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AN EXTENSION OF FORCE APPROPRIATION TO THE IDENTIFICATION OF NON-LINEAR MULTI-DEGREE OF FREEDOM SYSTEMS

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Classical force appropriation methods are used in the identification of linear systems to determine the multi-point force vector that will induce single-mode behaviour, thus allowing each normal mode to be identified in isolation. This paper presents an extension to this linear approach that will enable the approximate identification of multi-degree-of-freedom (d.o.f.) systems with weak non-linearity on a similar basis. The classical linear modal model is used, with additional terms included to represent the direct non-linear restoring forces of the system. Using this force appropriation for non-linear systems (FANS) method, a force vector with harmonics present is derived using an optimization approach such that the response of the system is restricted to that of a target mode, but in the non-linear region. The response obtained from several force levels is then curve fitted using the restoring force method applied in linear modal space so as to yield the direct linear and non-linear modal parameters for the target mode. The method is applied to a simulated two-d.o.f. example and good agreement is found between estimated and true parameters. An extension to the identification of critical non-linear modal cross-coupling terms is proposed.

1. INTRODUCTION

Some form of modal testing [1] is commonly performed on a wide range of structures in order to validate the mathematical dynamic model used for prediction of response, stability etc. Classical linear approaches for the identification of the model may be categorised as *phase separation* or *phase resonance* methods. Phase separation methods involve some form of curve fit to experimental data presented in the time or frequency domains, whereas phase resonance methods seek to excite and identify each undamped normal mode of the structure, one at a time. In the aerospace industry, the use of phase resonance methods is still common, though the process is often termed normal mode tuning or force appropriation [2–4]. Force appropriation methods permit the determination of a set of monophase forces for multiple exciters that will induce single mode behaviour when applied at the relevant undamped natural frequency. This process is otherwise known as "tuning" a mode. Once a normal mode has been tuned, the mode shape, modal damping, modal mass and undamped natural frequency may be estimated. Methods for determination of the

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appropriated force vector are iterative or direct, the latter being based upon the measured frequency response function matrix.

The force appropriation approach works fairly well for linear multi-degree-of-freedom (d.o.f.) systems. However, in practice, structures have non-linear characteristics that can influence their dynamic behaviour significantly. In the aircraft industry, for which this paper is particularly relevant, non-linearity can influence response, loads and flutter (stability). Problems emerge in using force appropriation when the structure is non-linear because the method fails to tackle the change of dynamic behaviour with force level and the model derived is still pseudo-linear, giving different results at different levels. In the aerospace industry, a common current practice is for non-linearity to be explored experimentally by seeking to "tune" modes at some pseudo-resonance condition using different force levels and to observe the change of tuned frequency with force level. This approach is inadequate, because it fails to give information that could be used reliably for prediction of non-linear effects. Also, the response shape alters because other modes become involved in the response once the behaviour becomes non-linear. Some information about the *type* of non-linearity present may be found from such an approach, but no *model* of the non-linearity is obtained. It would be extremely beneficial if a non-linear dynamic multi-d.o.f. model of an aircraft could be derived experimentally; the impact of non-linearity upon response, loads and flutter could then be assessed. However, any non-linear model derived should ideally be easy for the engineers to relate to and should fit into existing predictive methodologies. In this paper, the objective is to present such a method. Its application could extend beyond the aerospace sector to other structures with weak non-linear characteristics.

Whilst the identification of linear multi-d.o.f. systems is well established, that for non-linear systems is still relatively under-developed. A number of methods permit the *presence* of non-linearity to be detected and others give an indication of its *type*. However, few methods yield a non-linear mathematical *model* that can be used to reproduce the response of the system, and most of these models are not readily used and understood in practice by engineers.

The NARMAX method [5] yields a non-parametric discrete time model and is time-consuming for multi-input/multi-output use due to the combinatorial explosion of model terms. Also, it does not lend itself to physical interpretation of the model. A multi-input/multi-output multi-d.o.f. non-linear NARMAX model of an aircraft would be enormous and would not relate readily to other methodologies used in the aerospace industry. The higher order frequency response method [6] allows the translation of NARMAX models into continuous time and also eases the problem of obtaining non-linear system responses; however, it is also difficult to use and interpret physically. The linearized frequency domain approach [7] is promising and can yield a physical parameter model with additional linearized stiffness and/or damping terms; it has only been applied to relatively small systems so far. The selective sensitivity approach [8] can be used to reduce the dimension of the model space through the use of selective excitation, somewhat akin to force appropriation, and the method has recently been extended from linear systems to non-linear systems up to third order. The dimensional reduction permits the entire system to be identified via a sequence of low-dimensional estimation problems, but the method awaits experimental validation. A further approach is that of identifying the so-called non-linear normal modes [9, 10]; these arise from a non-linear transformation applied in order to uncouple the known equations of motion and these "modes" assist in understanding the non-linear behaviour of systems. However, they are difficult to relate to physically because they are not the same as the normal modes of the linear system. Some methods to identify non-linear modes are under development.

However, the restoring force surface (RFS) method, sometimes called the force-state mapping method, [11–16] allows a parametric model for a non-linear single or multi-d.o.f. system to be estimated. One benefit of the RFS approach for multi-d.o.f systems is that it permits use of the *classical linear modal model*, in which modal mass and linear stiffness terms are uncoupled and the linear modes may be coupled non-linearly by additional terms in the equations. The linear part of this model is precisely what is used in the aerospace industry, amongst many others. It is arguable that none of the other non-linear identification methods provide a model as recognizable and useable by the engineer because classical linear modal space is so commonly used.

The RFS method has been demonstrated experimentally only on systems with low numbers of d.o.f. Unfortunately, in practice, structures have a large number of d.o.f. often with a high modal density. A classical RFS identification of such structures in modal space would involve data in which all modes and cross-couplings were present potentially. It would then lead to a very complex mathematical model with an unknown non-linear model structure and with very many parameters to determine.

For these reasons it would be advantageous to extend the classical linear force appropriation approach to non-linear systems in order to reduce the scale of the identification problem. In this paper, an approach is presented [17–19]—force appropriation for non-linear systems (FANS)—which allows a special appropriated force vector to be derived that will result in a non-linear response. The structure is made to respond dominantly in the target linear mode shape, thereby permitting the direct non-linear characteristics of that mode to be identified in the absence of cross-coupling effects; any important cross-coupling terms would then be identified subsequently using a variant of the approach.

It will be shown that the special appropriated force vector can be derived using the Volterra series [20] if the system parameters are known *a priori*, or using an optimization routine in the case of a general identification. Once the contribution of the coupled modes has been significantly reduced by judicious choice of excitation, the non-linear characteristics of the target mode can be identified using any single-d.o.f non-linear identification method. In this paper, the RFS method is used to examine and identify the non-linear behaviour of the target mode.

The proposed approach therefore permits the scenario in which modes can be divided into those that behave nominally linearly, and can therefore be estimated using the classical linear approach, and those which are influenced by non-linearity, and whose direct modal terms may be identified using the FANS methodology. Any key non-linear cross-coupling terms between particular modes may be treated separately. The outcome of the approach is therefore an extension to the classical tuning process.

The FANS approach is demonstrated in concept using a d.o.f. simulated system with a spring grounded cubic stiffness non-linearity, such that the linear modes are coupled non-linearly. The modes may be made close in frequency. Results are compared to those achieved using a classical linear force appropriation approach when it is applied at higher force levels. Comments as to the range of applicability of the method are included.

2. THEORETICAL BASIS OF NON-LINEAR FORCE APPROPRIATION (FANS)

2.1. BASIC MULTI-DEGREE OF FREEDOM APPROACH

If a general N-d.o.f. non-linear dynamic system is considered, the equations of motion in *physical space* (i.e., in terms of coordinates measured by translational sensors) can be written as

$$[\mathbf{m}]\{\ddot{\mathbf{x}}\} + [\mathbf{c}]\{\dot{\mathbf{x}}\} + [\mathbf{k}]\{\mathbf{x}\} + \{\mathbf{g}_{nl}(\{\mathbf{x}\}, \{\dot{\mathbf{x}}\})\} = \{\mathbf{f}\}, \tag{1}$$

where $[\mathbf{m}]$, $[\mathbf{c}]$ and $[\mathbf{k}]$ represent the mass, damping and stiffness matrices for the *linear* part of the system, $\{\mathbf{x}\}$ is the $(N \times 1)$ vector of the physical displacements, the dot represents the time derivative, $\{\mathbf{f}\}$ is the vector of physical forces, and $\{\mathbf{g}_{nl}(\{\mathbf{x}\}\{\dot{\mathbf{x}}\})\}$is the vector of the *non-linear* restoring forces.

Any N-dimensional co-ordinate system related to these physical co-ordinates by an appropriate transformation can be used. However, the main systems of interest for structural dynamics are the above-mentioned physical co-ordinates, and *modal co-ordinates* that are commonly used to decouple linear systems, i.e., to diagonalize the parameter matrices. Modal co-ordinates are obtained by a linear transformation that is sufficient, for linear systems only, to convert to N-S.d.o.f. systems, provided the damping is proportional (i.e., the damping matrix is a linear combination of the mass and stiffness matrices). For non-linear systems, a decoupling transformation is generally not available. However, approximations can be obtained using the method of normal forms [21]. The difference between that method and the FANS method proposed here is that FANS produces an exact decoupling of the equations of motion but only for a specific input and at the expense of losing some of the non-linear interaction terms.

Equation (1) can then be transformed to *linear modal space*, using the modal matrix of the underlying linear system, and the classical modal transformation

$$\{\mathbf{x}\} = [\boldsymbol{\phi}]\{\mathbf{u}\}, \tag{2}$$

where $[\boldsymbol{\phi}]$ is the $(N \times n)$ modal matrix and $\{\mathbf{u}\}$ is the $(n \times 1)$ vector of modal co-ordinates. Note that $n < N$, so allowing for a reduction in the size of the model. The transformation yields the following equations expressed in linear modal space,

$$[\mathbf{M}_u]\{\ddot{\mathbf{u}}\} + [\mathbf{C}_u]\{\dot{\mathbf{u}}\} + [\mathbf{K}_u]\{\mathbf{u}\} + \{\mathbf{G}_{unl}(\{\mathbf{u}\}, \{\dot{\mathbf{u}}\})\} = \{\mathbf{p}\}, \tag{3}$$

where $[\mathbf{M}_u]$ and $[\mathbf{K}_u]$ are the diagonal $(n \times n)$ modal mass and stiffness matrices, $[\mathbf{C}_u]$ is the modal damping matrix, which is diagonal if the linear component of the damping is assumed to be proportional, and $\{\mathbf{p}\}$ is the applied modal force vector. $\{\mathbf{G}_{unl}\}$ is the vector of non-linear modal restoring forces expressed in linear modal space co-ordinates. In general, $\{\mathbf{G}_{unl}\}$ will be a function of several of the modal displacements and/or velocities; for example, for the jth mode, the non-linear modal restoring force is represented by the jth element of $\{\mathbf{G}_{unl}\}$ that can be written as

$$G_{unlj}(u_1, u_2 \ldots u_n, \dot{u}_1, \dot{u}_2 \ldots \dot{u}_n). \tag{4}$$

Thus equations (3) are only coupled by the non-linear modal restoring forces. The linear parts of the equations are not coupled provided that damping is proportional.

It is important to recognize that when a "mode" is referred to in this paper, what is meant is the mode of the linear part of the system. These "linear modes" will in general be coupled non-linearly by the non-linear restoring forces. There is no approximation involved in this representation, apart from the modal truncation, commonly found in modal analysis.

The aim of the non-linear force appropriation method (FANS) described in this paper is to design a modal excitation vector $\{p\}$, and eventually a physical vector $\{f\}$, that will excite a linear mode of interest (say the jth mode) into the non-linear region. The excitation will also eliminate the response of any other modes that are non-linearly coupled to it. The non-linear behaviour of a single mode may then be examined and identified. The non-linear modal restoring force in mode j will then be represented simply by

$$G_{unlj}(u_j, \dot{u}_j). \tag{5}$$

Then the system will respond non-linearly in the jth linear mode shape since all other modal contributions are absent, i.e., $u_i = 0$ when $i \neq j$.

Under this type of excitation, the system modal equations could then be written as

$$\begin{aligned} &\vdots \qquad\qquad \vdots \\ &G_{unlj-1}(u_j, \dot{u}_j) = p_{j-1} \\ m_j\ddot{u}_j + c_j\dot{u}_j + k_ju_j + {}&G_{unlj}(u_j, \dot{u}_j) = p_j \\ &G_{unlj+1}(u_j, \dot{u}_j) = p_{j+1} \\ &\vdots \qquad\qquad \vdots \end{aligned} \tag{6}$$

It may be seen that the response in all modes apart from the jth mode has been eliminated. The jth mode equation has the usual direct linear terms, together with direct non-linear terms. If a modal force pattern, $\{p\}$, could be determined that would cause the system to behave in this way, a single-d.o.f. RFS identification could then be carried out to identify the direct linear and non-linear terms for the jth mode. The jth mode of a non-linear system will then have been "appropriated". Any significant non-linear coupling terms could be identified later using a variant of the approach.

For a system with *known* parameters, the approach for obtaining the appropriated vector $\{p(t)\}$ to excite the jth mode would be to assume that $p_j(t)$ was sinusoidal at frequency ω (rad/s), obtain the response $u_j(t)$ from the jth equation in equation (6) (e.g. using the Volterra series), and then evaluate $p_i(t)$ $(i \neq j)$ by substituting $u_j(t)$ into the other equations in equation (6). Note that $u_j(t)$ and $p_i(t)$ $(i \neq j)$ would include a series of harmonics of ω that would need to be truncated. For a system with *unknown* parameters, the excitation would need to be derived using an optimization approach.

2.2. ILLUSTRATION OF APPROACH FOR 2 d.o.f. SYSTEM

The application of such an approach will now be illustrated *algebraically* using a 2d.o.f. non-linear system for which the first mode is to be excited and identified. It is assumed that all the parameters are *known a priori*, which of course will not be the case in practice. Such an example is included to allow the idea of the FANS method to be seen more clearly.

The system chosen is shown in Figure 1 and is deliberately designed to be symmetric in its linear components so as to yield very simple mode shapes. The mass, damping and stiffness values are m, c and k. The factor a can be altered to influence the degree of physical coupling between the masses and therefore the separation of the natural frequencies, so allowing very close modes to be represented. The system has a so-called "spring grounded" non-linear cubic stiffness governed by the parameter β.

The equations of motion in *physical space* can be shown to be

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (1+a)c & -ac \\ -ac & (1+a)c \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} (1+a)k & -ak \\ -ak & (1+a)k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} \beta x_1^3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}. \tag{7}$$

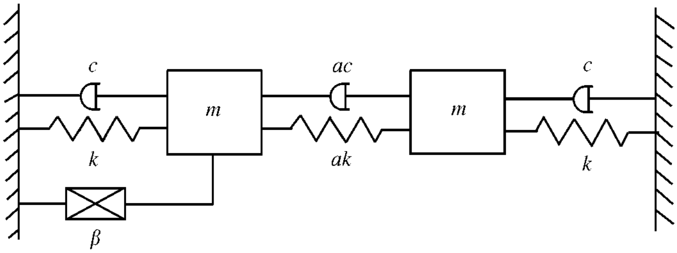

Figure 1. Two-degree-of-freedom system.

The modal matrix of the underlying linear system is

$$\phi = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{8}$$

where the columns define the two linear mode shapes.

Transforming these equations to *linear modal space* gives

$$\begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2c & 0 \\ 0 & 2(1+2a)c \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & 2(1+2a)k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$+ \begin{Bmatrix} \beta(u_1 - u_2)^3 \\ -\beta(u_1 - u_2)^3 \end{Bmatrix} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}. \tag{9}$$

Note that a non-linear identification based directly on this simple model would involve several non-linear coupling terms, which may or may not be significant in affecting the system response [11, 14, 15].

Now consider isolating mode 1 by constraining the response in mode 2 to be zero by suitable choice of an appropriated force vector. The determination of this excitation for an *unknown* system will be covered in a later section. Thus, setting $u_2 = 0$ in equation (9) leads to the two equations (similar to equation (6))

$$m\ddot{u}_1 + c\dot{u}_1 + ku_1 + (\beta/2)u_1^3 = p_1/2 \tag{10}$$

and

$$\beta u_1^3 = p_2. \tag{11}$$

Now assume that the excitation p_1 is harmonic at frequency ω (rad/s), and is given by

$$p_1(t) = P_{11}\cos(\omega t) \tag{12}$$

where P_{11} is the modal force amplitude, assumed to be known at this stage. For the non-linear single-d.o.f. system described by equation (10), it is well known that the steady state response to this excitation will be given by a series of harmonics, namely

$$u_1(t) = U_{11}\cos(\omega t + \vartheta_{11}) + U_{13}\cos(3\omega t + \vartheta_{13}) + \cdots, \tag{13}$$

where U_{jk} and ϑ_{jk} are the amplitude and phase of the kth harmonic for the jth modal response ($j = 1$ in this case). In this example, the system contains only a cubic stiffness non-linearity so only odd harmonics need be included in the response; if a quadratic non-linearity was present then even harmonics would also appear.

The amplitude and phase values for each harmonic in u_1 could be determined by, for example, a Harmonic Balance carried out on equation (10), assuming that the system parameters and excitation amplitude and frequency are known.

Once the response $u_1(t)$ is determined, it may be substituted into equation (11) and the second modal force $p_2(t)$, required to meet the condition $u_2 = 0$, may also be expressed in the form of a series of harmonics by performing a trigonometric expansion. The result is of the form,

$$p_2(t) = P_{21}\cos(\omega t + \psi_{21}) + P_{23}\cos(3\omega t + \psi_{23}) + \cdots, \tag{14}$$

where P_{jk} and ψ_{jk} are the amplitude and phase of the kth harmonic for the jth modal force.

Thus, the force vector $\{p\}$ defined by equations (12) and (14) will yield a response only in the first linear mode shape. Note that the p_2 excitation involves a number of harmonic terms in addition to the fundamental component. In classical linear force appropriation with proportional damping, the modal force would be $p_2 = 0$. (It should be noted that at this stage, only the modal force vector is being considered; later, the conversion to a physical force vector will be covered.)

If the system were *unknown a priori*, and if the modal forces could be estimated by some other means, then the system could be made to respond in u_1 only. Measurements of this response at several excitation amplitudes could then be used to identify the direct non-linear term $\beta/2$ for mode 1 in equation (10) as well as the direct linear modal terms. The exercise could then be repeated for mode 2. An uncoupled non-linear model for the two modes of the system in linear modal space would then be available. If the non-linear modal couplings were deemed to be important, then they would be identified separately (see later).

The benefits of this approach are not obvious for a 2d.o.f. system. However, for a system with a larger number of modes, a full non-linear identification could involve an enormous number of terms. Instead, it would be possible to use the FANS method to identify the direct non-linear behaviour of a limited number of modes where non-linear behaviour is known to be important. Such an approach is consistent with a normal mode tuning philosophy being extended into the non-linear region. Any key non-linear cross-coupling terms would be estimated separately using an extension of the approach.

2.3. VOLTERRA SERIES APPROACH FOR DETERMINATION OF EXCITATION

In this section, the FANS idea will be illustrated *numerically* for the above 2d.o.f. system using an approach based upon the Volterra series [20]. It is still assumed that the system parameters are known *a priori*, so the modal force pattern may be determined from the system model.

The Volterra series is essentially a functional Taylor series which can represent a large class of non-linear mappings $x(t)$ to $y(t)$. The expansion begins

$$y(t) = \int_{-\infty}^{\infty} h_1(\tau)x(t-\tau)\,\mathrm{d}\tau + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)\,\mathrm{d}\tau\,\mathrm{d}\tau_2 + \cdots, \tag{15}$$

where h_i are the Volterra kernels (or generalized coefficients) of the ith order. The general term in the series is

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)x(t-\tau_1)\ldots x(t-\tau_n)\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n. \tag{16}$$

As for linear systems, non-linear systems have a parallel frequency domain representation in terms of the higher order frequency response functions (HFRFs) which are defined as the Fourier transforms of the Volterra series kernels

$$H_n(\omega_1, \ldots, \omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \mathrm{e}^{-\mathrm{i}(\omega_1\tau_1 + \cdots + \omega_n\tau_n)} \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n. \tag{17}$$

The Volterra series can be used to predict the response of a known non-linear system. The response can be expressed, using the series in the frequency domain, as a combination of the HFRFs. For example, the response to a harmonic input $x(t) = X \cos \omega t$ is

$$y(t) = X|H_1(\omega)| \cos(\omega t + \angle H_1(\omega)) + \frac{X^2}{2}\{|H_2(\omega, \omega)| \cos(2\omega t + \angle H_2(\omega, \omega)) + H_2(\omega, -\omega)\} + O(X^3). \tag{18}$$

The values of the HFRFs for a particular system may be calculated using harmonic probing [22] if the system parameters are known. If these HFRFs and the input force are substituted into the frequency domain response representation, an expression for the response can be derived as a series of harmonics as in equation (18).

The Volterra series was used to calculate mode 1 response and hence a force pattern that would cause the response of the two-d.o.f. non-linear system shown in Figure 1 to be dominated by mode 1. System parameters of $m = 1{\cdot}0$ kg, $k = 3947{\cdot}84$ N/m, $c = 3{\cdot}77$ N s/m, $\beta = 5{\cdot}0 \times 10^9$ N/m^3 and $a = 0{\cdot}22$ were used, giving natural frequencies of 10 and 12 Hz for the underlying linear system; the modal matrix is the same as that in equation (8). A harmonic force $p_1(t)$, as defined in equation (12), was chosen to have an amplitude of $P_{11} = 0{\cdot}05$ N and a frequency of 10 Hz; the low excitation level was used to ensure the convergence of the Volterra series.

The response u_1 of the effective single-d.o.f. system described in equation (10) was calculated in the frequency domain using Harmonic Probing; in particular, the amplitudes and phases of the fundamental, third and fifth harmonics of the response were determined. The second modal force p_2 was then calculated using a trigonometric expansion of this response, as defined in equation (11); only terms up to the fifth harmonic were retained. The modal force vector $\{p\}$ for non-linear force appropriation was then known.

The response to this modal force vector was then simulated in the time domain using the equations of motion of the system written in linear modal space, namely equation (9). These equations were solved using a fourth order Runge–Kutta routine where the time step was 0·0001 s; the relative contributions of the two modes were then evaluated to demonstrate that the second mode response was nominally zero.

By way of comparison, the force vector for tuning mode 1 using the classical linear force appropriation approach would include a finite modal force in mode 1 but a zero value in mode 2; this vector was also applied to the same system. It should be noted that this excitation has no harmonics. In this case, a response involving only mode 1 would be expected if the system were linear. The presence of non-linearity will detract from the effectiveness of the classical linear method.

To demonstrate how well the force vector for the non-linear appropriation has reduced the contribution of the second mode compared to the classical linear appropriation approach, the ratios of the root mean square (r.m.s.) displacement of the modal responses of

TABLE 1

Comparison of ratio of mode 1/Mode 2 response to linear and non-linear appropriation for the 2 d.o.f. Non-linear system

		Force vector				
Approach	r.m.s. u_1/r.m.s. u_2	P_{11} (N)	P_{21} (N)	ψ_{21} (rad)	P_{23} (N)	ψ_{23} (rad)
Linear force appropriation	88·68	0·05	0·0	0·0	—	—
Non-linear force appropriation (Volterra)	121 684 (includes 5th harmonic)	0·05	0·0043	1·674	0·0014	5·024

modes 1 and 2 are shown in Table 1. For brevity, only the fundamental and third harmonic components of the force patterns are shown. It can be seen from this table that the special excitation derived using the Volterra series has reduced the response of mode 2 significantly, when compared to the classical approach. The r.m.s. ratio with the fifth harmonic omitted was 121 680, an almost identical value. Similar results were obtained at different excitation frequencies and force levels, up to the limit of convergence for the Volterra series.

Clearly, the FANS approach has succeeded in all but eliminating mode 2 response in comparison to the linear approach. The performance of the linear approach would deteriorate yet further as the force level, and the degree of non-linearity in the response, increased.

3. OPTIMIZATION APPROACH TO FORCE VECTOR DETERMINATION

It has been shown in the previous section that the Volterra series can successfully be used to calculate a modal force vector that will enhance the mode of interest and reduce the contribution of any coupled modes. However, as discussed earlier, it is only possible to use the Volterra series to calculate this modal force pattern if the linear and non-linear system parameters are known. However, for a real identification problem, a model of the system will not exist *a priori*, so such an approach will not be feasible.

However, it should be possible to use an *optimization approach* to estimate the amplitude and phase components of the modal force vector that would minimize the contribution of any coupled modes and so enhance the response of the target mode. In this section, such a methodology is considered. It is interesting to note that there is a corresponding family of equivalent linear force appropriation methods, each based upon *iterative* adjustments of the force vector [3].

3.1. OBJECTIVE FUNCTION IN MODAL SPACE

The objective function for the optimisation must be an expression that will be minimized when the contribution of any coupled modes is minimized and that of the mode of interest maximized. Again consider the 2d.o.f. example given above. In this case, an objective function that was a ratio of the two RMS modal displacements would fulfil this

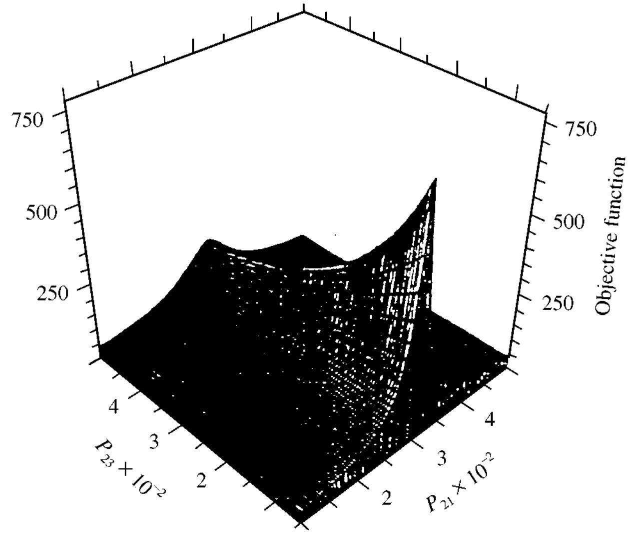

Figure 2. Variation of objective function with P_{21} and P_{23}.

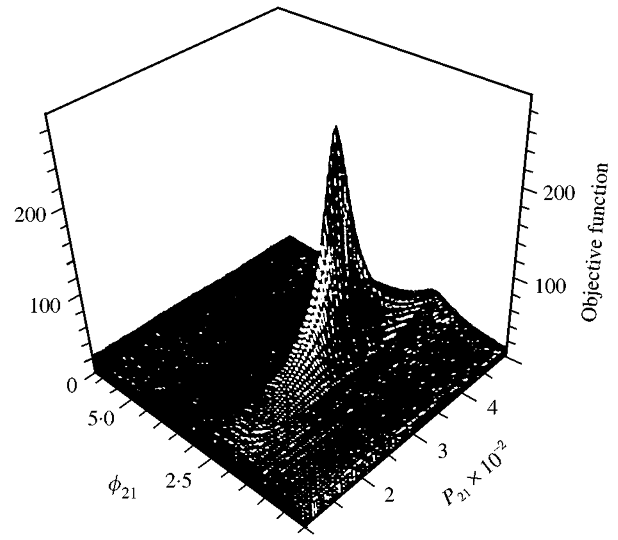

Figure 3. Variation of objective function with P_{21} and ϕ_{21}.

requirement, namely

$$J_1 = \text{r.m.s. } u_2/\text{r.m.s. } u_1 \tag{19}$$

where the r.m.s. quantities would be calculated from one or more cycles of the fundamental response components. This function will be minimized if the response is dominated by mode 1.

If it is assumed that the fundamental and third harmonic components of p_2 would be sufficient to represent the force vector, then the number of variables in the optimization for

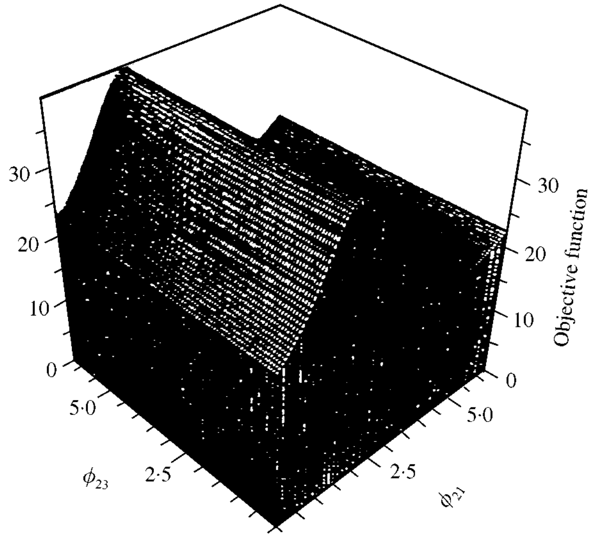

Figure 4. Variation of objective function with ϕ_{21} and ϕ_{23}.

the 2d.o.f. example may be reduced considerably. A four dimensional optimization will result, with the variables being P_{21}, ψ_{21}, P_{23} and ψ_{23}.

As an example to show the objective function for this problem, P_{11} has been fixed at 0·1 N to give noticeable non-linear behaviour, and the fundamental excitation frequency was chosen to be 10 Hz, the natural frequency of mode 1 of the linear system. Surfaces that demonstrate the variation of the objective function with these variables are shown in Figures 2–4; for each surface, two variables were held constant whilst the remaining two were varied. As the objective function in this case was a function of four variables, these surfaces only represent part of its behaviour.

3.2. OPTIMIZATION METHODS

Several optimization methods are possible and were evaluated in this work, namely the Variable Metric [23], Downhill Simplex [23], Genetic Algorithm [24], and Simulated Annealing [25] approaches. These methods were chosen to represent different types of optimization scheme. They were applied to the 2d.o.f. system in order to investigate which method would give the best results in terms of minimizing the objective function and the number of iterations required.

The *Variable Metric* approach is a *gradient method*, so it requires that the function to be minimized is continuous and differentiable. It will also converge only to the nearest minimum, which may be global or local. The other methods are *directed random searches* and as such can find a global minimum because they search over varying areas of the solution space. The *Downhill Simplex* method requires only that the function is continuous whereas the *Genetic Algorithm* and *Simulated Annealing* methods can even optimize a function that only exists at discrete locations in the variable space. (This can be advantageous when non-linear systems are considered because the optimization will not fail if a discontinuity occurs, such as that caused by a “jump” for a system with a cubic stiffness non-linearity.)

TABLE 2

Comparison of results from optimization methods and volterra series for the 2d.o.f. non-linear system

Optimization method	No. of function evaluations	Optimized objective function r.m.s. u_1/r.m.s. u_2	Optimized modal force vector			
			P_{21} (N)	P_{23} (N)	ψ_{21} (rad)	ψ_{23} (rad)
Genetic algorithm	40×10^3	691	0·00426	0·00074	1·7995	5·2943
Simulated annealing	1127	3300	0·00428	$2{\cdot}25 \times 10^{-5}$	1·6646	3·6544
Downhill simplex	526	7584	0·00434	0·001250	1·6746	4·4193
Variable metric	457	$4{\cdot}2 \times 10^6$	0·00434	0·001452	1·6746	5·0240
Volterra series	—	$1{\cdot}87 \times 10^5$	0·00434	0·001457	1·6746	5·0240

It should be noted that the type of minimum found is not important in this case. It is only important that the minimum found is sufficiently deep since the depth of the minimum is related to the reduction in the contribution of the coupled mode(s). Full explanations of these well-known optimization methods may be found in the references cited above.

The four approaches were compared using the same model. The contribution of mode 2 to the response of the non-linear 2d.o.f. system was to be minimized and the response of mode 1 enhanced. The first modal force was chosen to be 0·05 N and the excitation frequency was 10 Hz. The linear force appropriation results were used as a starting point for each optimization. A Runge–Kutta simulation was carried out using the force vector results for each iteration of the optimization, in order to obtain the steady state response of the system to the modal force vector.

The minimum objective function reached by the optimization, the resulting optimized modal force vector, and the number of function evaluations, are shown in Table 2 for all four approaches; the force vector derived using the Volterra series in section 2 above is included for comparison.

It can be seen from this table that the optimized force vectors are similar to those calculated using the Volterra series. The force pattern derived using the variable metric method was actually extremely close to that calculated from the Volterra series. In fact, the only significant deviation is in the fourth significant figure on P_{23} but this difference is still sufficient to boost the objective function by an order of magnitude.

This process was repeated [17] for a range of force levels, damping levels and closeness of natural frequencies (varying a to yield natural frequencies of 10 and 10·12 Hz). In all cases the characteristics of the results in Table 2 were seen. The Genetic Algorithm and Simulated Annealing methods performed badly and required far more function evaluations. The Downhill Simplex method sometimes used somewhat fewer evaluations than the Variable Metric, but the latter method consistently yielded by far the best objective function. The Variable Metric method was therefore used in the examples that follow.

4. ESTIMATING SYSTEM PARAMETERS FROM APPROPRIATED RESPONSES

After the application of the optimization approach, a single-d.o.f. identification method may be used since the response will then be dominated by the target mode. In this work, the RFS method [11, 14, 15] was used, based on a modal space model. The idea of the method is that the modal restoring force may be estimated from knowledge of the modal force, the modal acceleration and the modal mass (usually assumed or estimated from some other approach). The modal restoring force may then be plotted against the modal velocity and displacement and a least-squares curve fit performed to provide a mathematical model.

The effect of coupled mode(s) will have been minimized so the modal restoring force surface is that for a single mode and will yield visual information about the type of non-linearity contained in the mode of interest, provided a suitably high level of excitation is used. This information can be used to reduce the number of terms in the curve fit of the modal restoring force surface.

The nature of the FANS method means that the excitation must be harmonic. This type of excitation can lead to poor curve fit results, due to the problem of linear dependence [25], if the level of excitation is not high enough to induce harmonics in the response. To ensure that suitable levels of excitation were included, a multiple sine type of excitation was used in which harmonic excitation at several amplitude levels was applied. An optimization was performed to yield a response dominated by the target mode at each excitation level. The modal restoring force time histories from all amplitude levels were then used simultaneously in a least-squares curve fit. The method of mass estimation developed by Worden and Tomlinson [16] was used; here an initial modal mass value is estimated and an error term is included in the curve fit model.

4.1. SIMULATED EXAMPLE IN MODAL SPACE

The proposed method was applied to the first mode of the non-linear 2d.o.f. system. Optimized excitation patterns were calculated for several values of P_{11}, namely 0·05, 0·075, 0·1, 0·25, 0·5, 0·75 and 1·0 N with the simulations being performed in modal space. All the minimum objective functions all had values above 10 000. The composite restoring force surface for mode 1, obtained from all the force level data, and a stiffness projection of the surface, are shown in Figures 5 and 6. The shape of the surface projection is consistent with a cubic stiffness non-linearity for mode 1.

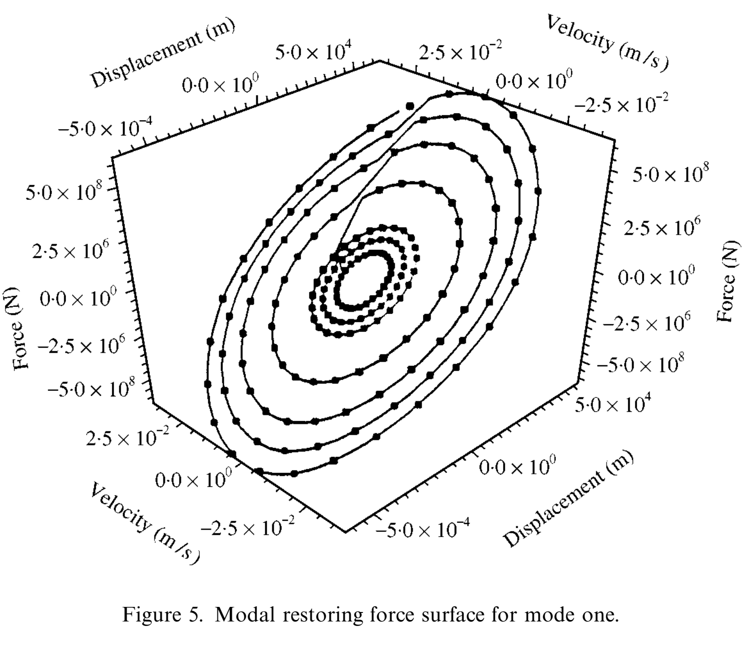

Figure 5. Modal restoring force surface for mode one.

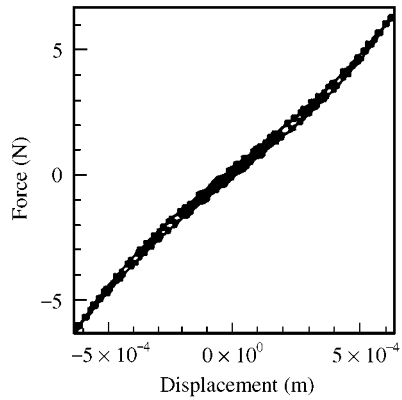

Figure 6. Stiffness projection of model restoring force for mode one.

TABLE 3

Parameter estimates for mode 1 for the 2 d.o.f. non-linear system

Parameter	True parameter	Estimated parameter
k (N/m)	3947·84	3962·9
c (Ns/m)	3·77	3·53
β (N/m^3)	$5{\cdot}0 \times 10^9$	$4{\cdot}99 \times 10^9$
m (kg)	1·0	1·0

Mode 1 restoring force was then curve fitted against modal displacement and velocity to give the direct linear and non-linear parameters for that mode, and hence the physical parameters deduced from equation (10). A comparison between the estimated and true parameters obtained is shown in Table 3.

It can be seen that good agreement was achieved between estimated and true parameters. Slight differences would be due to (a) not allowing a steady state condition to be reached to a sufficiently close tolerance and (b) only including third harmonic terms in the series representations for the modal force. The former is believed to be the prime reason. Damping is always the parameter that is most sensitive to error in any identification and so it is not surprising that this is where the error is greatest.

5. NON-LINEAR APPROPRIATION IN PHYSICAL SPACE

5.1. POSSIBLE APPROACHES

So far, for simplicity, the analysis has been performed in *modal space*, with modal forces derived and modal displacements used in the objective function. However, in practice, measurement transducers only provide data in physical space and forces must be applied via physical exciters.

The approach proposed here would be to force the system to respond in the physical linear mode shape for the target mode, using a revised objective function that only requires knowledge of the linear physical mode shape for that mode. The optimization would then be performed using amplitude and phase parameters for the physical forces. The number of physical forces to be applied would, in the first instance, be equal to the number of modes that are effective in the frequency range of interest (as for linear appropriation). However, this choice of the number and location of exciters for larger and more complex systems needs further consideration, given that other modes at multiples or fractions of the excitation frequency may be excited for a non-linear system.

Clearly, constraining the response of the system to be in one of its linear mode shapes requires that this mode shape be known before the optimisation is performed. In the general experimental case, the mode shapes will not be known. However, for most types of non-linearity, the responses of the system at low amplitude levels will tend to the response of the underlying linear system (the problem non-linearity would be friction, where the non-linearity is most evident at low response levels). Thus, if each mode shape of interest for the system could be measured at low excitation levels using a classical force appropriation method, this would provide a reasonable approximation to the desired linear mode shape. The appropriation may then be extended into the non-linear range by letting the forces include harmonic terms that are then optimized to minimize the objective function.

When the optimization was carried out in modal space, harmonics were only required on the second modal force. When physical forces are transformed to modal space, the transformation $\{\mathbf{p}\} = [\boldsymbol{\phi}]^{\mathrm{T}}\{\mathbf{f}\}$ is used. However, inverting this transformation using $\{\mathbf{f}\} = [\boldsymbol{\phi}^{\mathrm{T}}]^{-1}\{\mathbf{p}\}$ gives the physical force vector in terms of the modal force vector.

It was shown earlier for the 2d.o.f. example that $p_1(t)$ consists of a fundamental component only, whereas $p_2(t)$ is made up of a series of harmonics. Thus the physical forces will, in general, *both* be represented by a series of harmonics, so typically

$$f_1 = F_{11}\cos\omega t + F_{13}\cos(3\omega t + \chi_{13}) + \cdots, \tag{20}$$

$$f_2 = F_{21}\cos(\omega t + \chi_{21}) + F_{23}\cos(3\omega t + \chi_{23}) + \cdots, \tag{21}$$

where F_{jk} and χ_{jk} are the amplitude and phase of the kth harmonic for the jth *physical* force. F_{11} is the reference force and its value is set in order to determine the amplitude level of the excitation, in rather the same way as for P_{11} in the modal space approach. The fundamental and third harmonics were again included so the optimization in physical space for this 2d.o.f. system will now have six variables, compared to four when performed in modal space. It might be argued that this increase in number of variables means that the method would be more efficient performed in modal space. However, the difference in number of variables becomes less significant as the number of modes and exciters increases. The issue of maximizing the efficiency and robustness of the non-linear appropriation method will require further research for larger systems.

5.2. OBJECTIVE FUNCTION IN PHYSICAL SPACE

When the optimization is performed in physical space, normally for systems with more than 2d.o.f., the objective function used in equation (19) must be modified.

The Euclidean distance, or l_2 norm [26], is often used to compare two or more vectors. It was therefore thought that this could be used to compare the deviation of the actual response shape from the target mode shape. Each displacement response could be divided by the relevant element from the mode shape of interest so that when the response was exactly the same as the required mode shape, the norm would be zero. The norm for *n* transducers would then be given by

$$J_2 = \sqrt{\frac{\sum_{i=1}^{npts} (x_{1i}/\phi_1 - x_{2i}/\phi_2)^2 + (x_{1i}/\phi_1 - x_{3i}/\phi_3)^2 + \cdots + (x_{1i}/\phi_1 - x_{ni}/\phi_n)^2}{npts}}, \quad (22)$$

where ϕ_1, ϕ_2 etc., are the elements of the linear mode shape vector for the target mode. This summation is carried out over one cycle of the fundamental response component, where *npts* is the number of data points per cycle and x_{ki} is the *k*th physical displacement response at the *i*th time sample. In this equation, x_1 was chosen as the reference displacement, but any measurement point could be chosen. This objective function allows the response to contain harmonics, but all points will respond in the proportions of the required mode shape. The use of the modal assurance criterion (MAC) was also tried, but it was found to be less sensitive, and therefore less effective, than the l_2 norm.

5.3. SIMULATED 2 d.o.f. EXAMPLE IN PHYSICAL SPACE

The objective function J_2 was used in the physical space optimization in order to enhance the contribution of mode 1 of the 2d.o.f. system. The optimization was performed by adjusting amplitude and phase parameters of the physical force vector, using a range of values for the reference force F_{11}. The resulting time histories were then transformed to linear modal space for the restoring force identification, using the inverse of the modal matrix (the orthogonal properties of the modal matrix could be used instead if preferred).

The optimized physical force time histories for the highest excitation level are shown in Figure 7, with the resulting physical displacement response time histories shown in Figure 8. It is clear that the ratio of the two responses agrees visually with the required mode shape, i.e., in the proportion $\{1\ 1\}$. The transformed modal displacement responses for modes 1 and 2 are shown in Figure 9. It can be seen that the contribution of mode 2 is significantly less than that of mode 1; the peak displacement in mode 1 is around 5×10^{-4} m, whereas that for mode 2 is approximately 9×10^{-12} m. The modal restoring force surface for mode 1 is indistinguishable by eye from that in Figure 5 as might be expected.

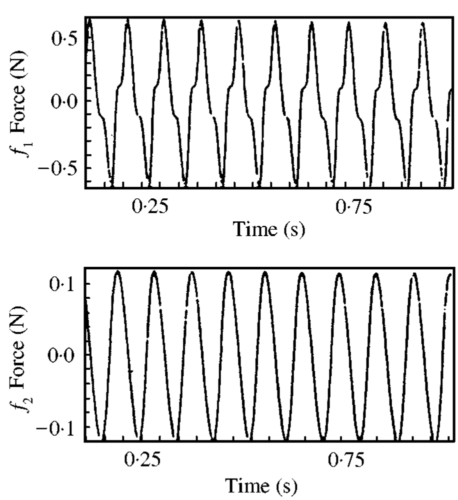

Figure 7. Optimized physical input forces.

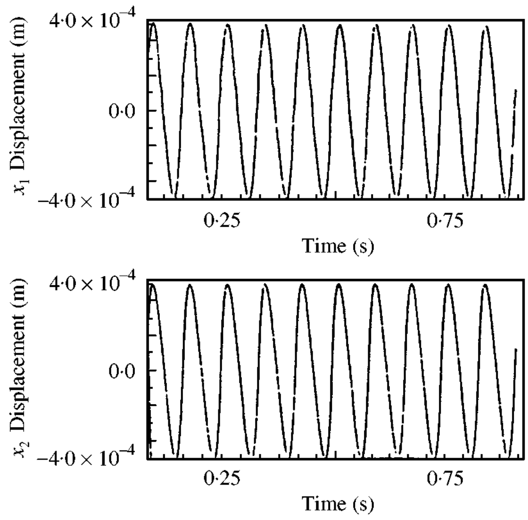

Figure 8. Physical displacement time histories.

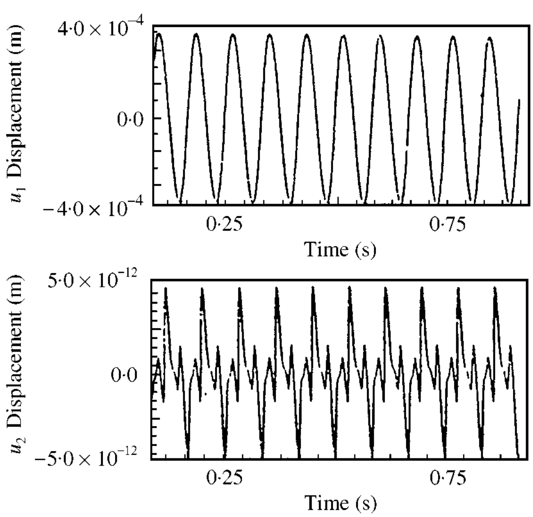

Figure 9. Modal displacement time histories.

The modal restoring force time history for mode 1 was then curve fitted, and the direct linear and non-linear estimated parameters are shown in Table 4.

It can be seen that the general level of the percentage error was reasonably low, but results were not quite as good as the results in Table 3. This is probably because the optimization involves more variables and is therefore more sensitive but failure to achieve the steady state condition would also have an effect. The equivalent mode 2 results were equally good.

Further simulations were performed with the natural frequencies being closer, namely 10 and 10·12 Hz as before. Results were found to be of a similar quality.

In addition to the 2d.o.f. example presented in this paper, systems with 3d.o.f., and containing damping non-linearity, have been identified. The results of these identifications

TABLE 4

Parameter estimates for mode 1 *for the* 2 *d.o.f. non-linear system-physical space approach*

Parameter	True parameter	Estimated parameter
k (N/m)	3974·84	3874
c (N s/m)	3·77	3·09
β (N/m^3)	$5{\cdot}0 \times 10^9$	$4{\cdot}96 \times 10^9$
m (kg)	1·0	0·97

are given in Reference [17], and generally good parameter estimates were found. An exception was the case where a zero element appeared in one of the mode shapes for the 3d.o.f. case. If the expression for the objective function in equation (22) is considered, it is clear that the presence of such an element will cause a "division by zero" in the calculation of this objective function. A possible solution would be to replace the zero element with a very small element; however, it can be seen that this will cause the expression for the objective function to be very badly scaled and thus to work less effectively. A better solution may be to disregard the response for any zero or very small element, enforcing the mode shape using the remaining co-ordinates. Alternatively, some weighting chosen to be proportional to the amplitude could be employed to suppress the effect of small responses.

6. COMPARISON TO CLASSICAL LINEAR FORCE APPROPRIATION

Having developed the FANS approach, how much better is it than the classical linear force appropriation in handling the tuning and identification of non-linear multi-d.o.f. systems?

To illustrate the value of the proposed non-linear method, the classical force vector for tuning mode 1 was derived from the Multivariate Mode Indicator Function method for linear force appropriation [2] using frequency response function data for the 2d.o.f. system at low excitation levels. The force vector $\{1\ 1\}$ was then applied at the natural frequency of mode 1 and at the same input force levels as used for the non-linear approach (note that no harmonics were present). The resulting modal restoring forces for mode 1 were then evaluated and curve fitted, ignoring any effect of mode 2. This identification was carried out at different separations of the linear natural frequencies of the system; the natural frequency of the second mode was chosen to be 10·12, 12 and 21 Hz in turn. As the frequency separation of the modes increases, the effect of the non-linear modal coupling will reduce and the application of the force vector from the linear appropriation method should produce better results. The results for each case are shown in Table 5.

It can be seen from this table that this appropriated force vector only begins to give good estimates for the direct linear and non-linear terms when the second linear natural frequency is 21 Hz. In contrast, the FANS method has been found to give good parameter estimates regardless of the frequency separation of the modes.

The inaccuracy in the parameter estimates for the system with close natural frequencies is because the non-linear cross-coupling terms were not counteracted sufficiently by this simple force vector. The stiffness projection of the modal restoring force surface of mode 1 for the classical appropriation is shown in Figure 10. It can be seen that the cubic profile of

Table 5

Parameter estimates for mode 1 for the 2d.o.f. non-linear system using a classical linear force appropriation approach

Parameter	True parameter	Estimated parameters		
		Natural frequency 10·00, 10·12 Hz	Natural frequency 10·00, 12·00 Hz	Natural frequency 10·00, 21·00 Hz
k (N/m)	3947·84	4181·20	4114·59	3961·27
c (Ns/m)	3·77	6·21	3·83	3·04
β (N/m^3)	$5{\cdot}0\times10^9$	$9{\cdot}88\times10^7$	$4{\cdot}12\times10^9$	$5{\cdot}12\times10^9$
m (kg)	1·0	1·0	1·0	1·0

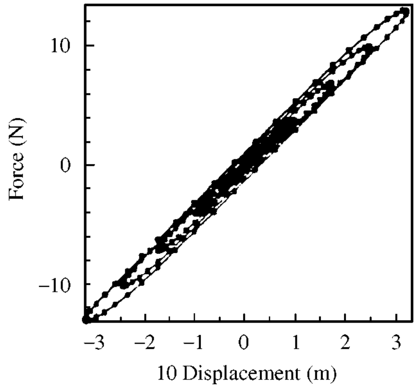

Figure 10. Stiffness protection of modal restoring force surface resulting from application of appropriated force vectors for mode one, very close modes.

the stiffness projection has been distorted by the presence of modal coupling terms from mode 2 contribution to the response.

7. ESTIMATION OF NON-LINEAR COUPLING TERMS

In the previous identifications, only direct linear and non-linear modal terms are identified by the proposed method, since any coupling terms are counteracted when the contribution of the coupled modes is minimized. Any significant non-linear couplings for a system could be identified using a similar optimization approach.

For example, if the coupling between mode j and mode k of a system is required, then the appropriation could be altered so as to excite both modes simultaneously, with all other modes suppressed. Excitation could be at one or other natural frequency, or both, and the identification focused on the cross-coupling terms since the direct terms will already be known. Such an approach would minimize the size of the identification at each stage of the test.

This combined response condition could be enforced using a modified objective function. This topic will be the subject of further research, as will be the development of a criterion to indicate which modes are significantly cross-coupled non-linearly.

8. CONCLUDING REMARKS

In this paper, an extension of the classical linear force appropriation method has been proposed for non-linear multi-d.o.f. systems. Using an optimization approach, a force vector can be determined such that a target linear mode of the system will respond in isolation by counteracting non-linear couplings to other modes. If this process is repeated at several force levels, a restoring force surface identification may be used to identify the direct linear and non-linear terms in the equation of motion for that mode. Any significant non-linear cross-coupling terms may be identified separately. The new method was demonstrated on a 2d.o.f. non-linear system. The method was found to provide significantly better estimates than the classical method when close modes were considered.

The analysis given in this paper is restricted to weakly non-linear systems where the term "weak" means here that a sinusoidal force will only produce superharmonic components in the response. In the situation, where subharmonics are present in the response, the method presented should be applicable if the amplitudes and phases of the subharmonics are included in the appropriated forces with coefficients determined by the optimization. There is an implicit assumption throughout that an acceptable degree of accuracy can be obtained using a finite number of terms in the force representation. In the limit of a period doubling bifurcation, where there is essentially a continuum of subharmonics, the method will fail.

An issue that must be addressed in future concerns the stability of the solutions under small perturbations but discussion is postponed until a later publication because the success of the experimental study [17] lends support to the robustness of the method.

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APPENDIX: ABBREVIATIONS

DOF	degrees of freedom
FANS	force appropriation of non-linear systems
HFRF	higher order frequency response function
MAC	modal assurance criterion
NARMAX	Non-linear auto regressive moving average using xogenous inputs
RFS	restoring force surface
RMS	root mean square