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QUENCHING STICK–SLIP CHAOS WITH DITHER

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1. INTRODUCTION

Dither refers to the application of high-frequency excitation to affect the low-frequency behavior of a system. It has been used to meld the behavior of various oscillatory systems, such as in quenching limit cycle behavior (Minorsky [1] cites E. A. Appleton in 1922), stabilizing equilibria (such as in the inverted pendulum [2]), and improving friction behavior (Zames and Schneydor [3] cite V. A. Besekerskii in 1947). Experimental applications have included servomechanisms [4], DC motors [5], planar sliders [6], spacecraft reaction–wheel compensation [7], stabilization of lasers (see, e.g., references [8, 9]), and the control of a flexible arm with gear-box friction [10].

Early efforts toward analyzing systems with dither included the use of describing functions [4, 3]. Since the frequency of the dithering excitation is well above the characteristic frequency of the system, this brings about the presence of two time scales—a fast scale associated with the dither, and a slow scale associated with the oscillator itself—making a system amenable to averaging [11], which has been used often in expressing the effect on non-linearities. Analyses show that non-smooth functions tend to get smoothened by dither [3, 12–15]. Matunaba and Onoda [6] experimentally showed the smoothening effect of dither on a friction characteristic.

In a recent JSV paper, Thomsen [16] presented an analysis of dither in a belt-driven mechanical system, for which the belt-drive model consisted of a friction characteristic with a negative slope for a range of relative velocities, as well as a discontinuity, or near discontinuity, at zero relative velocity. Such a friction characteristic leads to an oscillatory instability that can grow into a stick–slip limit cycle. Averaging showed that the effective friction in the dithered system could have no negative rate dependence, and no effective discontinuity. The former effect was found to stabilize the belt-driven oscillator. The latter effect, i.e., the smoothening of the discontinuity, seems to imply that the averaged behavior will not involve stick–slip motion. This would have substantial impact, as the non-linear phenomena associated solely with stick–slip would be absent.

This latter impact catches our attention. Stick–slip was shown to be critical in generating chaos in a forced friction oscillator of our earlier studies [17, 18]. (In fact, "near stick–slip" was sufficient; smoothened representations of the discontinuous friction law allowed "nearly sticking" behavior, which still accommodated chaos.) We performed experiments showing that dither could alleviate the stick–slip chaos [19]. (Dither has also been used to quench chaos in circuits [13, 20].)

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In this letter, we bring forth the results of our dither experiments. We then apply averaging to a simple model of our system, and show that the friction discontinuity is smoothened.

2. AN EXPERIMENTAL OSCILLATOR WITH DRY FRICTION

The experiment consisted of a mass attached to the end of a cantilivered elastic beam. A diagram of the associated mechanics model is showed in Figure 1. The mass had titanium plates on both sides, providing surfaces for dry friction. Spring loaded titanium pads rested against the titanium plates. The titanium plates were not parallel and formed a wedge in the direction of sliding, and thus a displacement of the mass caused a change in the normal force on the spring loaded pads. Hence the friction force varied linearly with displacement. The amplitude of the friction force during oscillations was estimated to be in the range of 0·05–0·12 N. More on the force–displacement relationship can be found in reference [17]. The elastic beam mass and pressure pads were attached to a common frame which was excited harmonically by an electromagnetic shaker. Strain gages attached to the elastic beam were used to sense the displacement of the mass relative to the oscillating frame. The beam and mass had a fundamental natural frequency of 2·4 Hz with the friction removed. The frequency of the second mode was 37 Hz.

The strain-gage signal, representing the displacement of the mass, passed through an active electronic filter (Ithaca Corp., Model 4213) with a cut-off frequency of 20 Hz. The signal was then differentiated to provide a velocity measurement. While the dithering frequency was well above the 20 Hz of the filter, our interest was not on responses at the dither frequency, rather it was focused on the effect of dither on subharmonic components of the motion, and the low-frequency chaotic bahavior.

Without dither, chaotic responses were possible when excitations were in the range of 2·5–6 Hz, with amplitudes ranging from 8 to 12 mm. Response amplitudes typically ranged from 3 to 10 mm.

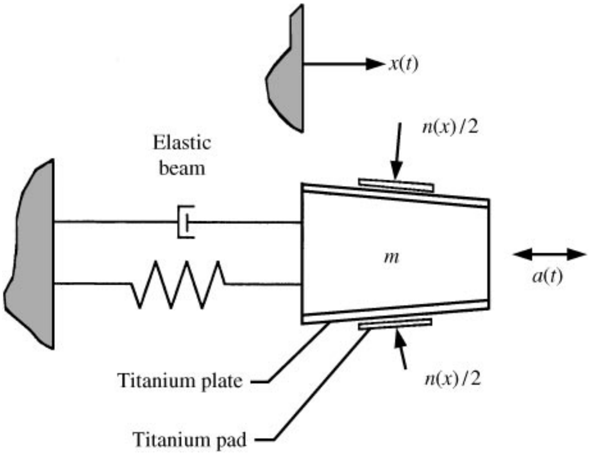

Figure 1. A mechanics model of a mass–beam system. The elastic beam is represented by a spring and dashpot. The base excitation produces an inertial force, represented by $a(t)$.

The high-frequency dithering excitation was added to the low-frequency excitation by summing two signal generators into a single source for the electromagnetic shaker. High-frequency excitations were applied in the range of 500–700 Hz.

The low-frequency response of the shaker, with the oscillator attached, was measured using a linear variable differential transformer (LVDT) attached to the oscillator frame. The high-frequency response of the same system was recorded using an accelerometer. The frequency responses were matched at 100 Hz (not near a resonance), so that the gain between the accelerometer signal and the LVDT signal was estimated. This gain was used to estimate the order of magnitude of the amplitude ratio of the high- and low-frequency components of the excitation in the dither experiments.

3. EXPERIMENTAL RESPONSES

We present a typical result. A driving frequency of 3·625 Hz, with a shaker amplitude of 10 mm, put the oscillator into chaos. The phase portrait and power spectral density of the strain-gage signal are shown in Figure 2. After adding a 700 Hz excitation component with an amplitude estimated as 3×10^{-3} mm, the response changed to a stable period two. The dynamics with dither is represented by a phase portrait and power spectral density of the

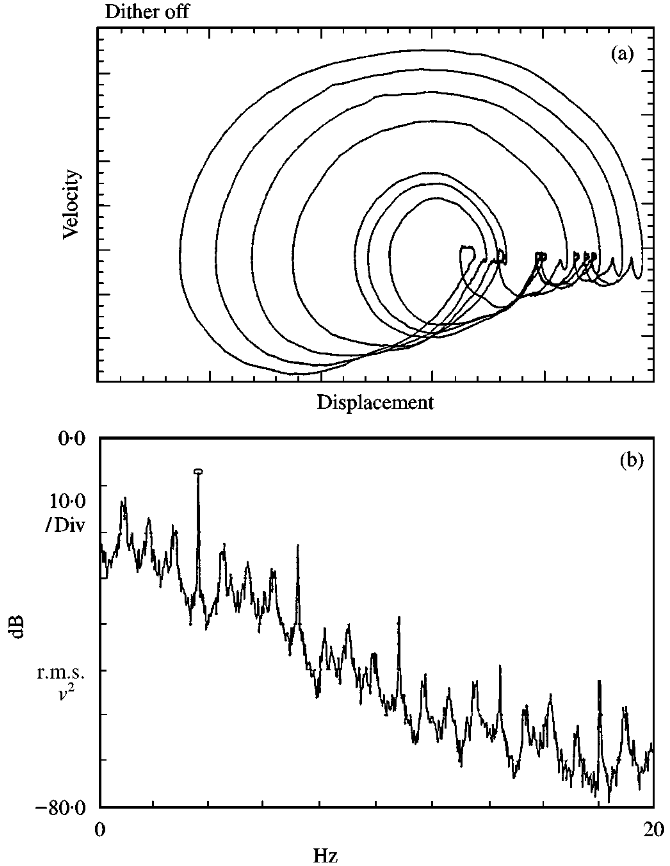

Figure 2. (a) A phase portait of the chaotic oscillator when no high-frequency excitation is applied; the trajectories evolve clockwise, (b) the spectrum of the response signal.

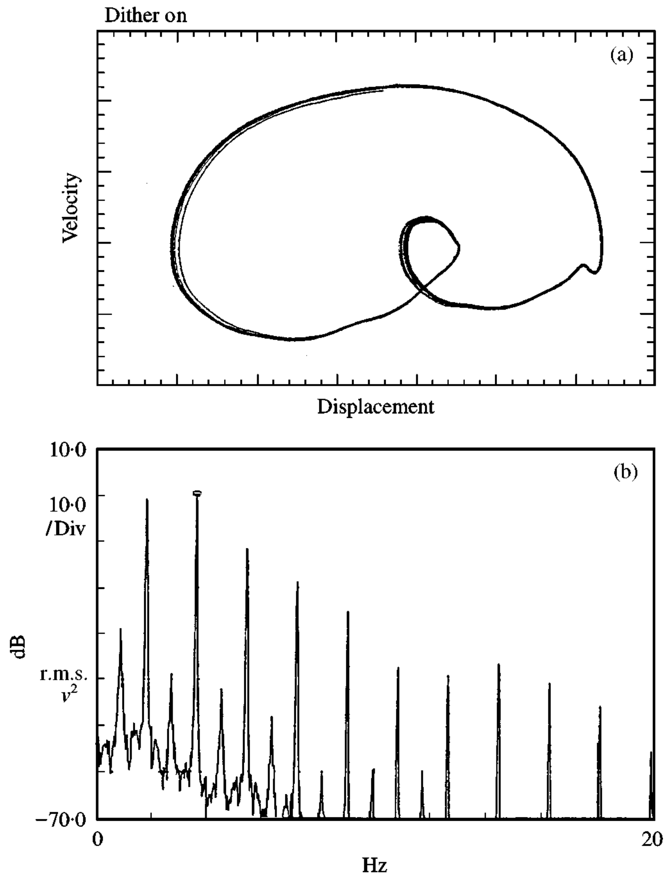

Figure 3. (a) A phase portait of the oscillator when high-frequency excitation is applied, (b) the spectrum of the response signal.

strain-gage signal in Figure 3. The broadband component of the power spectral density of the dithered signal was 30 dB lower than that of the chaotic undithered signal in the low-frequency range. The primary spike and the first subharmonic spike of the oscillation and dither were 7 and 6 dB higher than the primary harmonic of the chaotic oscillation without dither. Hence, much of the energy added to the excitation by the high-frequency component, and energy associated with the broadband frequency component of the chaotic oscillation, were channelled into the response at the driving frequency and the first subharmonic.

In most cases, the application of dither did not quench chaos until the magnitude of the high-frequency excitation reached a certain threshold. This threshold was not measured accurately, but for the example presented, it was of the order of magnitude of the dithering signal used. Sometimes a small change in the dithering frequency brought the system back to chaos, in which case an adjustment in the dithering amplitude usually removed the chaos again.

Chaos quenching was easily repeated in this system. Sometimes period-one motion resulted. There were settings for which we were unable to quench chaos with dither, probably because the dithering amplitude required was too large for our instrument settings.

4. MODEL AND ANALYSIS

In this section, we describe how the dither smoothens the discontinuity in the friction law, which likely removes the low-frequency stick–slip effects, by averaging the system over a high-frequency period. To keep the model simple, we focus on the simple Coulomb-plus-static friction law. We neglect any state-variable friction effects or contact compliance. It is possible that the presence of these effects would be significant when dither is applied, as such effects have been seen to influence dynamical responses (see, e.g., references [21–24]). Also, contact compliance has been shown to increase in significance as a single-source driving frequency increases [25].

Modelling the system with a single degree of freedom, despite the range of excitation frequencies (consistent with observations in the reaction–wheel system [7], we did not see significant excitation of higher modes), the non-dimensionalized equation of motion [17] is

$$\ddot{x} + 2\zeta\dot{x} + x + (1 + kx)f(\dot{x}) = a\omega^2 \sin(\omega t + \psi) + b\Omega^2 \sin \Omega t, \tag{1}$$

where x is the non-dimensional mass displacement relative to the moving support, constants a and b represent the amplitudes of base excitation components at the low and high frequencies ω and Ω, k quantifies the displacement dependence of the normal load, and the coefficient of friction is defined as $f(\dot{x}) = -1$, $\dot{x} < 0$ and $f(\dot{x}) = -1$, $\dot{x} > 0$, for sliding friction, and $|f(0)| \leqslant f_s$ for static friction. ψ accommodates an arbitrary phase angle between the two excitation components. Viscous damping is neglected ($\zeta = 0$).

We assume that $\Omega \gg \omega$, and ω is $O(1)$. This scaling of parameters, and the way they effect the excitation amplitudes, are consistent with the analysis performed by Thomsen [16]. As such, we can follow Thomsen's averaging analysis. To this end, we seek a response of the form

$$x(t) = z(t) + \Omega^{-1}\phi(t, \Omega t), \tag{2}$$

where $z(t)$ is the low-speed component of the response, and $\Omega^{-1}\phi(t, \Omega t)$ is the high-speed component of the response, which is modulated by a small parameter Ω^{-1}. We enforce $\phi(t, \Omega t)$ to have a zero mean over the high-frequency period, such that

$$\tilde{\phi} = \frac{1}{2\pi}\int_0^{2\pi} \phi(t, \Omega t)\, \mathrm{d}(\Omega t) = 0. \tag{3}$$

Treating the two time scales, t and Ωt, as independent variables, we have $\mathrm{d}x/\mathrm{d}t = \dot{z} + \Omega^{-1}\dot{\phi} + \phi'$, where $\dot{\phi} = \partial\phi/\partial t$, and $\phi' = \partial\phi/\partial(\Omega t)$. However, the high-speed component turns out not to exhibit slow-time behavior in the first order of approximation. So at this point we simplify things and let $\phi(t, \Omega t) \approx \phi(\Omega t)$. Inserting this response into equation (1), we obtain

$$\ddot{z} + \Omega\phi'' + z + \Omega^{-1}\phi + F(z + \Omega^{-1}\phi, \dot{z} + \phi') = b\omega^2 \sin(\omega t + \psi) + a\Omega^2 \sin \Omega t, \tag{4}$$

where $F(x, \dot{x}) = (1 + kx)f(\dot{x})$. Reordering equation (4) yields

$$\phi'' = a\Omega \sin \Omega t - \Omega^{-1}[\ddot{z} + z + F(z + \Omega^{-1}\phi, \dot{z} + \phi') - b\omega^2 \sin(\omega t + \psi)] + O(\Omega^{-2}).$$

Thus, at the first order, $\phi'' \approx a\Omega \sin \Omega t$, and so $\phi \approx -a\Omega \sin \Omega t$. No constants of integration are retained since ϕ has a zero average over a high-frequency cycle.

The total response, based on equation (2), is

$$x = z(t) - a \sin \Omega t + O(\Omega^{-1}).$$

We can determine the averaged slow-time behavior by examining z in the $O(1)$ terms of equation (4):

$$\ddot{z} + z + F(z + \Omega^{-1}\phi, \dot{z} + \phi') = b\omega^2 \sin(\omega t + \psi).$$

We average this equation by integrating over a high-frequency period. As such,

$$\ddot{z} + z + \bar{F} \approx b\omega^2 \sin(\omega t + \psi),$$

where

$$\bar{F} = \frac{1}{2\pi}\int_0^{2\pi} (1 + k(z + \Omega^{-1}\phi)) f(\dot{z} + \phi')\, \mathrm{d}(\Omega t).$$

Under the assumption that Ω is very large, we have approximated the slow-varying terms z, $\dot{z}$, and $\sin(\omega t + \psi)$ as constant during the integration interval.

Since f is discontinuous, we have two cases to examine for the averaging integrations. Looking at $f(\dot{z} + \phi') = f(\dot{z} - a\Omega \cos(\Omega t))$, we have $f = \pm 1$ throughout the integral if $|\dot{z}| > a\Omega$. Otherwise, $f = -1$ for $\Omega t < \Omega t_1$ and $\Omega t > \Omega t_2$, and $f = +1$ for $\Omega t_1 < \Omega t < \Omega t_2$, where $\Omega t_1 = \arccos(\dot{z}/a\Omega)$, and $\Omega t_2 = 2\pi - \Omega t_1$. Using these relationships, we perform the integrals and obtain

$$\bar{F} = +(1 + kz), \quad \dot{z} > a\Omega, \tag{5}$$

$$\bar{F} = -(1 + kz), \quad z < -a\Omega, \tag{6}$$

$$\bar{F} = (1 + kz)\left[1 - \frac{2}{\pi}\arccos\left(\frac{\dot{z}}{a\Omega}\right)\right], \quad |\dot{z}| < a\Omega. \tag{7}$$

So as a result of averaging over a high-speed period, the friction term, which had the discontinuous form $F(x, \dot{x}) = (1 + kx)f(\dot{x})$, is transformed to the continuous form $\bar{F}(z, \dot{z}) = (1 + kz)f(\dot{z})$, $|\dot{z}| > a\Omega$, and $\bar{F}(z, \dot{z}) = (1 + kz)(1 - (2/\pi)\operatorname{arc\,cos}(\dot{z}/a\Omega))$, $|\dot{z}| \leqslant a\Omega$. The averaged friction coefficient $\bar{f}(\dot{z})$ is shown in Figure 4. The figure shows that the averaged friction relationship has no discontinuity, and that an increase in $a\Omega$ (meaning an increase in dither amplitude or frequency) accentuates the removal of the discontinuity. The dithered friction characteristic is consistent with analyses of similar non-linearities [12, 13, 15]. In the case of friction, the result is impervious to the static friction f_s.

This result is also similar to that of Thomsen [16], whose friction lacked the displacement dependence but had further velocity dependence. In addition to the morphing of the friction from discontinuous of continuous, Thomsen had also noted a qualitative change in the velocity dependence, which effected the stability of steady sliding. For our purposes, the significant change in the friction characteristic is the loss of the discontinuity. The friction behaves like a non-linear viscous friction. Based on our experience with this oscillator, we expect the smoothening of the discontinuity to remove the non-linear behaviors associated with stick–slip (or "near stick–slip") for some threshold value of $a\Omega$, as observed in the experiments. As the damping is still non-linear, a period-two response is a reasonable possibility. We admit the possibility that unmodelled frictional or modal effects may also be involved in dither quenching.

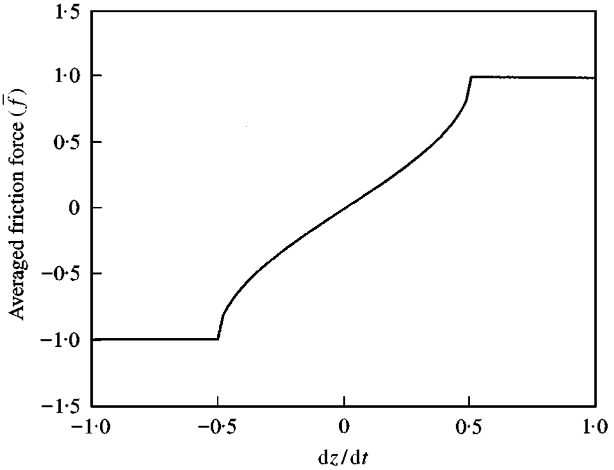

Figure 4. A plot of the effective friction coefficient $\bar{f}$ after averaging. In this case, $a\Omega = 0{\cdot}5$. For slow-scale speed, $|\dot{z}| < a\Omega$; the friction is non-linearly viscous. The friction saturates for velocities of larger magnitude.

5. CONCLUSION

We have applied high-frequency excitation, or dither, to significantly alter the low-frequency forced behavior of an experimental stick–slip friction oscillator. In particular, we have quenched chaos inherent to a stick–slip oscillator by effectively removing the stick–slip aspect of the behavior. This observed behavior is supported by averaging analysis which suggests that high-frequency excitation effectively removes the discontinuity for the low-frequency behavior.

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