Journal of Sound and Vibration (2000) **236**(5), 901–903
doi:10.1006/jsvi.1999.2726, available online at http://www.idealibrary.com on IDEAL®

ANALYSIS OF A FINITE-DIFFERENCE SCHEME FOR A LINEAR ADVECTION–DIFFUSION–REACTION EQUATION

R. E. Mickens

Department of Physics, Clark Atlanta University, Atlanta, GA 30314, *U.S.A.*

(*Received* 7 *October* 1999)

An important class of physical phenomena in acoustics, fluid dynamics, and the transport of contaminants can be modelled by the partial differential equation [1–3]

$$u_t + u_x + \lambda u = \delta u_{xx}, \tag{1}$$

where λ and δ are positive parameters, and the velocity of propagation has been normalized to unity. The main purpose of this letter is to extend the previous results of Mickens [4] which corresponds to placing $\lambda = 0$ in equation (1). In particular, a new finite-difference scheme is constructed using the concept of "exact" and "best" difference models as formulated in Mickens [5]. An analysis of stability for the scheme is done based on the application of a positivity constraint [6]. The details of the construction procedure are not provided since they follow directly from the results given in references [4–6].

Denote the space and time step sizes, respectively, by Δx and Δt, and the discrete approximation to $u(x, t)$ by $u_m^n \simeq u(x_m, t_n)$, where $x_m = (\Delta x)m$, $t_n = (\Delta t)n$, and (m, n) are integers. The finite-difference scheme selected for equation (1) is

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} + \frac{u_m^n - u_{m-1}^n}{\Delta x} + \lambda u_{m-1}^n = \delta \left[\frac{u_{m+1} - 2u_m + u_{m-1}}{(\Delta x)^2} \right]. \tag{2}$$

This discrete model was obtained by first constructing the "exact" finite-difference scheme for

$$u_t + u_x + \lambda u = 0; \tag{3}$$

see Mickens [5]; the result is

$$\frac{u_m^{n+1} - u_m^n}{\phi(\Delta t)} + \frac{u_m^n - u_{m-1}^n}{\phi(\Delta x)} + \lambda u_{m-1}^n = 0, \qquad \Delta t = \Delta x, \tag{4}$$

where

$$\phi(z) = \frac{1 - \mathrm{e}^{-\lambda z}}{\lambda}, \tag{5}$$

and $\Delta t = \Delta x$ is required. Observe that for small λz, equation (5) becomes

$$\phi(z) = z + O(\lambda z^2). \tag{6}$$

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The term on the right-hand side of equation (1) can be discretely modelled by a central-difference scheme for the second derivative [5, 6], i.e.,

$$\delta u_{xx} \to \delta \left[\frac{u^n_{m+1} - 2u^n_m + u^n_{m-1}}{(\Delta x)^2} \right]. \tag{7}$$

Combining the results of equations (4) and (7) gives the following non-standard finite-difference model for equation (1):

$$\frac{u^{n+1}_m - u^n_m}{\phi(\Delta t)} + \frac{u^n_m - u^n_{m-1}}{\phi(\Delta x)} + \lambda u^n_{m-1} = \delta \left[\frac{u^n_{m+1} - 2u^n_m + u^n_{m-1}}{(\Delta x)^2} \right]. \tag{8}$$

To simplify the analysis, replace the function $\phi(z)$ by its first approximation, $\phi(z) = z$; this holds if $0 < \lambda z \ll 1$. Thus, equation (8) takes the form given by equation (2).

Making the definitions

$$\beta \equiv \frac{\Delta t}{\Delta x}, \qquad R \equiv \frac{\delta \Delta t}{(\Delta x)^2}, \tag{9}$$

equation (8) can be rewritten to the form

$$u^{n+1}_m = R u^n_{m+1} + (1 - \beta - 2R) u^n_m + (\beta - \lambda \Delta t + R) u^n_{m-1}. \tag{10}$$

All the solutions to equation (10) will be stable and satisfy a max–min condition if the coefficients to the u^n_m and u^n_{m-1} terms are non-negative [6]. Such a "positivity" condition implies that

$$1 - \beta - 2R \geqslant 0, \qquad \beta - \lambda \Delta t + R \geqslant 0. \tag{11}$$

One way to satisfy the two conditions of equation (11) is to first require

$$1 - \beta - 2R = R. \tag{12}$$

Substituting the results of equation (9) into equation (12) gives a relationship between the step sizes

$$\Delta t = \frac{(\Delta x)^2}{3\delta + \Delta x}. \tag{13}$$

Note that when $\delta = 0$, the proper result $\Delta t = \Delta x$ is obtained.

Given equation (13), does the second inequality of equation (11) hold? Note that

$$\beta - \lambda \Delta t + R = \frac{\delta + \Delta x - \lambda (\Delta x)^2}{3\delta + \Delta x} \equiv \frac{y(\Delta x)}{3\delta + \Delta x}, \tag{14}$$

where the last equality defines the function $y(\Delta x)$. A direct calculation shows that $y(\Delta x) > 0$ for

$$0 < \Delta x < (\Delta x)_+ , \tag{15}$$

where $(\Delta x)_+$ is the positive root of $y(\Delta x)$ and is given by

$$(\Delta x)_+ = \left(\frac{1}{2\lambda}\right)[1 + \sqrt{1 + 4\lambda\delta}]. \tag{16}$$

In summary, a non-standard finite-difference scheme [5] has been constructed for a linear advection–diffusion–reaction partial differential equation. The numerical solutions of the scheme are stable and satisfy a max–min condition, just as the original differential equation. Written out, this explicit scheme for equation (1) is

$$u_m^{n+1} = R(u_m^n + u_{m+1}^n) + \left[\frac{\delta + \Delta x - \lambda(\Delta x)^2}{3\delta - \Delta x}\right]u_{m-1}^n , \tag{17}$$

where equation (13) gives the relation between the step sizes,

$$R = \frac{\delta \Delta t}{(\Delta x)^2} = \frac{\delta}{3\delta + \Delta x}, \tag{18}$$

and Δx is restricted by the condition of equations (15) and (16).

All of the analysis given above could be done with the approximation of equation (6). The only change would be a more complex relationship between the step sizes.

ACKNOWLEDGMENT

This research has been supported by DOE and the MBRS-SCORE Program at Clark Atlanta University.

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