Journal of Sound and Vibration (2000) **236**(5), 904–911
doi:10.1006/jsvi.2000.3044, available online at http://www.idealibrary.com on IDEAL®

THE MULTIPOLE EXPANSION: A NEW LOOK

RICARDO E. MUSAFIR

Department of Mechanical Engineering/COPPE and Hydraulics Department/EE, Universidade Federal do Rio de Janeiro, C.P. 68503, *Rio de Janeiro*, 21945-970, *Brazil.*
E-mail: *rem@serv.com.ufrj.br*

(*Received* 31 *January* 2000, *and in final form* 7 *April* 2000)

1. INTRODUCTION

The multipole expansion is a classical result, originally obtained by Maxwell in 1873 (see reference [1, p. 170, note]), which is currently used in acoustics, electromagnetism and fluid mechanics, areas where the Poisson or d'Alembert equations play a prominent role in the description of fields generated by source distributions or by appropriate sets of boundary conditions.

In acoustics, the study of sources received a great impulse with the establishment of Lighthill's brilliant analogy [2], pointing out the intrinsic quadrupolar nature of sources of internally generated sound in free flows and presenting a situation where a continuous tri-dimensional source distribution for the wave equation was quite meaningful. Although applications of the multipole expansion are not at all restricted to aeroacoustics, one can say that aeroacousticians use it, maybe, a little more than other acousticians, since source distributions, moving or stationary and frequently of quadrupolar type, are one of their main concerns. In this context, it is significant to note that in the first papers dealing explicitly with the multipole expansion in acoustics, Oestreicher, after formulating the theory [3], expanded it in order to describe the sound field of moving bodies [4] and that, today, the standard references for the subject are the texts by Pierce [1], Doak [5], Goldstein [6] and Ffowcs Williams [7], all but the first of them by authors whose names are strongly associated with aeroacoustics.

The expansion is commonly understood in the sense that, given an arbitrary source distribution Q, representing the right-hand side of the Poisson or d'Alembert equations, and regarded as a *monopole* source distribution, the corresponding generated *field* can be expressed, by means of a Taylor expansion in the space variable, as a series of fields of point sources located at a fixed arbitrary point. Due to the presence of the space derivatives proceeding from the Taylor expansion, the point sources can be identified as *point multipoles* [1, 3, 5–7], i.e., as sources obtained by particular combinations of pairs of point monopoles of canceling strengths. The number of space derivatives, n, denotes the multipole order, while 2^n represents the number of monopoles needed to generate a multipole component of order n. From the field expansion, the multipole strengths are obtained and the source function can be expressed by a series of point multipoles.

Being based on the expansion of the field generated by a source distribution, the multipole expansion tends to be regarded as dependent on the knowledge of the Green's function of the field equation. Probably as a consequence of this, the expansion is usually discussed in acoustics only for situations where the propagation can be adequately

0022-460X/00/400904 + 08 $35.00/0

described by the d'Alembert equation, i.e., for an homogeneous medium at rest, when the problem can be represented by

$$\{c_0^{-2}\partial^2/\partial t^2 - \nabla^2\}\phi(\mathbf{x}, t) = Q(\mathbf{x}, t), \tag{1}$$

where c_0 is the speed of the ϕ waves and the corresponding solution is given, in tri-dimensional unbounded space, by

$$\phi(\mathbf{x}, t) = \int_\infty \frac{Q(\mathbf{y}, t - |\mathbf{x} - \mathbf{y}|/c_0)}{4\pi|\mathbf{x} - \mathbf{y}|} \, \mathrm{d}V_\mathbf{y}, \tag{2}$$

a form that seems naturally to suggest the Taylor expansion of the integrand around a fixed source point. The fact that Green's function for a more general situation can also be expanded was used by Oestreicher [4] but has not, to the author's knowledge, been further pursued.

In the present paper, it is shown that the source distribution itself can be directly expanded into a series of point multipoles, without the need of referring to the expansion of the field. This result is more general than previous ones, being valid for any situation and supporting a generalization of the concept of multipoles in a non-homogeneous and/or moving medium which places emphasis on the physical meaning of the sources. The possibility of direct source expansion was left implicit in previous work [8], being here fully discussed.

2. THE MULTIPOLE EXPANSION

Given an arbitrary scalar function depending on space and time, $Q(\mathbf{x}, t)$, supposed null outside a closed domain C or, else, decaying faster than any power of $|\mathbf{x}|$ as $|\mathbf{x}| \to \infty$, so that it can represent a source distribution confined to a fairly bounded region, one can write,

$$Q(\mathbf{x}, t) = \int_\infty Q(\mathbf{y}, t)\delta(\mathbf{x} - \mathbf{y}) \, \mathrm{d}V_\mathbf{y}. \tag{3}$$

Expanding $\delta(\mathbf{x} - \mathbf{y})$ in a Taylor series around $y = 0$ as

$$\delta(\mathbf{x} - \mathbf{y}) = \sum_{n=0}^{\infty} \frac{1}{n!}(-\mathbf{y}\cdot\nabla)^n\delta(\mathbf{x}) = \delta(\mathbf{x}) - y_i\frac{\partial\delta(\mathbf{x})}{\partial x_i} + \frac{1}{2}y_i y_j\frac{\partial^2\delta(\mathbf{x})}{\partial x_i x_j} - \cdots \tag{4}$$

permits writing Q as the series of generalized functions

$$\begin{aligned} Q(\mathbf{x}, t) &= \left\{\int_\infty Q(\mathbf{y}, t)\,\mathrm{d}V_\mathbf{y} - \int_\infty y_i Q(\mathbf{y}, t)\,\mathrm{d}V_\mathbf{y}\frac{\partial}{\partial x_i} + \frac{1}{2}\int_\infty y_i y_j Q(\mathbf{y}, t)\,\mathrm{d}V_\mathbf{y}\frac{\partial^2}{\partial x_i x_j} - \cdots\right\}\delta(\mathbf{x}) \\ &= \sum_{n=0}^{\infty}(-1)^n\mathbb{Q}^{(n)}(t)(\cdot\nabla)^n\delta(\mathbf{x}), \end{aligned} \tag{5}$$

where $\mathbb{Q}^{(n)}(t)$ represents the tensor of order n, whose components, $Q^{(n)}_{ijk\ldots}$, are proportional to the moments of order n of Q with respect to the (arbitrary) reference point, $\mathbf{x} = 0$,

$$\mathbb{Q}^{(n)}(t) = \frac{1}{n!}\int_\infty (\mathbf{y})^n Q(\mathbf{y}, t)\,\mathrm{d}V_\mathbf{y}. \tag{6}$$

For d-dimensional space, there are d^n moments of order n, which can assume up to $(n + d - 1)!/[n!(d - 1)!]$ different values (since pairs of $Q^{(n)}_{ijk\ldots}$ whose indexes differ by a permutation are equal to each other). It can be shown, following the procedure of Oestreicher [3], that in general only the lowest order non-zero $\mathbf{Q}^{(n)}$ is independent of the choice of the reference point.

If Q is taken as the right-hand side of the d'Alembert equation (1) (or of the Poisson equation), the above result yields directly the same multipole expansion of the source distribution Q that would be obtained by the more lengthy procedure, described earlier, of recovering the multipole strengths from the expansion of the field of Q, the strengths of the 2^n-pole components being given by the corresponding $Q^{(n)}_{ijk\ldots}$. It must be stressed that equality (5) implies that both sides are to be regarded as equivalent generalized functions.

It should be noted that both methods rely on the existence of the integrals in equation (6) and, as long as wave equations in the time domain in tri-dimensional space are concerned, on the expansion of a delta function, since the classical approach is equivalent to expanding the wave equation's Green's function, G, in a Taylor series. The necessity of dealing explicitly with the expansion of a generalized function is avoided by expanding either the time domain form of the resulting field, as in references [6, 7], or instead, the Green's function of a problem formulated in the frequency domain, as done in references [1, 3–5].

The advantage of the present procedure is that it permits the generalization of the multipole expansion for any function suitable of being regarded as a source distribution to any equation, provided the integrals in equation (6) exist. The expansion is more meaningful, however, for source terms in linear equations since, in this case, the fact that the original and the expanded source distributions can be regarded as equivalent generalized functions means that they generate the same field. In a non-linear problem the expansion can be used to produce an approximation for the source function. The equivalence with the classical method, when applicable, is shown in Appendix A.

3. THE PHYSICS OF THE MULTIPOLES

The physical and mathematical interpretation of the multipoles obtained in the expansion is, of course, dependent on that of the original source function. If a scalar Q is seen as a monopole source distribution, the first term in the expansion, the scalar $\mathbf{Q}^{(0)}$, stands for a point monopole, the second, a vector with components $Q^{(1)}_i$, for a point dipole, and so on. If Q represents a scalar component of a multipole distribution of order m, $\mathbf{Q}^{(0)}$ will correspond to a point source of the same order, $\mathbf{Q}^{(1)}$ to one of order $m+1$, and so on. The generalization to tensor source functions of any order is straightforward, $\mathbf{Q}^{(n)}$ being, for a source function described by a tensor distribution of order m, a tensor of order $n+m$.

It is common to regard the source function Q for a scalar wave equation as a monopole distribution per unit volume or mass. This is consistent with equation (3) and suggests that localized excitation at $\mathbf{x} = 0$ would be properly represented by

$$Q(\mathbf{x}, t) = Q_0(t)\delta(\mathbf{x}), \tag{7}$$

where $Q_0(t)$, the point monopole source strength, describes the time history of the excitation. Although this is mathematically correct, it is not necessarily the best choice from a physical point of view: since a wave equation in acoustics is obtained by the combination of the equations of mass and momentum together with the entropy equation (which contains the information on energy that is not implicit in the other two and permits the establishment of a relationship between the material time derivatives of pressure and density), it follows that the source terms in the resulting wave equation can, ideally, always be traced back to source terms in the fundamental equations. The latter are the ones that correspond to the most physically meaningful forms of local excitation and thus, since they refer to the possible forms of basic singularities, should be treated as "monopoles".

The general form of the source function Q in an acoustic wave equation was discussed in references [8, 9]. Let q represent a source distribution in the equation resulting from the combination of the mass and entropy equations (i.e., in the isentropic continuity equation), being related to local changes in volume generated by the addition of mass or heat, and $\mathbf{f}$ a source distribution in the momentum equation. Thus, as the process of derivation of the wave equation involves the application of a scalar operator $\mathscr{M}$ to the first equation and the contraction of a vector operator $\mathscr{D}$ with the second one, usually after the equations are linearized in the fluctuations, the corresponding source function Q can be represented by

$$Q = \mathscr{M}(q) + \mathscr{D}\cdot(\mathbf{f}), \tag{8}$$

where the natural candidates to the expansion (on the understanding that they have a non-zero contribution to the sound field) are the fundamental source terms, q and $\mathbf{f}$, instead of the final source function Q.

The q and $\mathbf{f}$ terms are usually treated as a monopole and a dipole distribution, respectively, due to the fact that, in an homogeneous medium at rest, the natural form of Q in the linearized wave equation for pressure is

$$Q = \frac{\partial q}{\partial t} - \nabla\cdot\mathbf{f}, \tag{9}$$

where $\partial q/\partial t$, q, $-\nabla\cdot\mathbf{f}$ and Q can all be regarded as monopole distributions. The distinction between them can be made by calling q a volume *velocity* monopole distribution while $\partial q/\partial t$ and $-\nabla\cdot\mathbf{f}$, as well as Q, are seen as volume *acceleration* sources (see, e.g., reference [10]). As for $\mathbf{f}$, it can be regarded as a dipole distribution, of the volume acceleration type, for the d'Alembert equation, since its contribution to Q in equation (9) is effectively in dipole form: the first term in the expansion of $-\nabla\cdot\mathbf{f}$ is zero, while the coefficient of the second one is identical to that of the first term in the expansion of $\mathbf{f}$. Indeed, it is simple to prove that $\mathbf{Q}^{(n)}$ obtained in the expansion of $\mathbf{f}$ is identical to $\mathbf{Q}^{(n+1)}$ proceeding from the expansion of $-\nabla\cdot\mathbf{f}$ around the same source point.

The dipole character of $\mathbf{f}$ in equation (9) can also be made explicit by writing $\mathbf{f}$ as in equation (3), i.e., as

$$\mathbf{f}(\mathbf{x}, t) = \int_\infty \mathbf{f}(\mathbf{y}, t)\delta(\mathbf{x}-\mathbf{y})\,\mathrm{d}V_\mathbf{y} \tag{10}$$

and noting that its contribution to Q can be represented as

$$-\nabla\cdot\mathbf{f}(\mathbf{x}, t) = -\frac{\partial}{\partial x_i}\int_\infty f_i(\mathbf{y}, t)\delta(\mathbf{x}-\mathbf{y})\,\mathrm{d}V_\mathbf{y} = \int_\infty f_i(\mathbf{y}, t)\frac{\partial}{\partial y_i}\delta(\mathbf{x}-\mathbf{y})\,\mathrm{d}V_\mathbf{y}, \tag{11}$$

where each source element is associated with a space derivative of the delta function.

The identification of $\mathbf{f}$ with dipoles, however, is not exact in a general situation: if the sources are not in an homogeneous medium at rest (or in uniform movement), the operator $\mathscr{D}\cdot$, acting on the momentum equation, will not be given simply by the "dipole operator" $-\nabla\cdot$ and, as a consequence, the first term in the multipole expansion of the contribution of $\mathbf{f}$ to Q, i.e., of $\mathscr{D}\cdot\mathbf{f}$, will not, in general, be zero, so that $\mathbf{f}$ cannot be properly regarded as a dipole source distribution for the wave equation anymore.

This happens, for instance, in the case of a non-homogeneous medium at rest, with arbitrary mean density distribution $\rho_0(\mathbf{x})$ and under uniform mean pressure, for which the linearized wave equation for pressure fluctuations p, considering sources per unit volume q and $\mathbf{f}$, is given by

$$\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}p - \rho_0 \nabla\cdot\left(\frac{\nabla p}{\rho_0}\right) = \frac{\partial}{\partial t}q - \rho_0 \nabla\cdot\left(\frac{\mathbf{f}}{\rho_0}\right) = \frac{\partial}{\partial t}q - \nabla\cdot\mathbf{f} + \frac{\nabla\rho_0}{\rho_0}\cdot\mathbf{f}. \tag{12}$$

The presence of a non-zero monopole-like term in Q due to $\mathbf{f}$ can be physically justified as follows: if the medium is non-homogeneous or is in non-uniform movement, a point source of momentum cannot, in general, be modelled by two equal and opposite mass (or volume) sources, since the existence of a difference in the mean value of density (for sources per unit volume) or velocity (for all sources) along the line joining the mass sources requires, if they are expected to model the momentum source, a difference in their amplitudes which accounts for a monopole residue.

Nevertheless, although $\mathbf{f}$ will not necessarily contribute exclusively as a dipole distribution for the wave equation, it can always be regarded as a monopole distribution for the momentum equation, more specifically, as a *force monopole* distribution. This is the only multipole classification for $\mathbf{f}$ that does not depend on mean flow properties, being fully general. The imposed stress field $\mathbf{T}$, corresponding to $\mathbf{f} = -\nabla\cdot\mathbf{T}$ and usually regarded as a quadrupole distribution, would then be seen as a force dipole distribution.

It should be noted that the denominations *volume velocity* and *volume acceleration*, used previously to differentiate between the sources originated from q and $\mathbf{f}$ in equation (9), refer both to sources connected to the *continuity* equation (the only that can admit *volume* sources), but while the volume velocity ones, represented by q, stand for sources in the continuity equation itself, the volume acceleration sources actually stand for sources in the equation obtained by taking the time derivative of the continuity equation (the material time derivative, in the general case), so that the source function Q in a (second order) wave equation would be seen as a volume acceleration monopole distribution. This feature makes the volume acceleration sources entities whose physical meaning is entirely dependent on the proper association with volume velocity or momentum sources. Even so, as long as $\mathscr{D}\cdot\mathbf{f} = -\nabla\cdot\mathbf{f}$, the identification of $\mathbf{f}$ with a volume acceleration dipole poses no problem. For a general $\mathscr{D}$, however, the contribution to Q of a volume acceleration dipole source will be different from the actual contribution of $\mathbf{f}$. In equation (12), for instance, all three source terms in the rightmost representation can be regarded as volume acceleration monopole distributions, although neither of the two terms involving $\mathbf{f}$, if both are non-zero, can exist independently of the other.

If one chooses to call the external force term a volume acceleration dipole source distribution in a general situation, as done, e.g., by the author in previous work [8, 11], then the volume acceleration multipoles, from dipole onward, will always correspond to physical sources. This denomination, although providing a simple form of identifying the multipoles originated from $\mathbf{f}$, describes properly the source process in particular situations only.

The single option that is physically correct in any situation is to refer to $\mathbf{f}$ as a different type of monopole source distribution — a vector one, as discussed. More important, however, than the choice of the denomination is the requirement that all source terms in a wave equation be physically meaningful. Situations in aeroacoustics where some of the resulting source terms cannot be fully associated with sources in the fundamental equations were discussed in references [9, 11]. A fuller discussion of this aspect will appear elsewhere.

It must be remembered that depending on the complexity of the mean flow it is not possible to combine the linearized fundamental equations into a wave equation with time independent coefficients and, thus, to obtain the final source function Q (since in this case the operators $\mathscr{M}$ and $\mathscr{D}$ are not defined). Then, in order to compute the linear sound field, one is forced to work with the linearized fundamental equations and their source terms, as e.g., in reference [12]. The present analysis justifies that, in the compact limit, *as long as a farfield solution is sought*, these fundamental source terms — which can describe either external sources, boundary conditions or aerodynamical noise sources, being always amenable of being represented by q and $\mathbf{f}$ — can be approximated by the first non-zero term of the corresponding expansion. This procedure is, of course, valid independently of the complexity of the mean flow and can be used to produce the appropriate compact source term expressions without the need of forcing simplifying assumptions in order to reduce the wave equation to the d'Alembert equation and then expand the solution. Since these expressions involve an integral over the source region, they are subject to some of the manipulations that are usually performed on the solution, being thus expected to permit that interesting insight into characteristics of the far field be obtained directly from the source terms in the fundamental equations.

4. CONCLUSION

A generalization of the multipole expansion, based on a direct expansion of the source field, has been presented. The limitations concerning convergence of series and integrals are the same as in the classical approach. The advantage is the extension of the multipole expansion for source functions in any equation.

With the present method a number of results in aeroacoustics — particularly those concerning expressions for sources in the compact limit, when the source distribution can be approximated by the first non-zero term in the expansion — can be obtained in a more straightforward way, since it permits working directly with source terms in the linearized (preferably) or complete forms of the fundamental equations or in the different wave equations. A consequence of this feature is that many of these results can have their domain of validity extended, not being necessarily bound by the approximations involved in reducing wave equations to the d'Alembert equation.

ACKNOWLEDGMENT

The author is grateful to Prof. Moyses Zindeluk and to Dr Jules G. Slama for their suggestions and comments in the course of this work. Financial support was provided by the National Research Council of Brazil, CNPq.

REFERENCES

1. A. D. PIERCE 1989 *Acoustics — An Introduction to its Physical Principles and Applications*. New York: Acoustical Society of America, second edition.
2. M. J. LIGHTHILL 1952 *Proceedings of the Royal Society, London* **A211**, 564–587. On sound generated aerodinamically. I. General theory.
3. H. L. OESTREICHER 1957 *Journal of the Acoustical Society of America* **29**, 1219–1222. Representation of the field of an acoustic source as a series of multipole fields. Erratum in *Journal of the Acoustical Society of America* **30**, 481, 1958.
4. H. L. OESTREICHER 1957 *Journal of the Acoustical Society of America* **29**, 1223–1232. Field of a spatially extended moving source.
5. P. E. DOAK 1965 *Proceedings of the 5th ICA*, Liège, Vol. 1b, K56.1-4. Multipole analysis of acoustic radiation.
6. M. E. GOLDSTEIN 1976 *Aeroacoustics*. New York: McGraw-Hill.

7. J. E. FFOWCS WILLIAMS 1986 in *Frontiers in Physical Acoustics*, 53–101. Amsterdam: North-Holland. Physical origins of acoustical noise.
8. R. E. MUSAFIR 1994 *Journal de Physique* IV **4**, C5-943-946. Some thoughts on multipolar sources in a moving medium and the structure of the wave equation.
9. R. E. MUSAFIR 1997 *Proceedings of the 4th French Congress on Acoustics*, Marseille, Vol. 2, 923–926. A discussion on the structure of aeroacoustic wave equations.
10. MORFEY and H. K. TANNA 1971 *Journal of Sound and Vibration* **15**, 325–351. Sound radiation from a point force in circular motion.
11. R. E. MUSAFIR 1993 *Proceedings of the Institute of Acoustics*, Vol. 15 Part 3(4), 901–909. A note on the description of jet noise source terms.
12. P. A. DURBIN 1983 *Journal of Sound and Vibration* **91**, 519–525. High frequency Green function for aerodynamic noise in moving media, Part I: general theory.

APPENDIX A: THE DIFFERENT EXPANSION POSSIBILITIES

Let $\mathscr{L}_{(\mathbf{x},\,t)}$ represent a linear partial differential operator and $G(\mathbf{x}, t|\mathbf{y}, \tau)$ the solution of

$$\mathscr{L}_{(\mathbf{x},\,t)} G(\mathbf{x}, t|\mathbf{y}, \tau) = \delta(\mathbf{x}-\mathbf{y})\delta(t-\tau), \tag{A1}$$

so that the solution of the general problem

$$\mathscr{L}_{(\mathbf{x},\,t)} \phi(\mathbf{x}, t) = Q(\mathbf{x}, t) \tag{A2}$$

can be represented by

$$\phi(\mathbf{x}, t) = \int_{-\infty}^{+\infty} \int_{\infty} G(\mathbf{x}, t|\mathbf{y}, \tau) Q(\mathbf{y}, \tau)\, \mathrm{d}V_{\mathbf{y}}\, \mathrm{d}\tau. \tag{A3}$$

It is particularly interesting to rewrite equation (A3) as

$$\phi(\mathbf{x}, t) = \int_{-\infty}^{+\infty} \int_{\infty} \int_{\infty} G(\mathbf{x}, t|\boldsymbol{\zeta}, \tau)\delta(\boldsymbol{\zeta}-\mathbf{y}) Q(\mathbf{y})\, \mathrm{d}V_{\zeta}\, \mathrm{d}V_{\mathbf{y}}\, \mathrm{d}\tau \tag{A4a}$$

or, omitting the time dependence throughout, in order to simplify the notation, as

$$\phi(\mathbf{x}) = \int_{\infty} \int_{\infty} G(\mathbf{x}|\boldsymbol{\zeta})\delta(\boldsymbol{\zeta}-\mathbf{y}) Q(\mathbf{y})\, \mathrm{d}V_{\zeta}\, \mathrm{d}V_{\mathbf{y}}, \tag{A4b}$$

since these forms permit illustrating the different expansion possibilities.

Expanding $\delta(\boldsymbol{\zeta}-\mathbf{y})$ as

$$\delta(\boldsymbol{\zeta}-\mathbf{y}) = \sum_{n=0}^{\infty} \frac{1}{n!}(-\mathbf{y}\cdot\nabla)^n \delta(\boldsymbol{\zeta}), \tag{A5}$$

one can write

$$\phi(\mathbf{x}) = \int_{\infty} \int_{\infty} G(\mathbf{x}|\boldsymbol{\zeta}) \sum_{n=0}^{\infty} \frac{1}{n!}(-\mathbf{y}\cdot\nabla_{\zeta})^n \delta(\boldsymbol{\zeta}) Q(\mathbf{y})\, \mathrm{d}V_{\zeta}\, \mathrm{d}V_{\mathbf{y}}. \tag{A6}$$

By changing the order of integration, the above expression can be written as

$$\phi(\mathbf{x}) = \int_{\infty} G(\mathbf{x}|\boldsymbol{\zeta}) \int_{\infty} Q(\mathbf{y}) \sum_{n=0}^{\infty} \frac{1}{n!}(-\mathbf{y}\cdot\nabla_{\zeta})^n \delta(\boldsymbol{\zeta})\, \mathrm{d}V_{\mathbf{y}}\, \mathrm{d}V_{\zeta} \tag{A7}$$

which corresponds to the source expansion.

Transferring the space derivatives in equation (A6) to G, one obtains

$$\phi(\mathbf{x}) = \int_\infty \int_\infty \delta(\boldsymbol{\zeta}) \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{y} \cdot \nabla_\zeta)^n G(\mathbf{x}|\boldsymbol{\zeta}) Q(\mathbf{y}) \, \mathrm{d}V_\zeta \, \mathrm{d}V_y \tag{A8}$$

which can be written, after the $\boldsymbol{\zeta}$ integral is performed, as

$$\phi(\mathbf{x}) = \int_\infty \sum_{n=0}^{\infty} \left[\frac{1}{n!} (\mathbf{y} \cdot \nabla_\zeta)^n G(\mathbf{x}|\boldsymbol{\zeta}) \right]_{\zeta=0} Q(\mathbf{y}) \, \mathrm{d}V_y \tag{A9}$$

and expresses the expansion of G, corresponding to Oestreicher's result [4]. If G is such that $G(\mathbf{x}|\boldsymbol{\zeta}) = G(\mathbf{x} - \boldsymbol{\zeta}|0)$, then it follows that $\nabla_\zeta G = -\nabla_x G$, and the usual result,

$$\phi(\mathbf{x}) = \int_\infty \sum_{n=0}^{\infty} \frac{1}{n!} (-\mathbf{y} \cdot \nabla_x)^n G(\mathbf{x}|0) Q(\mathbf{y}) \, \mathrm{d}V_y, \tag{A10}$$

which can be regarded as the expansion of the field, is obtained.

As long as the Green's function G exists, it can always be expanded, the corresponding source multipole expansion for a given source function Q being also defined, provided the source integrals converge. This expansion does not depend on the knowledge of G. The direct expansion of the source is subject to the same restrictions concerning the integrals of Q and can be performed also for problems for which G is not defined.