha u^*_\lambda + u^*_{3,\alpha}|) + (2R^\alpha\zeta^*_\alpha + 3S^\alpha\phi^*_\alpha$$
$$+ S^\alpha C^\lambda_\alpha \zeta^*_\lambda + L^\alpha C^\lambda_\alpha\,\phi^*_\lambda)]\,\mathrm{d}\omega$$
$$-\int_\omega [N_z\beta^*_3 + 2M_z\zeta^*_3]\,\mathrm{d}\omega - \int_\omega [R^\alpha\beta^*_{3,\alpha} + S^\alpha\zeta^*_{3,\alpha}]\,\mathrm{d}\omega.$$

Using relation (2) and after integration by parts, it becomes

$$W_i = \int_\omega [N^{\alpha\beta}_{\perp\beta} - (M^{\delta\beta}C^\alpha_\beta)_{\perp\delta} - Q^\delta C^\alpha_\delta]\,u^*_\alpha\,\mathrm{d}\omega + \int_\omega [M^{\alpha\beta}_{\perp\beta} - Q^\alpha]\,\beta^*_\alpha\,\mathrm{d}\omega$$
$$+\int_\omega [P^{\alpha\beta}_{\perp\beta} - 2R^\alpha_1 - S^\delta C^\alpha_\delta)]\,\zeta^*_\alpha\,\mathrm{d}\omega + \int_\omega [\bar{M}^{\alpha\beta}_{\perp\beta} - 3S^\alpha - L^\delta C^\alpha_\delta]\,\phi^*_\alpha\,\mathrm{d}\omega$$
$$+\int_\omega [N^{\alpha\beta}C_{\alpha\beta} - (M^{\alpha\beta}C^\delta_\alpha C_{\delta\beta}) + Q^\alpha_{\perp\alpha}]\,u^*_3\,\mathrm{d}\omega + \int_\omega [R^\alpha_\alpha - N_z]\,\beta^*_3\,\mathrm{d}\omega$$
$$+\int_\omega [S^\alpha_{\perp\alpha} - 2M_z]\,\zeta^*_3\,\mathrm{d}\omega + \int_{\partial\omega} [-N^{\alpha\beta}v_\beta + (M^{\delta\beta}C^\alpha_\beta)v_\delta]\,u^*_\alpha\,\mathrm{d}\Gamma$$
$$-\int_{\partial\omega} M^{\alpha\beta}\beta^*_\alpha v_\beta\,\mathrm{d}\Gamma - \int_{\partial\omega} P^{\alpha\beta}v_\beta\zeta^*_\alpha\,\mathrm{d}\Gamma - \int_{\partial\omega}\bar{M}^{\alpha\beta}\phi^*_\alpha v_\beta\,\mathrm{d}\Gamma$$
$$-\int_{\partial\omega} Q^\alpha v_\alpha u^*_3\,\mathrm{d}\Gamma - \int_{\partial\omega}[R^\alpha v_\alpha\beta^*_3 + S^\alpha v_\alpha\zeta^*_3]\,\mathrm{d}\Gamma.$$

The application of the virtual work leads to equations (16) and (18).