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AUTHORS' REPLY

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For an historical evolution of the "reverse path" spectral method in the context of single-degree-of-freedom (d.o.f.) systems, the interested reader can and should refer to the extensive literature review provided by Bendat [1]. For instance, a quote from this citation [1] concerning the work of early investigations follows: "Rice and Fitzpatrick [1988, 1991] wrote two outstanding articles dealing with useful techniques for non-linear system analysis and identification that were developed following the works of Bendat and Piersol [1982, 1986a], Bendat [1983], and Vugts and Bouquet [1985], but independently of other related work by Bendat [1985], Bendat and Palo [1989, 1990], and Bendat *et al.* [1990, 1992]."

Since we developed our method for specific application to multi-d.o.f. systems [2], we did not feel the need to duplicate the extensive literature review on the identification of single-d.o.f. systems. Formulation of the "reverse path" spectral technique has been thoroughly covered by Bendat in a well-known textbook [3]; therefore, it is unnecessary to cite all parties involved with the development and application of the method. Consequently, only the most recent work on the development of the method has been referenced in our publications [2, 4] so that readers may adequately follow the analytical treatment. In summary, we have followed the accepted practices of citing and utilizing the scientific literature.

Concerning the extension of the "reverse path" spectral method towards multi-d.o.f. systems, there are two distinct approaches. Each approach is formulated from the generalized set of coupled differential equations of motion; see equations (1a–c) and (2) of reference [2] and equations (1)–(3) of reference [5]. These equations are essentially Newton's second law of motion applied to a generic non-linear vibration system. Consequently, initial formulation of the methods should be similar. However, beyond these basic governing equations that set the stage for further analysis, there exist fundamental differences between the two methods which concern the inputs and identified paths of the "reverse path" model. For example, Rice and Fitzpatrick formulate the "reverse path" approach by treating each response $X_i(\omega)$ of the response vector $\mathbf{X}(\omega) = [X_1(\omega)\, X_2(\omega)\, \dots\, X_i(\omega)\, \dots\, X_N(\omega)]^T$. Where N is the number of d.o.f., as separate inputs to the "reverse path" model. Additional inputs are non-linear scalar functions resulting from multiplying out and collecting terms of like form of functions describing the non-linear restoring forces. The paths of each input $X_i(\omega)$ are elements $B_{ij}(\omega)$ of the dynamic stiffness matrix $\mathbf{B}(\omega)$. Frequency response functions of each element $B_{ij}(\omega)$ are identified and correspond to systems of order 0, 1 and 2. The frequency response functions of the second order systems are then inverted resulting in frequency response functions similar to those of a single-d.o.f. mechanical oscillator. Elements of the mass, damping and stiffness matrices are then estimated from the single-d.o.f. frequency response functions using single-d.o.f. curve-fitting techniques. This procedure is discussed at the end of section 2.2 of reference [5] where it is stated that "Once the linear operators denoted by R are estimated and inverted they will appear as familiar linear single-d.o.f. frequency response functions. The constituent mass, damping and stiffness may be found using standard fitting procedures." This part of the procedure is what we describe in the introduction of our article [2] when referring to Rice and Fitzpatrick's work [5] as follows: "A similar approach has been used for the identification of two-d.o.f. non-linear systems where each response location is considered as a single-d.o.f. mechanical oscillator." We go on to mention that their technique requires excitations to be applied at every response location in order to fully identify the system. this limitation is overcome by our approach; and in fact, identification of non-linearities away from locations of applied excitations is a critical issue that is clearly emphasized and discussed in our articles [2, 6].

Unlike their method [5], we formulate the "reverse path" approach from the generalized set of coupled differential equations of motion by retaining the system response in vector form, i.e., $\mathbf{X}(\omega) = [X_1(\omega)\, X_2(\omega)\, \dots\, X_i(\omega)\, \dots X_N(\omega)]^T$. This response vector $\mathbf{X}(\omega)$ is then used as a vector input to the "reverse path" model. Therefore, the path whose input is $\mathbf{X}(\omega)$ is the entire dynamic stiffness matrix $\mathbf{B}(\omega)$. The multi-d.o.f. dynamic compliance matrix $\mathbf{H}(\omega)$ is then identified by first applying spectral conditioning [2, 7] to obtain an equivalent model with uncorrelated inputs, and then re-reversing the path corresponding to the dynamic stiffness matrix $\mathbf{B}(\omega)$. As a result, $\mathbf{H}(\omega)$ is independent of the dynamical effects from the non-linearities; for example, natural frequencies are independent of the excitation levels. Therefore, modal parameters can be estimated from $\mathbf{H}(\omega)$ using well-known multi-d.o.f. modal parameter identification methods [8], as opposed to the single-d.o.f. technique employed by Rice and Fitzpatrick as discussed earlier. Although not all of the elements $H_{ij}(\omega)$ of the matrix $\mathbf{H}(\omega)$ will be identified since excitations at all response locations are not required, reciprocity may be employed to obtain additional elements, i.e., $H_{ij}(\omega) = H_{ji}(\omega)$. This step allows for identification of non-linearities at locations away from applied excitations. One additional difference between the two approaches is that we retain the functions describing the non-linear restoring forces in their original form and these functions become the additional inputs to the "reverse path" model. Consequently, each spectrum is itself an approximation of a coefficient and not a combination of the many coefficients resulting from multiplying out and collecting terms of like form. This eliminates the need to resolve the original coefficients from algebraic expressions which may become rather cumbersome for higher order polynomials. Also, we retain functions of the same type (such as quadratic and cubic) in vector form; however, they could be separated and treated as individual inputs.

It should be noted that although we make the assumption in our initial formulation that we know the locations of the non-linearities [2], this assumption is not necessary. The "reverse path" model, as formulated from the assumed equations of motion for describing the physical system, is based on the knowledge an analyst may somehow have concerning the locations of the non-linearities and the associated mathematical equations for describing them. Therefore, one may include functions for describing non-linearities at any location on the physical system. From the identification process, estimated coefficients of insignificant non-linear functions will then be zero. However, computation involved with the identification process may become excessive. Therefore, any *a priori* knowledge of the locations and types of non-linearities should be employed to reduce the complexity of the model, i.e. non-linear functions should only be included where non-linearities are likely to be located and these functions should contain terms with high probability of describing the nature of the non-linear restoring forces. In additional work [4], we address the topic of identifying non-linearities of unknown forms.

Arguably, Rice and Fitzpatrick's method [5] appears to be advantageous since, ideally physical properties are preferred over modal properties. However, as pointed out above, excitations must be applied and measured at each response location for full identification. For systems with a large number of d.o.f. application of excitations at all response locations could be rather difficult task. Furthermore, identification of the physical properties of multi-d.o.f. systems should be questioned, since this is a difficult task even for linear systems as evident from the widespread use of modal techniques [8]. Extensive work has been applied to the linear system identification of multi-d.o.f. mechanical, fluid and structural systems from which the most practical and widely used methods have been formulated based on the modal domain.

In summary, our initial work [2] derives a procedur e to overcome the inability to identify non-linearities at locations away from applied and measured excitations. This process is carried out by treating the response as a vector input to the "reverse path" model and identifying multi-d.o.f. frequency response functions. Consequently, our formulation results in a unique method more suited for identification of non-linear multi-d.o.f. systems with a large number of d.o.f. A separate article [6] further addresses the critical differences between our method [2] and that developed by Rice and Fitzpatrick [5]. Both methods are compared analytically and an example is provided to illustrate the procedures. Finally, we believe that many unresolved research problems still exist in the area of non-linear system identification and expect that current methods [1–6] would form the basis of further work.

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